

## Extension theory and the $\Psi^\infty$ operator

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**Abstract.** We are going to define for each simplicial complex  $K$ , an operator  $\Psi^\infty$  on the subcomplexes of  $K$ . If one is given a collection of spaces, closed subspaces of them, and maps of the closed subspaces to a subpolyhedron of  $|K|$  that extend to maps into  $|K|$ , then we are going to use the  $\Psi^\infty$  operator to help determine a subcomplex of minimal cardinality into which the maps can be extended simultaneously.

The question (raised by A. Dranishnikov and J. Dydak) of whether the extension dimension,  $\text{extdim}_{(\mathcal{C}, \mathcal{T})} X$ , has a countable representative when  $X$  is compact and metrizable,  $\mathcal{C}$  is the class of compact metrizable spaces, and  $\mathcal{T}$  is the class of CW-complexes is an unsolved problem. We shall define an “anti-basis” for a CW-complex and use this along with the  $\Psi^\infty$  operator to allow one to view this problem from another perspective.

### 1. Introduction

Extension theory, which was first introduced by A. DRANISHNIKOV in 1994, is based on the following notion. If  $K$  is a CW-complex and  $X$  is a space, then one says that  $K$  is an *absolute extensor* for  $X$ ,  $K \in \text{AE}(X)$ , or  $X$  is an *absolute co-extensor* for  $K$ ,  $X \tau K$ , if for each closed subset  $A$  of  $X$  and map (i.e. continuous function)  $f : A \rightarrow K$ , there exists a map  $F : X \rightarrow K$  such that  $F$  is an extension of  $f$ . For example, if  $X$  is a normal space and  $K = I = [0, 1]$ , then Tietze’s extension theorem yields that  $I \in \text{AE}(X)$ , or  $X \tau I$ .

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*Key words and phrases:* absolute co-extensor, absolute extensor, cardinality of complex, cohomological dimension, CW-complex, compactification, covering dimension, extension theory, finitely-bounded, Hausdorff  $\sigma$ -compactum,  $\lambda$ -bounded,  $\sigma$ -compactum, Stone-Čech compactification, universal compactum, weight.

Suppose that  $X$  is either a metrizable space or a compact Hausdorff space. A classical result from the theory of covering dimension is that  $\dim X \leq n$  if and only if  $X \tau S^n$ . For cohomological dimension  $\dim_G$  over an abelian group  $G$ , a similar fact is true:  $\dim_G X \leq n$  if and only if  $X \tau K$  where  $K$  is an Eilenberg–Mac Lane CW-complex in the class  $K(G, n)$ . For these and other reasons, A. DRANISHNIKOV [2] defined the notions of extension theory and extension dimension. Given a class  $\mathcal{C}$  of spaces and a class  $\mathcal{T}$  of CW-complexes, one defines (see Section 5) an equivalence relation  $\sim_{(\mathcal{C}, \mathcal{T})}$  on the CW-complexes. For a given space  $X$ , not necessarily in  $\mathcal{C}$ , its *extension dimension*,  $\text{extdim}_{(\mathcal{C}, \mathcal{T})} X$ , may exist. The latter, when it exists, is a uniquely determined equivalence class under  $\sim_{(\mathcal{C}, \mathcal{T})}$ . He and J. DYDAK asked in [4] (Problem 5.4 below) whether with respect to the classes  $\mathcal{C}$  of compact Hausdorff spaces and  $\mathcal{T}$  of CW-complexes, the extension dimension of every metrizable compactum has a countable representative. We shall show, Proposition 5.5, that for certain universal compacta the answer is yes.

Whenever  $X$  is a Tychonoff space, then by  $\beta X$  we shall mean the Stone–Čech compactification of  $X$ . Let  $X = \sum \{X_s \mid s \in S\}$  be a topological sum of compact Hausdorff spaces,  $K$  be a CW-complex, and assume that  $X_s \tau K$  for each  $s \in S$ . Suppose that one is given a collection  $\{A_s \mid s \in S\}$  of closed subsets  $A_s$  of  $X_s$  along with maps  $f_s : A_s \rightarrow K$ . Under what conditions can these maps  $f_s$  be extended to maps  $F_s : X_s \rightarrow K$  so that for some finite subcomplex  $K_0$  of  $K$ ,  $F_s(X_s) \subset K_0$  for all  $s \in S$ ? Definitions needed to describe such a problem and others more general than it along with some rudimentary results, e.g., Corollary 2.9, about such extensions can be found in Section 2 below. A situation like this was encountered in [16] where the author (see Proposition 2.4 of that citation) determined a relationship between that kind of extension problem and whether  $\beta X \tau K$ .

To deal simultaneously with the problems outlined above, we shall introduce in Section 3, for each simplicial complex  $K$ , the operator  $\Psi^\infty$  on the subcomplexes of  $K$ . It will be true that  $\Psi^\infty$  is idempotent and that if  $X$  is a space,  $X \tau |K|$ , and  $L$  is a subcomplex of  $K$ , then  $X \tau |\Psi^\infty(L)|$ . Moreover, if  $L$  is of infinite cardinality, then the cardinality of  $\Psi^\infty(L)$  equals the cardinality of  $L$ . We apply the  $\Psi^\infty$  operator in Section 4. Our main result in that section is Theorem 4.7 which covers as a special case metrizable  $\sigma$ -compacta. One might also view Corollary 4.5 to see one of the fundamental properties of the  $\Psi^\infty$  operator.

In Section 5 we introduce the concept of an *anti-basis* for a polyhedron  $|K|$ . Roughly speaking, it consists of a set of subcomplexes of  $K$  that detect when a space  $Y$  is not an absolute co-extensor for  $K$ . Theorem 5.10 states that for certain classes of spaces the existence of a countable anti-basis consisting of finite

subcomplexes implies the existence of a countable representative of extension dimension as in the question of Dranishnikov and Dydak.

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## 2. $\lambda$ -bounded collections of maps

If  $K$  is a simplicial complex, then  $|K|$  will be endowed with the weak topology and we shall treat  $|K|$  as a CW-complex whose CW-structure is determined in the usual way by the triangulation  $K$ . By  $\text{card } K$  we of course mean the cardinal number of the set of simplexes of  $K$ . If  $K$  is a CW-complex, then by  $\text{card } K$  we mean the cardinal number of the set of cells of  $K$ . If  $X$  is a space, then  $\text{wt } X$  will designate the weight of  $X$ . The next lemma will be used implicitly below.

