

Near periodicity and Zhukovskij stability

By CHANGMING DING (Xiamen), ZHEN JIN (Taiyuan)
and JOSÉ M. SORIANO (Sevilla)

Abstract. In this paper, we prove pairwise equivalence between near periodicity, Poisson stability, recurrence and nonwandering of a point with Zhukovskij stability. In a locally compact phase space, a periodic point is nearly periodic if and only if it is Zhukovskij stable. Also, it is shown that for a nearly periodic system each orbit closure is minimal and stable.

1. Introduction

The notion of near periodicity was first introduced in [1]. The equivalence between near periodicity, recurrence, Poisson stability and nonwandering under Lipschitz stability was established in [11]. Obviously, Lipschitz stability is stricter than Lyapunov stability, also, Lyapunov stability is rather restrictive for its isochronous correspondence of orbits. In this paper we generalize the results in [11] under a relaxed concept of stability, i.e., Zhukovskij stability. Meanwhile, we deal with the relationship between periodicity and near periodicity. It is shown that a periodic point is nearly periodic if and only if it is Zhukovskij stable. If a nearly periodic point is Lyapunov asymptotically stable, then it is a rest point. We also show the minimality and stability of each orbit closure for a nearly periodic system. Therefore, near periodicity is not only a recursive notion but also of some stability.

Let (X, d) be a metric space with a prescribed metric d . Denote by $S(x, r) =$

Mathematics Subject Classification: 34C25, 37B20.

Key words and phrases: periodicity, near periodicity, Zhukovskij stability.

This paper is supported by the National Science Foundation of China.

The corresponding author is C. Ding.

$\{y \in X \mid d(x, y) < r\}$ and $S[x, r] = \{y \in X \mid d(x, y) \leq r\}$ the open ball and the closed ball with center x and radius $r > 0$ respectively. In addition, let $H(x, r) = \{y \in X \mid d(x, y) = r\}$ and for a set A in X , define $S(A, r) = \bigcup_{a \in A} S(a, r)$. A dynamical system or continuous flow (X, π) on X is a continuous map $\pi : X \times \mathbb{R} \rightarrow X$ such that $\pi(x, 0) = x$, $\pi(\pi(x, t), s) = \pi(x, t + s)$ for all $x \in X$ and $t, s \in \mathbb{R}$. We suppress the map π notationally and just write $x \cdot t$ in place of $\pi(x, t)$. Similarly, if $A \subset X$ and $I \subset \mathbb{R}$, then $A \cdot I$ is the set $\{x \cdot t \mid x \in A, t \in I\}$, in particular $x \cdot \mathbb{R} = \{x\} \cdot \mathbb{R}$ and $x \cdot \mathbb{R}^+ = \{x\} \cdot \mathbb{R}^+$ are the orbit and the positive semi-orbit, respectively, of a point $x \in X$. A set $Y \subset X$ is positively (negatively) invariant if $Y \cdot \mathbb{R}^+ = Y$ ($Y \cdot \mathbb{R}^- = Y$), and is invariant if $Y \cdot \mathbb{R} = Y$. A nonempty set Y is called (positively) minimal provided it is closed and (positively) invariant, but none of its nonempty proper subsets has these two properties. The limit set, prolongational limit set and prolongational set are defined by $\omega(x) = \{y \in X \mid x \cdot t_n \rightarrow y \text{ for some sequence } \{t_n\} \text{ in } \mathbb{R}^+ \text{ with } t_n \rightarrow +\infty\}$, $J^+(x) = \{y \in X \mid \text{there are sequences } \{x_n\} \text{ in } X \text{ and } \{t_n\} \text{ in } \mathbb{R}^+ \text{ such that } x_n \rightarrow x, t_n \rightarrow +\infty \text{ and } x_n \cdot t_n \rightarrow y\}$ and $D^+(x) = \{y \in X \mid \text{there are sequences } \{x_n\} \text{ in } X \text{ and } \{t_n\} \text{ in } \mathbb{R}^+ \text{ such that } x_n \rightarrow x \text{ and } x_n \cdot t_n \rightarrow y\}$. The negative versions $\alpha(x)$, $J^-(x)$ and $D^-(x)$ are defined similarly by reversing the direction of the time t .

A point $x \in X$ is said to be nonwandering if $x \in J^+(x)$. A point x is called positively (negatively) Poisson stable if $x \in \omega(x)$ ($x \in \alpha(x)$), and x is said to be Poisson stable if it is both positively and negatively Poisson stable. A point x is called recurrent if for each $\epsilon > 0$ there exists a $T = T(\epsilon) > 0$, such that $x \cdot \mathbb{R} \subset S(x \cdot [t - T, t + T], \epsilon)$ for all $t \in \mathbb{R}$. A compact set M in X is positively (negatively) stable provided every neighborhood U of M contains a positively (negatively) invariant neighborhood V of M , i.e., $M \subset V \subset U$ and $V \cdot \mathbb{R}^+ = V$ ($V \cdot \mathbb{R}^- = V$). A set M is called stable provided it is both positively and negatively stable, i.e., each neighborhood U of M contains an invariant neighborhood V of M . It is well-known that a compact set M in a locally compact space is positively (negatively) stable if and only if $D^+(M) = M$ ($D^-(M) = M$), where $D^\pm(M) = \bigcup_{x \in M} D^\pm(x)$.