**Lemma 2.1.** *Let  $f : K \rightarrow L$  be a map of CW-complexes and  $K_0$  a subcomplex of  $K$ :*

- (1) *if  $K_0$  is finite, then  $f(K_0)$  is contained in a finite subcomplex of  $L$ ;*
- (2) *if  $K_0$  is infinite, then  $f(K_0)$  is contained in a subcomplex  $M$  of  $L$  with  $\text{card } M \leq \text{card } K_0$ .  $\square$*

Let us recall Definition 2.1 of [16]; we use a slightly different terminology in order to conform to the needs of this paper.

*Definition 2.2.* Let  $K$  be a CW-complex and  $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$  a collection of pairs of spaces. Suppose that for each  $s \in S$ ,  $A_s$  is closed in  $X_s$  and a map  $f_s : A_s \rightarrow K$  has been given. We shall say that  $\{f_s \mid s \in S\}$  is  $\mathcal{X}$  *finitely-bounded* in  $K$  if there exists a finite subcomplex  $K_0$  of  $K$  such that each map  $f_s$  can be extended to a map of  $X_s$  into  $K_0$ . If  $A_s = A$ ,  $X_s = X$ , and  $f_s = f$  for all  $s \in S$ , then we shall refer to the map  $f$  as being  $(X, A)$  *finitely-bounded* in  $K$  with the obvious meaning.

Let us first note:

**Lemma 2.3.** *Let  $f : K \rightarrow L$  be a map between CW-complexes  $K$  and  $L$ . Suppose that  $\{X_s \mid s \in S\}$  is a set of spaces. Let  $\{A_s \mid s \in S\}$  and  $\{f_s \mid s \in S\}$  be collections such that for each  $s \in S$ ,  $A_s$  is a closed subspace of  $X_s$  and  $f_s : A_s \rightarrow K$  is a map. Put  $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$ .*

- (1) *If  $\{f_s \mid s \in S\}$  is  $\mathcal{X}$  finitely-bounded in  $K$ , then  $\{f \circ f_s \mid s \in S\}$  is  $\mathcal{X}$  finitely-bounded in  $L$ .*

- (2) Suppose that  $g : L \rightarrow K$  is a map,  $g \circ f \simeq 1_K$ , and for each  $s \in S$ ,  $X_s$  has the homotopy extension property with respect to CW-complexes. Assume also that there is a finite subcomplex  $M$  of  $K$  such that  $f_s(A_s) \subset M$  for all  $s \in S$  and that  $\{f \circ f_s \mid s \in S\}$  is  $\mathcal{X}$  finitely-bounded in  $L$ . Then  $\{f_s \mid s \in S\}$  is  $\mathcal{X}$  finitely-bounded in  $K$ .

PROOF. (1) Let  $K_0$  be a finite subcomplex of  $K$ , and  $\{F_s \mid s \in S\}$  a collection such that for each  $s \in S$ ,  $F_s : X_s \rightarrow K_0$  is a map having the property that  $F_s \mid A_s = f_s$ . There exists a finite subcomplex  $L_0$  of  $L$  (see Lemma 2.1(1)) such that  $f(K_0) \subset L_0$ . Then  $\{f \circ F_s \mid s \in S\}$  witnesses the fact that  $\{f \circ f_s \mid s \in S\}$  is  $\mathcal{X}$  finitely-bounded in  $L$ .

(2) Let  $L_0$  be a finite subcomplex of  $L$  and  $\{G_s \mid s \in S\}$  a collection of maps  $G_s : X_s \rightarrow L_0$  such that  $G_s \mid A_s = f \circ f_s$  for all  $s \in S$ . Again applying Lemma 2.1(1), choose a finite subcomplex  $K^*$  of  $K$  such that  $g(L_0) \subset K^*$ . We may assume that  $M \subset K^*$ . Hence,  $\{g \circ G_s \mid s \in S\}$  is a collection such that  $g \circ G_s : X_s \rightarrow K^*$  is a map for each  $s \in S$ .

Let  $F : K \times [0, 1] \rightarrow K$  be a homotopy such that  $F(x, 0) = x$  and  $F(x, 1) = g \circ f(x)$  for all  $x \in K$ . Put  $K' = F(K^* \times [0, 1])$ . Then  $K'$  is contained in a finite subcomplex  $K_0$  of  $K$ . Moreover,  $F(K^* \times \{0\}) = K^*$ , so  $K^* \subset K_0$ . Putting  $F^* = F \mid (K^* \times [0, 1]) : K^* \times [0, 1] \rightarrow K_0$  one gets a deformation  $F^*$  of  $K^*$  in  $K_0$  having the property that  $F^*(x, 1) = g \circ f(x)$  for all  $x \in K^*$ .

Notice that if  $s \in S$  and  $a \in A_s$ , then  $f_s(a) \in M \subset K^*$ . So there is a homotopy  $Q_s : A_s \times I \rightarrow K_0$  given by  $Q_s(a, t) = F^*(f_s(a), t)$ . We see that  $Q_s(a, 0) = F^*(f_s(a), 0) = f_s(a)$  and  $Q_s(a, 1) = F^*(f_s(a), 1) = g \circ f \circ f_s(a)$ . But  $g \circ G_s \mid A_s = g \circ f \circ f_s$  and  $g \circ G_s : X_s \rightarrow K^* \subset K_0$ . The homotopy extension property shows that  $f_s$  extends to a map of  $X_s$  into  $K_0$ . Since  $K_0$  is finite and is independent of the choice of  $s \in S$ , our proof of (2) is complete.  $\square$

We now extend Definition 2.2 to consider maps to CW-complexes whose images must land in subcomplexes of infinite cardinalities.

*Definition 2.4.* Let  $K$  be a CW-complex,  $\lambda$  be an infinite cardinal, and  $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$  a collection of pairs of spaces. Suppose that for each  $s \in S$ ,  $A_s$  is closed in  $X_s$  and a map  $f_s : A_s \rightarrow K$  has been given. We shall say that  $\{f_s \mid s \in S\}$  is  $\mathcal{X}$   $\lambda$ -bounded in  $K$  if there exists a subcomplex  $K_0$  of  $K$  such that  $\text{card } K_0 \leq \lambda$  and each map  $f_s$  can be extended to a map of  $X_s$  into  $K_0$ . If  $A_s = A$ ,  $X_s = X$ , and  $f_s = f$  for all  $s \in S$ , then we shall refer to the map  $f$  as being  $(X, A)$   $\lambda$ -bounded in  $K$  with the obvious meaning.

**Lemma 2.5.** *Let  $Y$  be a space,  $A$  a closed subspace of  $Y$ , and  $K$  a CW-complex. Let  $X$  be a space,  $g : X \rightarrow Y$  a map,  $B$  a closed subspace of  $g^{-1}(A)$ , and  $f : A \rightarrow K$  a map.*

- (1) *If  $f$  is  $(Y, A)$  finitely-bounded, then  $f \circ (g \mid B)$  is  $(X, B)$  finitely bounded.*
- (2) *If  $\lambda$  is an infinite cardinal and  $f$  is  $(Y, A)$   $\lambda$ -bounded in  $K$ , then  $f \circ (g \mid B)$  is  $(X, B)$   $\lambda$ -bounded in  $K$ .  $\square$*

As pointed out in [16], nontrivial examples of CW-complexes  $K$  along with a non-finitely-bounded collection of maps in  $K$  can be extrapolated from the proof of Theorem 1.5 of [12]. In that proof, the author produces a countably infinite set  $\mathcal{T}$  and a collection,  $\{X^T \mid T \in \mathcal{T}\}$  of metrizable compacta. Each  $X^T$  has a specified closed subspace  $S_T$  homeomorphic to  $S^2$ . It is true that  $\dim_G X^T \leq 2$  for every abelian group  $G$ . In the last paragraph of the proof, select  $K$  (designated  $P$  there) to be  $K(G, 2)$  for any nontrivial abelian group  $G$ . Then for each  $T \in \mathcal{T}$ , let  $f_T : S_T \rightarrow K$  be a map such that  $f_T(S_T) = f_{T'}(S_{T'})$  and  $(f_T)_*(H_2(S_T)) = (f_{T'})_*(H_2(S_{T'})) \neq 0$  for each  $T, T' \in \mathcal{T}$ . With this and an examination of the finale of the proof of Theorem 1.5 in [12], we have,

**Proposition 2.6.** *For every nontrivial abelian group  $G$  and  $K = K(G, 2)$ , there exist a countably infinite set  $S$ , collections  $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$ , and  $\{f_s \mid s \in S\}$  where for each  $s \in S$ ,  $X_s$  is a compact metrizable space with  $X_s \tau K$ ,  $A_s$  is a closed subspace of  $X_s$ , and  $f_s : A_s \rightarrow K$  is a map whose image lies in a fixed finite subcomplex of  $K$ , chosen in such a manner that,  $\{f_s \mid s \in S\}$  is  $\aleph_0$ -bounded in  $K$  but not  $\mathcal{X}$  finitely-bounded in  $K$ .  $\square$*

The next lemma can be proved using the same techniques found in our proof of Lemma 2.3.

**Lemma 2.7.** *Let  $f : K \rightarrow L$  be a map between CW-complexes  $K$  and  $L$ . Suppose that  $\{X_s \mid s \in S\}$  is a set of spaces. Let  $\{A_s \mid s \in S\}$  and  $\{f_s \mid s \in S\}$  be collections such that for each  $s \in S$ ,  $A_s$  is a closed subspace of  $X_s$  and  $f_s : A_s \rightarrow K$  is a map. Put  $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$  and suppose that  $\lambda$  is an infinite cardinal.*

- (1) *If  $\{f_s \mid s \in S\}$  is  $\mathcal{X}$   $\lambda$ -bounded in  $K$ , then  $\{f \circ f_s \mid s \in S\}$  is  $\mathcal{X}$   $\lambda$ -bounded in  $L$ .*
- (2) *Suppose that  $g : L \rightarrow K$  is a map,  $g \circ f \simeq 1_K$ , and for each  $s \in S$ ,  $X_s$  has the homotopy extension property with respect to CW-complexes. Assume also that there is a subcomplex  $M$  of  $K$  with  $\text{card } M \leq \lambda$  such that  $f_s(A_s) \subset M$  for all  $s \in S$  and that  $\{f \circ f_s \mid s \in S\}$  is  $\mathcal{X}$   $\lambda$ -bounded in  $L$ . Then  $\{f_s \mid s \in S\}$  is  $\mathcal{X}$   $\lambda$ -bounded in  $K$ .*

A proof similar to that of Lemma 3.1 of [6], which applies to polyhedra, can be used to obtain the following stronger result applying to CW-complexes:

**Lemma 2.8.** *Let  $X$  be a space with  $\text{wt } X \leq \lambda$  for some infinite cardinal  $\lambda$ . Suppose  $f : X \rightarrow K$  is a map where  $K$  is a CW-complex. Then  $f(X) \subset L$  for some subcomplex  $L$  of  $K$  where  $\text{card } L \leq \lambda$ .  $\square$*

**Corollary 2.9.** *Let  $K$  be a CW-complex and  $\lambda$  an infinite cardinal. Suppose that  $S$  is a set with  $\text{card } S \leq \lambda$ ,  $\{X_s \mid s \in S\}$ ,  $\{A_s \mid s \in S\}$  are collections of spaces with  $X_s \tau K$ ,  $\text{wt } X_s \leq \lambda$ , and  $A_s$  is a closed subset of  $X_s$  for each  $s \in S$ . Put  $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$ . Then every collection  $\{f_s \mid s \in S\}$  of maps  $f_s : A_s \rightarrow K$  is  $\mathcal{X}$   $\lambda$ -bounded in  $K$ .  $\square$*

This leads to a result showing that “gluing” together such a collection of spaces does not change the “ $\lambda$ -bounded in  $K$ ” condition.

**Corollary 2.10.** *Let  $\lambda$  be an infinite cardinal,  $S$  a set,  $\{X_s \mid s \in S\}$ ,  $\{A_s \mid s \in S\}$  collections of spaces, and  $A_s$  a closed subset of  $X_s$  for each  $s \in S$ . Put  $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$ . Assume that there is a space  $A$  and for each  $s \in S$ , a homeomorphism  $h_s : A_s \rightarrow A$ . Let  $X$  be the quotient set formed from  $\sum \{X_s \mid s \in S\}$  by gluing the sets  $X_s$  to  $A$  via the homeomorphisms  $h_s$  and let  $q$  be the quotient function. Let  $X$  be given a topology such that  $A$  is closed in  $X$  and  $q \mid X_s : X_s \rightarrow X$  is a map for each  $s \in S$ . Then for each CW-complex  $K$  the following are true:*

- (1) *If  $f : A \rightarrow K$  is a map that is  $(X, A)$   $\lambda$ -bounded, then  $\{f \circ (q \mid A_s) \mid s \in S\}$  is  $\mathcal{X}$   $\lambda$ -bounded in  $K$ .*
- (2) *If  $\text{card } S \leq \lambda$  and for each  $s \in S$ ,  $\text{wt } X_s \leq \lambda$  and  $X_s \tau K$ , then every map  $f : A \rightarrow K$  is  $(X, A)$   $\lambda$ -bounded in  $K$ .  $\square$*

An example of a space  $X$  as in Corollary 2.10 could be obtained as follows. Suppose that  $\{X_s \mid s \in S\}$  is a collection of Hausdorff spaces each containing a closed subspace  $A_s$  homeomorphic to say  $S^n$ . Then form  $X$  by gluing these spaces together along  $S^n$  and applying the weak topology to  $X$ .