Definition 1.1. A point $x \in X$ is said to be positively (negatively) nearly periodic if $D^+(x) = \omega(x)$ ($D^-(x) = \alpha(x)$). It is said to be nearly periodic if both $D^+(x) = \omega(x)$ and $D^-(x) = \alpha(x)$ hold.

In [11], Lee introduced the concept of weak near periodicity, i.e., $\overline{x \cdot \mathbb{R}^+} = J^+(x)$ and $\overline{x \cdot \mathbb{R}^-} = J^-(x)$. In fact, it can be verified that weak near periodicity is equivalent to near periodicity. We point out that in [1], [11] recurrence and almost periodicity respectively correspond to Poisson stability and recurrence here.

Remark. AUSLANDER [3], [4] used prolongations to define generalized recurrence and α -stabilities. So, the notion of near periodicity should imply recurrence as well as some stability (see Theorem 3.4). Recall that a dynamical system is topologically transitive if for any two nonempty open sets U and V of X there exists some $t \in \mathbb{R}$ with $U \cap V \cdot t \neq \emptyset$, and a point is transitive if its orbit is dense in X . For a topological dynamical system defined by a continuous map on a compact metric space, we similarly define the notion of positively near periodicity. In that case, it is easy to show that in a topologically transitive system, the transitive points are exactly the positively nearly periodic points. As to the case of real flows, we need many careful and deep considerations to find more interior properties. Actually, the referee suggests an extension to general group actions which include real flows as well as topological dynamical systems about maps. We will do such a generalization in a subsequent paper, since there are interesting researches for connecting our considerations with the existing works.

A dynamical system is said to have a property (pointwise) if every point in it possesses the corresponding property. For example, a nearly periodic system (X, π) means that for each $x \in X$, x is nearly periodic.

2. Zhukovskij stability

Historically, Lyapunov stability, Poincaré (orbital) stability and Zhukovskij stability are pairwise different and perhaps the most important stabilities of solutions of differential equations. The paper [12] presents excellent comparisons and analyses for these kinds of stabilities. Now, we recall the concept of Zhukovskij Stability (see [12] and [9]).

Definition 2.1. A point x in X is called positively (negatively) Zhukovskij stable provided that given any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that for any $y \in S(x, \delta)$, one can find a time parametrization τ_y such that $d(x \cdot t, y \cdot \tau_y(t)) < \epsilon$ holds for $t \geq 0$ ($t \leq 0$), where τ_y is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$ ($(-\infty, 0]$ to $(-\infty, 0]$) with $\tau_y(0) = 0$. Moreover, if $d(x \cdot t, y \cdot \tau_y(t)) \rightarrow 0$ as $t \rightarrow +\infty$ ($t \rightarrow -\infty$) also holds, then the point x is said to be positively (negatively) Zhukovskij asymptotically stable. A point x is said to be Zhukovskij stable if it is both positively and negatively Zhukovskij stable.

For convenience, in order to prove our results, we sometimes use an equivalent statement of Zhukovskij stability. In Definition 2.1, let $\bar{t} = \tau_y(t)$ and $h(\bar{t}) = \tau_y^{-1}(\bar{t}) = t$; it follows that $d(x \cdot t, y \cdot \tau_y(t)) = d(x \cdot h(\bar{t}), y \cdot \bar{t})$. Thus, we restate

positively the Zhukovskij stability of x as follows: For each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $y \in S(x, \delta)$, then one can find a homeomorphism τ_y from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_y(0) = 0$ to keep $d(x \cdot \tau_y(t), y \cdot t) < \epsilon$ for $t \geq 0$. In addition, this is equivalent to the condition that for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in S(x, \delta)$, then one can find two homeomorphisms $h_y(t)$ and $h_x(t)$ from $[0, +\infty)$ to $[0, +\infty)$ with $h_x(0) = h_y(0) = 0$ to keep $d(x \cdot h_x(t), y \cdot h_y(t)) < \epsilon$ for $t \geq 0$.

In Definition 2.1, for two different points y_1 and y_2 in $S(x, \delta)$, $|\tau_{y_1}(t) - \tau_{y_2}(t)|$ ($t \in [0, +\infty)$) may be unbounded even if $d(y_1, y_2)$ is sufficiently small (see Example 3.8 in Section 3). That is, in general, τ_y is not continuous at y . If x is Zhukovskij stable, so is $x \cdot t$ for each $t \in \mathbb{R}$, thus the orbit $x \cdot \mathbb{R}$ can be called Zhukovskij stable.

Remark. Zhukovskij stability is closely related to equicontinuity (see AKIN [2] and GLASNER [10]) in the topological dynamical systems. In fact, observe that if $\tau_y(t) \equiv t$ for each $y \in S(x, \delta)$, then Zhukovskij stability is just Lyapunov stability, and the latter corresponds to equicontinuity.

Theorem 2.2. *If a system (X, π) is positively (negatively) Zhukovskij stable at a point $x \in X$, then $J^+(x) = \omega(x)$ ($J^-(x) = \alpha(x)$).*

PROOF. Let (X, π) be positively Zhukovskij stable at x . Since $\omega(x) \subset J^+(x)$ always holds, it is sufficient to show that $J^+(x) \subset \omega(x)$. Let $p \in J^+(x)$. Then there exist a sequence $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}^+$ and a sequence $\{x_n\}_{n=1}^\infty$ in X such that $x_n \rightarrow x$, $t_n \rightarrow +\infty$ and $x_n t_n \rightarrow p$. Now, given any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that for each $y \in S(x, \delta(\epsilon))$, one can find a homeomorphism τ_y from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_y(0) = 0$ satisfying $d(x \cdot \tau_y(t), y \cdot t) < \epsilon$ for $t \geq 0$. We select an $N > 0$ such that if $n \geq N$, then $d(x, x_n) < \delta(\epsilon)$ and $d(x_n \cdot t_n, p) < \epsilon$. Further, it follows that for $n \geq N$, $d(x \cdot \tau_{x_n}(t), x_n \cdot t) < \epsilon$ holds for $t \geq 0$, where the homeomorphisms τ_{x_n} are defined similarly as τ_y . Thus, we obtain that for $n \geq N$,

$$d(x \cdot \tau_{x_n}(t_n), p) \leq d(x \cdot \tau_{x_n}(t_n), x_n \cdot t_n) + d(x_n \cdot t_n, p) < 2\epsilon,$$

where $\tau_{x_n}(t_n) \rightarrow +\infty$ as $t_n \rightarrow +\infty$. This implies that $p \in \omega(x)$. A similar argument works for the case in brackets. The proof is complete. $\square$

Corollary 2.3. *Let x be positively (negatively) Zhukovskij stable. The point x is nonwandering if and only if it is positively (negatively) nearly periodic. In particular, if a periodic point x is positively (negatively) Zhukovskij stable, then it is positively (negatively) nearly periodic, but the converse is not true.*

PROOF. We only prove the case outside the brackets. If x is nonwandering,

i.e., $x \in J^+(x)$, then we have that $x \cdot \mathbb{R}^+ \subset J^+(x)$ since $J^+(x)$ is invariant, thus $D^+(x) = x \cdot \mathbb{R}^+ \cup J^+(x) = \omega(x)$. Conversely, by $x \in D^+(x) = \omega(x)$, a nearly periodic point is always Poisson stable (i.e., $x \in \omega(x)$), thus it follows that x is also nonwandering. The irrational flow on the torus (see [13, p. 345]) implies that a nearly periodic point may not be periodic, even under Lipschitz stability ([11]). $\square$

In the following, we only prove the results of positive versions of stability and near periodicity, etc. Similar arguments work for the negative versions.

Theorem 2.4. *Let X be locally compact and x be positively Zhukovskij stable. Then, if x is positively Poisson stable, it is positively Lagrange stable, i.e. the closure of its positive semi-orbit $\overline{x \cdot \mathbb{R}^+}$ is compact.*