### 3. $\Psi$ Operators

For each simplicial complex  $K$ , denote by  $\mathcal{F}_K$  the set of nonempty finite subcomplexes of  $K$ . Fix a simplicial complex  $K$ . Suppose that  $M \in \mathcal{F}_K$ ; let  $\mathcal{D}_{(M,K)}$  be the set of  $D \in \mathcal{F}_K$  such that  $M \subset D$ . Define a relation  $\sim_{(M,K)}$  on  $\mathcal{D}_{(M,K)}$  by declaring that if  $D, C \in \mathcal{D}_{(M,K)}$ , then  $D \sim_{(M,K)} C$  if there exists a

simplicial isomorphism of  $D$  to  $C$  which is the identity on  $M$ . Plainly  $\sim_{(M,K)}$  is an equivalence relation on  $\mathcal{D}_{(M,K)}$ , and we shall write the equivalence class of an element  $D$  of  $\mathcal{D}_{(M,K)}$  as  $[D]_{(M,K)}$ . The equivalence class  $[M]_{(M,K)}$  is just  $\{M\}$ .

Let  $\mathcal{E}_{(M,K)}$  be the set of equivalence classes of  $\mathcal{D}_{(M,K)}$  under the relation  $\sim_{(M,K)}$  and  $q_{(M,K)} : \mathcal{D}_{(M,K)} \rightarrow \mathcal{E}_{(M,K)}$  the quotient function. The set  $\mathcal{E}_{(M,K)}$  is countable. Using the axiom of choice, fix once and for all a function  $\theta_{(M,K)} : \mathcal{E}_{(M,K)} \rightarrow \mathcal{D}_{(M,K)}$  such that  $\theta_{(M,K)}([D]_{(M,K)}) \in [D]_{(M,K)}$  for each  $D \in \mathcal{D}_{(M,K)}$ , i.e.,  $\theta_{(M,K)}(E) \in q_{(M,K)}^{-1}(E)$  for each  $E \in \mathcal{E}_{(M,K)}$ . We point out that  $M \subset \theta_{(M,K)}(E)$ . Assume that the preceding construction has been applied to each  $M \in \mathcal{F}_K$ .

For  $M \in \mathcal{F}_K$  and  $E \in \mathcal{E}_{(M,K)}$ ,  $\theta_{(M,K)}(E)$  is a subcomplex of  $K$ . Thus,

(i) for all  $M \in \mathcal{F}_K$ ,  $\bigcup \theta_{(M,K)}(\mathcal{E}_{(M,K)})$  is a subcomplex of  $K$  containing the subcomplex  $M$ .

We now define the function  $\Psi$  from the set of subcomplexes  $L$  of  $K$  to the set of subcomplexes of  $K$  by,

$$\Psi(L) = \bigcup \left\{ \bigcup \theta_{(M,K)}(\mathcal{E}_{(M,K)}) \mid M \in \mathcal{F}_L \right\}.$$

An application of (i) shows that for each pair  $L \subset L'$  of subcomplexes of  $K$ ,

(ii)  $\Psi(L)$  is a subcomplex of  $K$ , and  $L \subset \Psi(L)$ , and

(iii)  $\Psi(L) \subset \Psi(L')$ .

Let us denote  $\Psi^0(L) = L$ ; inductively for each  $k \in \mathbb{N}$  if  $\Psi^{k-1}(L)$  has been defined, then by  $\Psi^k(L)$  we mean  $\Psi(\Psi^{k-1}(L))$ . Put,

$$\Psi^\infty(L) = \bigcup \{ \Psi^k(L) \mid k \in \mathbb{N} \}.$$

Of course  $\Psi^\infty(L)$  is a subcomplex of  $K$ . We shall show that  $\Psi^\infty$  is an idempotent operator on the set of subcomplexes of a given simplicial complex  $K$ .

**Lemma 3.1.** *Let  $K$  be a simplicial complex and  $L$  a subcomplex of  $K$ . Then  $\Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$ , and hence  $\Psi^\infty(\Psi^\infty(L)) = \Psi^\infty(L)$ .*

PROOF. Suppose that  $M$  is a finite subcomplex of  $\Psi^\infty(L)$ . Then for some  $k \in \mathbb{N}$ ,  $M \subset \Psi^k(L)$ , so by (iii),  $\Psi(M) \subset \Psi(\Psi^k(L)) = \Psi^{k+1}(L) \subset \Psi^\infty(L)$ .  $\square$

From the construction of  $\Psi$ , it is not difficult to see that,

(iv) in case  $L$  is infinite, then  $\text{card}(\Psi^1(L)) = \text{card}(L)$ .

We therefore may state the following lemma.

**Lemma 3.2.** *Let  $K$  be a simplicial complex and  $L$  a subcomplex of  $K$ . If*

- (1)  $L = \emptyset$ , then  $\Psi^1(L) = \emptyset$ ,
- (2)  $L$  is finite, then  $\text{card}(\Psi^1(L)) \leq \aleph_0$ ,
- (1)  $L$  is infinite and  $k \in \mathbb{N}$ , then we may conclude that  
 $\text{card}(\Psi^k(L)) = \text{card}(\Psi^\infty(L)) = \text{card}(L)$ . □

We now provide an example to illustrate the operator  $\Psi^1$ . Let  $K$  be an infinite wedge of one-simplexes with vertex  $v$  and  $L$  the subcomplex consisting of the vertex  $v$ . The finite subcomplexes of  $K$  that contain  $L$  consist of finite wedges of 1-simplexes along with a finite set of vertices, including  $v$ , not in that wedge. With this in mind, one may choose  $\theta_{(L,K)}$  in such a manner that  $\Psi^1(L)$  is a countable wedge of 1-simplexes. Simply fix in advance a subcomplex  $M$  of  $K$  that is a countably infinite wedge of 1-simplexes. We may write  $M = \bigcup \{M_n \mid n \in \mathbb{N}\}$  where for each  $n \in \mathbb{N}$ ,  $M_n$  is a wedge of  $n$  1-simplexes and  $M_n \subset M_{n+1}$ . Make all choices of values of  $\theta_{(L,K)}$  to be subcomplexes of  $M$  and so that for each  $n \in \mathbb{N}$ ,  $\theta_{(L,K)}([M_n]) = M_n$ . Then  $\Psi^1(L) = M$ .

On the other hand, we may also choose  $\theta_{(L,K)}$  in a way that  $\Psi^1(L)$  consists of a countable wedge of 1-simplexes along with a nonintersecting discrete, nonempty, finite or countably infinite set of vertices. Indeed, for any choice of  $\theta_{(L,K)}$ ,  $\Psi^1(L)$  will always consist of a countable wedge of 1-simplexes along with a nonintersecting countable set of vertices.

The situation with  $\Psi^2(L)$  will again depend on  $\theta_{(L,K)}$ . If  $\Psi^1(L) = M$  as above, then  $\Psi^2(L)$  consists of  $M$  along with a countable (possibly empty) set of vertices outside  $M$ . If  $\Psi^1(L)$  contains some discrete nonempty set of vertices, then  $\Psi^2(L)$  could consist of  $\Psi^1(L)$  along with some additional 1-simplexes and perhaps an additional countable discrete set of vertices.

When a homotopy  $F : |M| \times I \rightarrow |M|$  is treated then we in addition define the  $\Psi_F^\infty$  operator, derive its properties (see Lemma 3.3), and use it to give a short proof of Proposition 3.4.

Let  $M$  be a simplicial complex and  $F : |M| \times I \rightarrow |M|$  a homotopy having the property that if  $x \in |M|$ ,  $\sigma \in M$ , and  $x \in \text{int } \sigma$ , then  $F(x, 0) \in \sigma$ . For each finite subcomplex  $Q$  of  $M$ , note that  $F(|Q| \times \{0\}) \subset |Q|$ . Let  $S_Q$  be the smallest subcomplex of  $M$  such that  $F(|Q| \times I) \subset |S_Q|$ . Then  $Q$  is a subcomplex of  $S_Q$  and  $S_Q$  is finite. For any subcomplex  $L$  of  $M$ , put  $\Psi_F(L) = \bigcup \{S_Q \mid Q \in \mathcal{F}_L\}$ . Then  $\Psi_F(L)$  is a subcomplex of  $M$ . Let  $\Psi_F^0(L) = L$ , and for each  $k \in \mathbb{N}$ , if  $\Psi_F^{k-1}(L)$  has been defined, then we let  $\Psi_F^k(L) = \Psi_F(\Psi_F^{k-1}(L))$ . Finally, let  $\Psi_F^\infty(L) = \bigcup \{\Psi_F^k(L) \mid k \in \mathbb{N}\}$ . Then it is easy to check the next result.

**Lemma 3.3.** *Let  $M$  be a simplicial complex and  $F : |M| \times I \rightarrow |M|$  a homotopy having the property that if  $x \in |M|$ ,  $\sigma \in M$ , and  $x \in \text{int } \sigma$ , then*

$F(x, 0) \in \sigma$ . Then for each subcomplex  $L$  of  $M$ ,  $\Psi_F^1(\Psi_F^\infty(L)) = \Psi_F^\infty(L)$ , and hence  $\Psi_F^\infty$  is an idempotent operator on the set of subcomplexes of  $M$ . Moreover,

- (1) if  $\text{card } L$  is finite, then  $\text{card}(\Psi_F^\infty(L)) \leq \aleph_0$ ,
- (2) if  $\text{card } L$  is infinite, then  $\text{card}(\Psi_F^\infty(L)) = \text{card}(L)$ , and
- (3)  $F(|\Psi_F^\infty(L)| \times I) \subset |\Psi_F^\infty(L)|$ . □

In Proposition 3.4, for completeness we state (1) without proof since this is a standard fact in the theory of CW-complexes, and our current techniques are useful only for proving (2).

**Proposition 3.4.** *Let  $K$  be a CW-complex of cardinality  $\alpha$ .*

- (1) *If  $\alpha$  is finite, then there exists a finite simplicial complex  $T$  and a homotopy equivalence between  $K$  and  $|T|$ .*
- (2) *If  $\alpha$  is infinite, then there exists a simplicial complex  $T$  of cardinality  $\leq \alpha$  and a homotopy equivalence between  $K$  and  $|T|$ .*

PROOF. As mentioned above, we only prove (2). There exists a simplicial complex  $M$  and a homotopy equivalence  $h : K \rightarrow |M|$ . Let  $f : |M| \rightarrow K$  be a homotopy inverse of  $h$  and  $F : |M| \times I \rightarrow |M|$  a homotopy from the identity of  $|M|$  to the map  $h \circ f$ . Choose a subcomplex  $L$  of  $M$  with  $\text{card } L \leq \alpha$  such that  $h(K) \subset |L|$ . Let  $T = \Psi_F^\infty(L) \subset M$  and  $f^* = f \mid |T| : |T| \rightarrow K$ . Apply Lemma 3.3(3) to see that  $h \circ f^*$  is homotopic to the identity on  $|T|$ . It is routine to check that  $f^* \circ h$  is homotopic to the identity on  $K$ . Apply Lemma 3.3(1,2) to see that  $\text{card } T \leq \alpha$ . □

#### 4. $X$ -connectedness and $\lambda$ -boundedness

By a pair  $(U, V)$  of spaces we mean a space  $U$  along with a subspace  $V$  of  $U$ . Next is Definition 6.1 of [16]. As mentioned there, this should be compared with similar ones given in [9], [10], and [11].

*Definition 4.1.* Let  $X$  be a space and  $(U, V)$  a pair of spaces. We shall say that  $(U, V)$  is  $X$ -connected if for each closed subset  $A$  of  $X$  and map  $f : A \rightarrow V$ , there exists a map  $F : X \rightarrow U$  that extends  $f$ .

The term  $\sigma$ -compactum usually refers to a metrizable space that can be written as a countable union of compact subspaces of itself. Such a space is obviously normal and Hausdorff; moreover, every CW-complex is an absolute neighborhood extensor for it. Let us generalize that definition.