PROOF. Suppose the contrary: if $\overline{x \cdot \mathbb{R}^+}$ is not compact, then we may find a sequence $\{x \cdot t_n\}_{n=1}^\infty$ ($t_n \rightarrow +\infty$) that has no convergent subsequences. Now, x is a regular point, so there exists a τ -tube U ([5, p. 49]) containing x . Since X is locally compact, let $S[x, 2r]$ ($r > 0$) be a compact ball in U . Thus we may assume that $x \cdot t_n$ does not lie in $S[x, 2r]$ for every n . Because x is positively Poisson stable, there exists a sequence $\{\tau_n\}_{n=1}^\infty$ ($\tau_n \rightarrow +\infty$) with $\tau_1 < t_1 < \tau_2 < t_2 < \dots < \tau_n < t_n < \dots$ such that $x \cdot \tau_n \rightarrow x$, where we suppose that $x \cdot \tau_n \in S(x, r)$. Note that if necessary, we use subsequences. Further, by the continuity of π we can take a sequence $\{\theta_n\}_{n=1}^\infty$ such that for each n , $x \cdot \theta_n \in H(x, 2r)$, $\tau_n < \theta_n < t_n$ and $x \cdot (\theta_n, t_n] \subset X - S[x, 2r]$. We assert that $t_n - \theta_n \rightarrow +\infty$, otherwise there is a subsequence $\{n_k\}$ such that $t_{n_k} - \theta_{n_k} \rightarrow T \geq 0$. Then $x \cdot t_{n_k} = (x \cdot \theta_{n_k}) \cdot (t_{n_k} - \theta_{n_k})$ and $x \cdot \theta_{n_k} \in H(x, 2r)$ imply that $\{x \cdot t_n\}$ has a convergent subsequence, a contradiction. Next, since x is positively Zhukovskij stable, for the above r there exists a $\delta > 0$ ($\delta < r$) such that if $d(x, x \cdot \tau_n) < \delta$, then $d(x \cdot t, (x \cdot \tau_n) \cdot h_n(t)) < r$ holds for $t \geq 0$ and a homeomorphism h_n from $[0, +\infty)$ to $[0, +\infty)$ with $h_n(0) = 0$. Without loss of generality, we assume that for each n , $d(x, x \cdot \tau_n) < \delta < r$. Since $S[x, 2r]$ lies in the τ -tube U , it follows that $\theta_n - \tau_n \leq 2\tau$ for every n , thus there exists a $T = T(\tau) > 0$ such that if $t > T$, then $h_n(t) \geq 2\tau \geq \theta_n - \tau_n$ for each n . In addition, by the continuity of π and Definition 2.1, for each fixed $t > T$ it follows from $t_n - \theta_n \rightarrow +\infty$ that one can take an $n_0 = n_0(t)$ to satisfy $t \leq h_{n_0}^{-1}(t_{n_0} - \theta_{n_0})$. This implies that $h_{n_0}(t) \leq t_{n_0} - \theta_{n_0} \leq t_{n_0} - \tau_{n_0}$, i.e., for $t > T$ there exists an n_0 such that $\theta_{n_0} \leq \tau_{n_0} + h_{n_0}(t) \leq t_{n_0}$. Thus we obtain that $d(x, x \cdot (t_{n_0} + h_{n_0}(t))) \geq 2r$. Hence,

$$d(x, x \cdot t) \geq d(x, x \cdot (\tau_{n_0} + h_{n_0}(t))) - d(x \cdot (\tau_{n_0} + h_{n_0}(t)), x \cdot t) \geq 2r - r = r,$$

and this contradicts that x is positively Poisson stable. This completes the proof. $\square$

Theorem 2.5. (a) *Let (X, π) be periodic system with no rest points. Then it is Zhukovskij stable.* (b) *If the periods of all points in a periodic system (X, π) are bounded, then the system is Zhukovskij stable.*

PROOF. (a) For each $x \in X$, since x is not a rest point, let $T_x > 0$ be its minimal period. Given any $\epsilon > 0$, it follows from the continuity of π that there exists a $\delta > 0$ such that if $y \in S(x, \delta)$, then $d(y \cdot t, x \cdot t) < \epsilon$ for $t \in [0, T_x + 1]$. Also, we may assume that $|T_y - T_x| < 1$ holds for $y \in S(x, \delta)$, where T_y is the period of y . Now, we define a homeomorphism $\tau_y : [0, +\infty) \rightarrow [0, +\infty)$ as follows: For each positive integer n and $t \in [(n-1)T_x, nT_x]$, let

$$\tau_y(t) = (n-1)T_y + \frac{T_y}{T_x}[t - (n-1)T_x].$$

Thus it is easy to see that x is positively Zhukovskij stable. Similarly, we obtain that x is negatively Zhukovskij stable.

(b) There exists a $T > 0$ such that for each $x \in X$, its period $T_x \leq T$ or x is a rest point. By the continuity of π , there exists a $\delta > 0$ such that if $y \in S(x, \delta)$, then $d(y \cdot t, x \cdot t) < \epsilon$ for $t \in [0, T + 1]$. Thus, an argument similar to that of (a) works for this case. $\square$

Corollary 2.6. *If a periodic system (X, π) has no rest points or the periods of all points in X are bounded, then the system is nearly periodic.*

This corollary is obtained in [1, p. 247, Theorem 1.10] and [11, p. 401, Theorem 2.5]. Here its proof follows immediately from Corollary 2.3 and Theorem 2.5.

Theorem 2.7. *If there is a Zhukovskij stable point in a compact minimal set, then each point in the minimal set is Zhukovskij stable.*