*Definition 4.2.* Let  $X$  be a space. Then we shall say that  $X$  is a *Hausdorff  $\sigma$ -compactum* if  $X$  is a normal Hausdorff space, every CW-complex is an absolute neighborhood extensor for  $X$ , and  $X$  can be written as a countable union of compact Hausdorff subspaces.

**Proposition 4.3.** *Let  $K$  be a simplicial complex and  $L$  a subcomplex of  $K$ . Suppose that  $\{X_s \mid s \in S\}$  is a collection of Hausdorff  $\sigma$ -compacta and that for each  $s \in S$ ,  $X_s \tau |K|$ . The following are true.*

- (1) *The pair  $(|\Psi^\infty(L)|, |\Psi^\infty(L)|)$  is  $X_s$ -connected for each  $s \in S$ .*
- (2) *If  $s \in S$  and  $X_s$  is compact Hausdorff, then  $(|\Psi^{n+1}(L)|, |\Psi^n(L)|)$  is  $X_s$ -connected.*
- (3) *If  $\lambda$  is an infinite cardinal,  $\text{card } L \leq \lambda$ , for each  $s \in S$ ,  $A_s$  is a closed subset of  $X_s$ ,  $f_s : A_s \rightarrow |L|$  is a map, and  $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$ , then  $\{f_s \mid s \in S\}$  is  $\mathcal{X}$   $\lambda$ -bounded in  $|K|$ .*

PROOF. Statement (3) of this proposition will follow from Statement (1) along with an application of Lemma 3.2(2,3). We proceed with a proof of (1).

Consider  $s \in S$  and a map  $f_s : A_s \rightarrow |\Psi^\infty(L)|$ . Write  $X_s = \bigcup \{Z_i \mid i \in \mathbb{N}\}$  with  $Z_1 \subset Z_2 \subset \dots$ , and for each  $i \in \mathbb{N}$ ,  $Z_i$  is a compact Hausdorff space. We shall proceed with an induction argument.

Since  $X_s \tau |K|$ , then  $Z_1 \tau |K|$ . Let  $g_1 : A_s \cup Z_1 \rightarrow |K|$  be a map such that  $g_1 \mid A_s = f_s$ . There exists  $M \in \mathcal{F}_{\Psi^\infty(L)}$  such that  $g_1(Z_1 \cap A_s) \subset |M|$ . Now  $g_1(Z_1) \subset |M'|$  for some finite subcomplex  $M'$  of  $K$ , where  $M \subset M'$ . By the definition of  $\Psi^1(\Psi^\infty(L))$ , we may as well assume that  $M' \subset \Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$ . Hence we may treat  $g_1 : A_s \cup Z_1 \rightarrow |\Psi^\infty(L)|$ .

Using the ANE property of  $|\Psi^\infty(L)|$ , there exists a closed neighborhood  $D_1$  of  $A_s \cup Z_1$  in  $X_s$  and a map  $h_1 : D_1 \rightarrow |\Psi^\infty(L)|$  that extends  $g_1$ .

Suppose that  $k \in \mathbb{N}$  and we have found  $D_1 \subset \dots \subset D_k$ , and  $h_1, \dots, h_k$  such that for  $1 \leq i \leq k$ :

- (a)  $D_i$  is a closed neighborhood of  $A_s \cup Z_i$  in  $X_s$ ,
- (b)  $h_i$  is a map of  $D_i$  to  $|\Psi^\infty(L)|$ ,
- (c)  $h_i \mid A_s = f_s$ , and
- (d) if  $1 \leq i < j \leq k$ , then  $D_i \subset D_j$  and  $h_j \mid D_i = h_i$ .

Choose a map  $g_{k+1} : D_k \cup Z_{k+1} \rightarrow |K|$  such that  $g_{k+1} \mid D_k = h_k$ . Now  $h_k(Z_k \cup (Z_{k+1} \cap D_k)) = g_{k+1}(Z_k \cup (Z_{k+1} \cap D_k)) \subset |\Psi^\infty(L)|$ . There exists  $N \in \mathcal{F}_{\Psi^\infty(L)}$  such that  $g_{k+1}(Z_k \cup (Z_{k+1} \cap D_k)) \subset |N|$ .

Now  $g_{k+1}(Z_{k+1}) \subset |N'|$  for some finite subcomplex  $N'$  of  $K$ , where  $N \subset N'$ . By the definition of  $\Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$ , we may as well assume that  $N' \subset \Psi^\infty(L)$ .

There exists a closed neighborhood  $D_{k+1}$  of  $D_k \cup Z_{k+1}$  in  $X_s$  and a map  $h_{k+1} : D_{k+1} \rightarrow |\Psi^\infty(L)|$  that extends  $g_{k+1}$ .

This completes the induction. Observe that  $\bigcup \{\text{int}_{X_s} D_k \mid k \in \mathbb{N}\} = X_s$ . Define a function  $F_s : X_s \rightarrow |\Psi^\infty(L)|$  to be  $\bigcup \{h_k \mid k \in \mathbb{N}\}$ . Clearly  $F_s$  is a map, and  $F_s \mid A_s = f_s$ . Hence for all  $s \in S$ , there exists a map of  $X_s$  into  $|\Psi^\infty(L)| \subset |K|$  that extends  $f_s$ . This completes our proof of (1).

In the special case that  $X_s$  is a compact Hausdorff space, start with a map  $f_s : A_s \rightarrow |\Psi^n(L)|$ . Just apply the first step of the above inductive argument and see that  $F_s(X_s) \subset |\Psi^1(\Psi^n(L))| = |\Psi^{n+1}(L)|$ , so (2) is true.  $\square$

**Corollary 4.4.** *Let  $K$  be a simplicial complex,  $\lambda$  an infinite cardinal, and  $L$  a subcomplex of  $K$  with  $\text{card } L \leq \lambda$ . Suppose that  $X$  is a Hausdorff  $\sigma$ -compactum such that  $X\tau|K|$ ; then for each closed subspace  $A$  of  $X$  every map  $f : A \rightarrow |L|$  is  $(X, A)$   $\lambda$ -bounded in  $|K|$ .*  $\square$

**Corollary 4.5.** *Let  $K$  be a simplicial complex and  $X$  a Hausdorff  $\sigma$ -compactum with  $X\tau|K|$ . Then for every subcomplex  $L$  of  $K$ ,  $X\tau|\Psi^\infty(L)|$ .*

We shall use Proposition 4.6 in our proof of Theorem 4.7. It appears as Proposition 3.1 of [8] where we explain that this result differs from E. MICHAEL's Proposition 3.6(a) of [14], but is an improved version based on Lemma 1 of [15].

**Proposition 4.6.** *Let  $X$  be a paracompact space and  $\mathcal{G}$  a collection of subsets of  $X$ . Suppose that the following are true:*

- (1)  $\mathcal{G}$  contains an open cover of  $X$ ,
- (2) if  $U \in \mathcal{G}$  and  $W$  is open in  $U$ , then  $W \in \mathcal{G}$ ,
- (3) if  $U, Q$  are open elements of  $\mathcal{G}$ , then  $U \cup Q \in \mathcal{G}$ , and
- (4) if  $\mathcal{K} \subset \mathcal{G}$  is a discrete collection of open subsets of  $X$ , then  $\bigcup \mathcal{K} \in \mathcal{G}$ .

*Then the entire space  $X$  is in  $\mathcal{G}$ .*  $\square$

**Theorem 4.7.** *Let  $K$  be a simplicial complex,  $\lambda$  an infinite cardinal, and  $L$  a subcomplex of  $K$  with  $\text{card } L \leq \lambda$ . Suppose that  $X$  is a paracompact space,  $\{X_s \mid s \in S\}$  a locally finite cover of  $X$  consisting of closed subspaces that are Hausdorff  $\sigma$ -compacta, and  $X_s\tau|K|$  for each  $s \in S$ . Then  $(|\Psi^\infty(L)|, |\Psi^\infty(L)|)$  is  $X$ -connected, and hence every map  $f : A \rightarrow |\Psi^\infty(L)|$  of a closed subset  $A$  of  $X$  is  $\lambda$ -bounded in  $|K|$ . If for each  $s \in S$ ,  $X_s$  is compact and Hausdorff, then we may state additionally that for all  $n \in \mathbb{N}$ ,  $(|\Psi^{n+1}(L)|, |\Psi^n(L)|)$  is  $X$ -connected.*

PROOF. Let  $\mathcal{G}$  be the collection of open subsets  $G$  of  $X$  such that if  $A$  is a closed subset of  $\text{cl}_X G$ , and  $f : A \rightarrow |\Psi^\infty(L)|$  is a map, then  $f$  extends to a

map of  $\text{cl}_X G$  to  $|\Psi^\infty(L)|$ . We will show that  $\mathcal{G}$  satisfies conditions (1)–(4) of Proposition 4.6. Then we will be assured that  $X \in \mathcal{G}$ . The proof of the first part will be concluded by referring to Lemma 3.2(2,3).

Let  $\mathcal{U}$  be an open cover of  $X$  with the property that if  $U \in \mathcal{U}$ , then  $\text{cl}_X(U)$  intersects  $X_s$  for only finitely many  $s \in S$ . Fix  $U \in \mathcal{U}$ , let  $A$  be a closed subset of  $\text{cl}_X U$ , and  $f : A \rightarrow |\Psi^\infty(L)|$  a map.

Let  $T \subset S$  be the finite subset having the property that  $X_s \cap \text{cl}_X U \neq \emptyset$  if and only if  $s \in T$ . Define  $T_1$  to be the subset of  $T$  such that if  $s \in T$  and  $X_s$  is a compact Hausdorff space, then  $s \in T_1$ . Let  $T_2 = T \setminus T_1$ .

Put  $Y = \bigcup \{X_s \mid s \in T_1\}$ . Then  $Y$  is a compact Hausdorff space. By Proposition 4.3, there exists a map  $h : Y \rightarrow |\Psi^1(\Psi^\infty(L))| = |\Psi^\infty(L)|$  that extends  $f|(A \cap Y) : A \cap Y \rightarrow |\Psi^\infty(L)|$ . Let  $h^* : A \cup Y \rightarrow |\Psi^\infty(L)|$  be the map such that  $h^*|A = f$  and  $h^*|Y = h$ .

Let  $s \in T_2$ . By Proposition 4.3, there exists a map  $f_s : X_s \rightarrow |\Psi^\infty(\Psi^\infty(L))| = |\Psi^\infty(L)|$  that extends  $h^*|(A \cup Y) \cap X_s : ((A \cup Y) \cap X_s) \rightarrow |\Psi^\infty(L)|$ . Put  $f_1^* : A \cup Y \cup X_s \rightarrow |\Psi^\infty(L)|$  such that  $f_1^*|(A \cup Y) = h^*$  and  $f_1^*|X_s = f_s$ . Using the fact from Lemma 3.1 that  $\Psi^\infty(\Psi^\infty(L)) = \Psi^\infty(L)$ , one may, step by step, add the remaining  $\sigma$ -compacta indexed by  $T_2$  to end up with a map of  $\text{cl}_X U$  to  $|\Psi^\infty(L)|$  that extends  $f$ . This shows that  $\mathcal{G}$  contains an open cover of  $X$ . Part (2) of Proposition 4.6 is easily seen to be true.

Since for open elements  $U$  and  $Q$  of  $\mathcal{G}$ ,  $\text{cl}_X(U \cup Q) = \text{cl}_X U \cup \text{cl}_X Q$ , the reader can see how to prove (3) of Proposition 4.6 by using the same techniques we just employed above. That  $\mathcal{G}$  satisfies Part (4) of Proposition 4.6 is obvious.

In case  $X_s$  is compact and Hausdorff for each  $s \in S$ , then one may simply change the definition of  $\mathcal{G}$  to require that  $f$  extends to a map of  $\text{cl}_X G$  to  $|\Psi^{n+1}(L)|$ .  $\square$

## 5. Extension Dimension and Anti-Bases

We shall now recall the notion of extension dimension. Let  $\mathcal{C}$  be a class of spaces,  $\mathcal{T}$  a class of CW-complexes, and  $K, K' \in \mathcal{T}$ . If it is true that for all  $X \in \mathcal{C}$ ,  $X\tau K$  implies that  $X\tau K'$ , then we write  $K \leq_{(\mathcal{C}, \mathcal{T})} K'$ . This defines a preorder on  $\mathcal{T}$  (see [4] or [7]). One specifies  $K \sim_{(\mathcal{C}, \mathcal{T})} K'$  if and only if  $K \leq_{(\mathcal{C}, \mathcal{T})} K'$  and  $K' \leq_{(\mathcal{C}, \mathcal{T})} K$ ; then  $\sim_{(\mathcal{C}, \mathcal{T})}$  is an equivalence relation on  $\mathcal{T}$ . An equivalence class  $[K]_{(\mathcal{C}, \mathcal{T})}$  under this relation is called an *extension type* relative to  $(\mathcal{C}, \mathcal{T})$ . For any space  $X$ , we write  $X\tau[K]_{(\mathcal{C}, \mathcal{T})}$  to mean that  $X\tau K'$  for all  $K' \in [K]_{(\mathcal{C}, \mathcal{T})}$ .