PROOF. Let A be a compact minimal set. A point p in A is Zhukovskij stable. Suppose $q \in A$ and let $\epsilon > 0$ be given. Since p is Zhukovskij stable, there is a $\delta = \delta(\epsilon) > 0$ such that for any $y \in S(p, \delta)$, one can find a time parametrization τ_y such that $d(p \cdot t, y \cdot \tau_y(t)) < \epsilon/2$ holds for $t \geq 0$, where τ_y is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_y(0) = 0$. Since A is compact and minimal, it is covered by the family of open sets $\{S(p, \delta) \cdot t \mid t \in \mathbb{R}\}$. Thus, there exists a $\tau \in \mathbb{R}$ such that $q \in S(p, \delta) \cdot \tau$. Choose an $r \in (0, \epsilon)$ such that $S(q, r) \subset S(p, \delta) \cdot \tau$. Now, for each $x \in S(q, r)$, $x \cdot (-\tau)$ and $q \cdot (-\tau)$ lie in $S(p, \delta)$. Then, there are two homeomorphisms $h_x(t)$ and $h_q(t)$ from $[0, +\infty)$ to $[0, +\infty)$ with $h_x(0) = h_q(0) = 0$ such that $d(x \cdot (h_x(t) - \tau), p \cdot t) < \epsilon/2$ and $d(q \cdot (h_q(t) - \tau), p \cdot t) < \epsilon/2$ for $t \geq 0$. It follows that $d(x \cdot (h_x(t) - \tau), q \cdot (h_q(t) - \tau)) < \epsilon$ holds for $t \geq 0$. Thus it is easy to see that q is Zhukovskij stable. The proof is complete. $\square$

3. Near periodicity

In this section, we first consider the equivalence between several recursion notions and generalize the results in [11].

Lemma 3.1 ([5, p. 39]). *For an $x \in X$, let $\overline{x \cdot \mathbb{R}}$ be compact. Then the orbit $x \cdot \mathbb{R}$ is recurrent if and only if for each $\epsilon > 0$ the set $K_\epsilon = \{t \mid d(x, x \cdot t) < \epsilon\}$ is relatively dense.*

Theorem 3.2. *Let X be locally compact and x be Zhukovskij stable. The following conditions on the point x are pairwise equivalent.*

- (1) nearly periodic; (2) recurrent; (3) Poisson stable; (4) nonwandering.

PROOF. It is clear that (2) $\implies$ (3) $\implies$ (4). By Theorem 2.2, we have that $\omega(x) = J^+(x)$ and $\alpha(x) = J^-(x)$, thus (3) $\iff$ (4) holds. (1) $\iff$ (4) is obtained in Corollary 2.3. Then it is sufficient to show that (3) $\implies$ (2). Let x be Poisson stable. By Theorem 2.4, $\overline{x \cdot \mathbb{R}}$ is compact. If x is a rest point, then we are done. So, let x be a regular point. For a given $\epsilon > 0$, suppose that the set $K_\epsilon = \{t \mid d(x, x \cdot t) < \epsilon\}$ is not relatively dense. We assume that there exists a τ -tube U such that $S[x, 2r] \subset S(x, \epsilon) \subset U$ for some $r > 0$ and $S[x, 2r]$ is compact. Thus the set $\{t \mid d(x, x \cdot t) \leq 2r\} (\subset K_\epsilon)$ is not relatively dense either. Now, by the continuity of π and $x \in \omega(x)$ there are sequences $\{\tau_n\}$, $\{\theta_n\}$ and $\{t_n\}$ with $\tau_1 < \theta_1 < t_1 < \dots < \tau_n < \theta_n < t_n < \dots$ such that $x \cdot \tau_n \rightarrow x$, $x \cdot \theta_n \in H(x, 2r)$, $x \cdot t_n \notin S[x, 2r]$, $t_n \rightarrow +\infty$ and $t_n - \theta_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Using the technique in the proof of Theorem 2.4, we get a contradiction to the compactness of $\overline{x \cdot \mathbb{R}}$. It follows that K_ϵ is relatively dense. Therefore, by Lemma 3.1 the proof is complete. $\square$

Remark. Under Lyapunov stability, it follows from [5, p. 108, Theorem 6.10] that we can add almost periodicity to the equivalence in Theorem 3.2. It is not known to the author whether this remains valid under Zhukovskij stability of x . Note that the local compactness of X is only used in proving (3) $\implies$ (2).

In the following, we shall see that near periodicity not only implies recursiveness but also some stability. Meanwhile, we consider the relationship between near periodicity and periodicity.

Lemma 3.3 ([6, p. 136]). *In a locally compact phase space, every positively minimal set is compact.*

Theorem 3.4. *Let (X, π) be a nearly periodic system in a locally compact space. Then for each $x \in X$ the closure $\overline{x \cdot \mathbb{R}}$ is a minimal and stable set.*

PROOF. First, we prove the minimality of $\overline{x \cdot \mathbb{R}}$. Otherwise, $\overline{x \cdot \mathbb{R}}$ has a proper closed invariant subset $A \subset \overline{x \cdot \mathbb{R}}$ with $A \neq \emptyset$. Of course, x does not lie in A . Take a $y \in A$, the near periodicity of x implies that $\omega(x) = D^+(x) = \overline{x \cdot \mathbb{R}}$, hence $y \in \omega(x) \subset J^+(x)$. So it is easy to see that $x \in J^-(y)$. Since A is a closed invariant set, it follows that $y \cdot \mathbb{R}^- \subset A$ and $\alpha(y) \in A$. Now, the near periodicity of the system implies $D^-(y) = \alpha(y) \subset A$ and we obtain that $x \in J^-(y) \subset D^-(y) \subset A$. This contradiction shows the minimality of $\overline{x \cdot \mathbb{R}}$. Further, we assert that $\overline{x \cdot \mathbb{R}}$ is compact. In fact, let $y \in \overline{x \cdot \mathbb{R}} = \omega(x)$, then $J^+(x) \subset J^+(y)$ (see [5, p. 60]). We obtain that $\omega(x) = J^+(x) \subset J^+(y) = \omega(y)$, thus $\omega(y) = \omega(x) = \overline{x \cdot \mathbb{R}}$ for each $y \in \overline{x \cdot \mathbb{R}}$. It follows from [6, p. 133, Lemma 12.3] that $\overline{x \cdot \mathbb{R}}$ is positively minimal, and therefore it is compact by Lemma 3.3. Next, by the near periodicity of x we have that $D^+(x) = \omega(x) = \overline{x \cdot \mathbb{R}} = D^-(x) = \alpha(x)$. It follows that $D^+(\overline{x \cdot \mathbb{R}}) \supset D^+(x) = \overline{x \cdot \mathbb{R}}$. In order to prove that $D^+(\overline{x \cdot \mathbb{R}}) \subset \overline{x \cdot \mathbb{R}}$, we take a $y \in \overline{x \cdot \mathbb{R}}$ and consider $D^+(y) = y \cdot \mathbb{R}^+ \cup J^+(y)$. The minimality of $\overline{x \cdot \mathbb{R}}$ implies that $x \in \omega(y)$, thus $J^+(y) \subset J^+(x)$. We obtain that $D^+(y) \subset y \cdot \mathbb{R}^+ \cup J^+(x) \subset \overline{x \cdot \mathbb{R}} \cup D^+(x) = \overline{x \cdot \mathbb{R}}$. Hence $D^+(\overline{x \cdot \mathbb{R}}) = \overline{x \cdot \mathbb{R}}$ holds, and similarly $D^-(\overline{x \cdot \mathbb{R}}) = \overline{x \cdot \mathbb{R}}$. Thus it follows that $\overline{x \cdot \mathbb{R}}$ is stable. $\square$

Theorem 3.5. *Let x be positively Lyapunov and asymptotically stable. If x is also positively nearly periodic, then it is a rest point.*

PROOF. If x is not a rest point, it follows that there exist positive numbers ϵ and t_0 such that $S(x, \epsilon) \cdot t_0 \cap S(x, \epsilon) = \emptyset$. Since x is positively nearly periodic, it follows that $x \in \omega(x)$, so there exists a sequence $\{t_i\}_{i=1}^\infty \subset \mathbb{R}^+$ such that $t_i \rightarrow +\infty$ and $x \cdot t_i \rightarrow x$, $x \cdot t_i \neq x$. By Lyapunov asymptotical stability of x , it follows that there is a $\delta \in (0, \epsilon)$ such that for any $y \in S(x, \delta)$, $d(x \cdot t, y \cdot t) < \epsilon$ ($t \geq 0$) and $d(x \cdot t, y \cdot t) \rightarrow 0$ as $t \rightarrow +\infty$. Now we choose a $t^1 \in \{t_i\}$ such that $t^1 > t_0$ and $d(x, x \cdot t^1) = \eta < \delta$ ($\eta > 0$). It follows from the continuity of π that there exists a $\theta \in (0, \eta/2)$ such that if $y \in S(x, \theta)$, then $d(x \cdot t, y \cdot t) < \eta/2$ for $t \in [0, t^1]$. Since $x_1 = x \cdot t^1 \in S(x, \delta)$, there is a $t^2 \in \{t_i\}$ with $t^2 > t^1$ such that $d(x, x \cdot t^2) < \theta/2$ and $d(x \cdot t, x_1 \cdot t) < \theta/2$ for $t \geq t^2$. Let $x_2 = x \cdot t^2$ and $x_3 = x_1 \cdot t^2$. Thus, $d(x, x_3) \leq d(x, x_2) + d(x_2, x_3) \leq \theta < \eta/2$. On the other hand, since x_2 lies in $S(x, \theta)$, it follows that $d(x \cdot t^1, x_2 \cdot t^1) < \eta/2$, i.e., $d(x_1, x_3) < \eta/2$ ($x_2 \cdot t^1 = x \cdot (t^2 + t^1) = x_1 \cdot t^2$). Hence, $d(x, x_1) \leq d(x, x_3) + d(x_3, x_1) < \eta$. This contradicts $d(x, x_1) = \eta$, so we conclude that x is a rest point. The proof is complete. $\square$

The irrational flow on the torus implies that a Lyapunov stable and nearly periodic point may not be periodic. So, it is an interesting problem to find a

condition such that a nearly periodic point is periodic. We think that the answer to the following problem is positive.

Problem. Let a dynamical system (X, π) be positively Zhukovskij and asymptotically stable at a positively nearly periodic point $x \in X$. Is the point x periodic?

Lemma 3.6. *Let X be locally compact. If a point $x \in X$ is periodic and positively nearly periodic, then its orbit $x \cdot \mathbb{R}$ is positively stable.*

PROOF. Since x is periodic and positively nearly periodic, we obtain that $D^+(x) = \omega(x) = x \cdot \mathbb{R}$. Further, if a point y lies in the periodic orbit $x \cdot \mathbb{R}$, then $y \cdot \mathbb{R}^+ = x \cdot \mathbb{R}^+$ and $J^+(y) = J^+(x)$. It follows that $D^+(y) = y \cdot \mathbb{R}^+ \cup J^+(y) = x \cdot \mathbb{R}^+ \cup J^+(x) = D^+(x)$. Thus we conclude that $D^+(x \cdot \mathbb{R}) = x \cdot \mathbb{R}$, i.e. $x \cdot \mathbb{R}$ is positively stable. $\square$

Theorem 3.7. *Let X be locally compact. If a periodic point x is also positively nearly periodic, then its orbit $x \cdot \mathbb{R}$ is positively Zhukovskij stable.*

PROOF. First, if x is not a rest point, let $T > 0$ be its period. From Lemma 3.6, it follows that the orbit $x \cdot \mathbb{R} = x \cdot [0, T]$ is positively stable. Given any $\epsilon > 0$, $U = S(x \cdot [0, T], \epsilon)$ is an open neighborhood of $x \cdot [0, T]$. By the tubular flow theorem [5, p. 50], there exists a transversal Σ in $S(x, \sigma) \cdot (-\theta, \theta)$ for some positive numbers σ and θ with $S(x, \sigma) \cdot (-\theta, \theta) \subset S(x, \epsilon)$. Thus it follows that there exists a $\mu \in (0, \sigma)$ such that $d(y \cdot t, x \cdot t) < \sigma$ for $y \in S(x, \mu)$ and $t \in [0, T + \theta]$. By Lemma 3.6, there is a positively invariant open set V such that $x \cdot [0, T] \subset V \subset S(x \cdot [0, T], \mu) \subset U$. Take a $\delta > 0$ such that $S(x, \delta) \subset S(x, \mu) \cap V$. Now, if $y \in S(x, \delta)$, it follows from the definition of μ that the semi-orbit $y \cdot \mathbb{R}^+$ goes back into $S(x, \sigma)$ at time T . Thus, we may suppose that $y \cdot \mathbb{R}^+$ crosses Σ at time $t_1 \in [T - \theta, T + \theta]$. Since V is positively invariant and $V \subset S(x \cdot [0, T], \mu)$, it is easy to see that $y_1 = y \cdot t_1 \in \Sigma \cap V \subset \Sigma \cap S(x, \mu)$, hence the orbit $y_1 \cdot \mathbb{R}^+$ also goes back to Σ at time $t_2 \in [T - \theta, T + \theta]$. Define $y_2 = y_1 \cdot t_2 = y \cdot (t_1 + t_2)$: then by induction we obtain two sequences $\{y_i\}_{i=1}^{+\infty}$ and $\{t_i\}_{i=1}^{+\infty}$ ($\subset [T - \theta, T + \theta]$) satisfying $y_i = y_{i-1} \cdot t_i$ for $i = 2, 3, 4, \dots$. Let $T_i = \sum_{k=1}^i t_k$ and $T_0 = 0$. We define a homeomorphism $\tau_y : [0, +\infty) \rightarrow [0, +\infty)$ as follows: For each positive integer n and $t \in [T_{n-1}, T_n]$, let $\tau_y(t) = (n-1)T + \frac{t - T_{n-1}}{T_n - T_{n-1}}T$. Thus, from $V \subset U$ it is easy to verify that $d(y \cdot \tau_y(t), x \cdot t) < \epsilon$ for $t \geq 0$, i.e. x is positively Zhukovskij stable. Secondly, if x is a fixed point, then for any $\epsilon > 0$ it follows from Lemma 3.6 that there is a positively invariant neighborhood U such that $x \in U \subset S(x, \epsilon)$. Let $\delta > 0$ be such that $S(x, \delta) \subset U$. If $y \in S(x, \delta)$, it is easy to see that $y \cdot \mathbb{R}^+ \subset S(x, \epsilon)$.

Thus we obtain that $d(y \cdot t, x \cdot t) < \epsilon$ for $t \geq 0$, i.e., x is positively Zhukovskij (Lyapunov) stable. The proof is complete. $\square$

Note that in Theorem 3.7 we cannot get Zhukovskij asymptotical stability. Consider a planar system in polar coordinates: $\dot{r} = f(r)$ and $\dot{\theta} = 1$, where $f(r) = r(1-r)^2 \sin \frac{1}{r-1}$ for $r \neq 1$ and $f(1) = 0$. It is easy to see that the orbit $r = 1$ is periodic and nearly periodic. Since there is an infinite number of periodic orbits in any neighborhood of the unit circle, the orbit $r = 1$ is stable but not asymptotically stable. However, if $x \cdot \mathbb{R}$ is also an isolated closed orbit of a planar system, we can prove the uniformly asymptotically Zhukovskij stability (see [8]). Note that for a planar system, if x is positively nearly periodic, then it is periodic, but the converse may not be true.

Now, we present an example that a periodic point is nearly periodic, but not Lyapunov stable and of course not Lipschitz stable.

Example 3.8. Consider a system in $\mathbb{R}^2$ defined by differential equations in polar coordinates: $\dot{r} = r(1-r)^3$ and $\dot{\theta} = r$. The unit circle is an isolated stable periodic orbit. We choose an $r_0 = r(0) > 1$ to fix a solution $r = r(\theta)$ outside of the unit circle $r = 1$. In [7], it is shown that $T = 2k\pi - \int_0^{2k\pi} \frac{d\theta}{\sqrt{2(\theta+\alpha)+1}}$ is the time that $r = r(\theta)$ surrounds the unit circle k times. Let $\beta(k) = \int_0^{2k\pi} \frac{d\theta}{\sqrt{2(\theta+\alpha)+1}}$. Then $\beta(k) \rightarrow +\infty$ holds as $k \rightarrow +\infty$. Thus $r = r(\theta)$ is not isochronously attracted to the solution $r = 1$, since the closed orbit $r = 1$ needs time $2k\pi$ for circling itself k times and the difference of time between solutions $r = r(\theta)$ and $r = 1$ tends to infinity. This example shows that a periodic orbit may attract its neighbors with unbounded time phase.

Theorem 3.9. *Let X be locally compact. A periodic orbit $x \cdot \mathbb{R}$ is nearly periodic if and only if it is Zhukovskij stable.*

PROOF. It follows immediately from Corollary 2.3 and Theorem 3.7. $\square$

ACKNOWLEDGEMENT. The authors sincerely thank the referees for many valuable suggestions. In fact, the referees have suggested Theorem 2.7 and many further important researches (see Remarks after Definitions 1.1 and 2.1).

References

- [1] K. H. AHMAD and S. ELAYDI, Nearly periodic dynamical systems, *Appl. Analysis* **27**(4) (1988), 243–251.

- [2] E. AKIN, On chain continuity, *Discrete Contin. Dynam. Systems* **2**(1) (1996), 111–120.
- [3] J. AUSLANDER, Generalized recurrence in dynamical systems, *Contributions to Differential Equations* (1964), 65–74.
- [4] J. AUSLANDER and P. SEIBERT, Prolongations and stability in dynamical systems, *Ann. Inst. Fourier, Grenoble* **14**(2) (1964), 237–268.
- [5] N. P. BHATIA and G. P. SZEGÖ, Stability Theory of Dynamical Systems, *Springer, Berlin*, 1970.
- [6] N. P. BHATIA and O. HAJEK, Local semi-dynamical systems, Lecture Notes in Mathematics, Vol. 90, *Springer, Berlin*, 1969.
- [7] C. DING, The global structure of uniformly asymptotically Zhukovskij stable systems, *Rocky Mount. J. Math.* **35**(4) (2005), 1115–1124.
- [8] C. DING and J. M. SORIANO, Uniformly asymptotically Zhukovskij stable orbits, *Comp. and Math. Appl.* **49**(1) (2005), 81–84.
- [9] C. DING, On the limit sets of orbits with some kinds of stabilities, *Publ. Math. Debrecen* **63**(3) (2003), 413–419.
- [10] E. GLASNER, Ergodic Theory Via Joinings, Math. Surveys and Monographs, 101, *AMS, Providence, RI*, 2003.
- [11] K. H. LEE, Weak near periodicity in dynamical systems, *Nonlinear Analysis, TMA* **15**(5) (1990), 399–404.
- [12] G. A. LEONOV, D. V. PONOMARENKO and V. B. SMIRNOVA, Local instability and localization of attractors, From stochastic generators to Chua’s systems, *Acta Appl. Math.* **40**(3) (1995), 179–243.
- [13] V. V. NEMYTSKII and V. V. STEPANOV, Qualitative theory of differential equations, *Princeton University Press*, 1966.

CHANGMING DING
DEPARTMENT OF MATHEMATICS
XIAMEN UNIVERSITY
XIAMEN, 361005
P.R. CHINA

E-mail: cding@xmu.edu.cn

ZHEN JIN
DEPARTMENT OF MATHEMATICS
NORTH UNIVERSITY OF CHINA
TAIYUAN, 030051
P.R.CHINA

JOSÉ M. SORIANO
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
UNIVERSIDAD DE SEVILLA
APTD. 1160
SEVILLA 41080
SPAIN

(Received June 11, 2006; revised April 18, 2008)