The relation  $\leq_{(\mathcal{C}, \mathcal{T})}$  induces a partial order, also denoted  $\leq_{(\mathcal{C}, \mathcal{T})}$ , on the extension types. We write that  $[K]_{(\mathcal{C}, \mathcal{T})} \leq_{(\mathcal{C}, \mathcal{T})} [K']_{(\mathcal{C}, \mathcal{T})}$  if it is true that  $L \leq_{(\mathcal{C}, \mathcal{T})} L'$  for all  $L \in [K]_{(\mathcal{C}, \mathcal{T})}$  and  $L' \in [K']_{(\mathcal{C}, \mathcal{T})}$ . One may check that  $[K]_{(\mathcal{C}, \mathcal{T})} \leq_{(\mathcal{C}, \mathcal{T})} [K']_{(\mathcal{C}, \mathcal{T})}$  if and only if  $L \leq_{(\mathcal{C}, \mathcal{T})} L'$  for some  $L \in [K]_{(\mathcal{C}, \mathcal{T})}$  and some  $L' \in [K']_{(\mathcal{C}, \mathcal{T})}$ .

For Lemma 5.1 to follow, one might take  $\mathcal{C}$  to be any class of compact Hausdorff spaces or metrizable spaces.

**Lemma 5.1.** *Let  $\mathcal{T}$  be a class of CW-complexes and  $\mathcal{C}$  be a class spaces  $X$  having the homotopy extension property with respect to  $K$  for any element  $K$  of  $\mathcal{T}$ . Whenever  $K, L$  are homotopy equivalent elements of  $\mathcal{T}$ , then  $[K]_{(\mathcal{C}, \mathcal{T})} = [L]_{(\mathcal{C}, \mathcal{T})}$ .  $\square$*

Let  $X$  be a space. Consider  $S = \{[K]_{(\mathcal{C}, \mathcal{T})} \mid X\tau[K]_{(\mathcal{C}, \mathcal{T})}\}$ . If  $S$  has an initial element<sup>1</sup> with respect to the relation  $\leq_{(\mathcal{C}, \mathcal{T})}$ , then that element is called the *extension dimension* of  $X$  relative to  $(\mathcal{C}, \mathcal{T})$ , written  $\text{extdim}_{(\mathcal{C}, \mathcal{T})} X$ .

In the sequel we shall use,

$\mathcal{T}_{\text{CW}}$  = the class of CW-complexes,

$\mathcal{T}_{\text{POL}}$  = the class of polyhedra,

$\mathcal{K}$  = the class of compact Hausdorff spaces,

$\mathcal{K}_m$  = the class of compact metrizable spaces.

Theorem 11 of [3] along with Lemma 1.1 of [6] can be used to obtain the next information.

**Theorem 5.2.** *For each  $L, K \in \mathcal{T}_{\text{CW}}$ , it is true that  $L \leq_{(\mathcal{K}, \mathcal{T}_{\text{CW}})} K$  if and only if  $L \leq_{(\mathcal{K}_m, \mathcal{T}_{\text{CW}})} K$ . Hence,  $[K]_{(\mathcal{K}, \mathcal{T}_{\text{CW}})} = [K]_{(\mathcal{K}_m, \mathcal{T}_{\text{CW}})}$ . Similarly, if  $K \in \mathcal{T}_{\text{POL}}$ , then  $[K]_{(\mathcal{K}, \mathcal{T}_{\text{POL}})} = [K]_{(\mathcal{K}_m, \mathcal{T}_{\text{POL}})}$ .  $\square$*

It is remarked in Theorem 5.5 of [6] (see also [5]) that for any compact Hausdorff space  $X$ ,  $\text{extdim}_{(\mathcal{K}, \mathcal{T}_{\text{CW}})} X$  exists. This extension dimension has a special type of representative. Let us cite Theorem 13 of [3].

**Theorem 5.3.** *For each  $X \in \mathcal{K}$ , there exists  $L = \bigvee \{L_a \mid a \in A\}$  where  $\text{card } A \leq 2^{\aleph_0}$ , for each  $a \in A$ ,  $L_a \in \mathcal{T}_{\text{CW}}$ ,  $L_a$  is countable, and,*

$$\text{extdim}_{(\mathcal{K}, \mathcal{T}_{\text{CW}})} X = [L]_{(\mathcal{K}, \mathcal{T}_{\text{CW}})}. \quad \square$$

Now we state Problem 2.19.2 of [4], noting that it has also been posed as Problem 2 of [3] and Problem 2.1 of [1].

<sup>1</sup>By an *initial element* of  $S$ , we mean  $s_0 \in S$  having the property that  $s_0 \leq_{(\mathcal{C}, \mathcal{T})} s$  for all  $s \in S$ . If such  $s_0$  exists, it is unique.

*Problem 5.4.* Determine whether for each compact metrizable space  $X$ , there is a countable CW-complex  $M$  such that  $\text{extdim}_{(\mathcal{K}, \mathcal{T}_{\text{CW}})} X = [M]_{(\mathcal{K}, \mathcal{T}_{\text{CW}})}$ .

The next fact is immediate from Corollary 1.3 of [7]

**Proposition 5.5.** *Let  $K$  be a countable CW-complex and  $\alpha$  an infinite ordinal. Suppose that  $X$  is a compact Hausdorff space with  $\text{wt } X \leq \alpha$  having the property that  $X \tau K$  and each compact Hausdorff space  $Y$  with  $Y \tau K$  and  $\text{wt } Y \leq \alpha$  embeds in  $X$ . Then  $\text{extdim}_{(\mathcal{K}, \mathcal{T}_{\text{CW}})} X = [K]_{(\mathcal{K}, \mathcal{T}_{\text{CW}})}$ .*

This provides many examples of compact Hausdorff spaces with “countable” extension dimension, since by Corollary 1.9 of [13], every finite CW-complex admits a universal Hausdorff compactum of a given weight.

**Lemma 5.6.** *Let  $K$  be a CW-complex and  $X$  a Hausdorff  $\sigma$ -compactum. Suppose that  $K$  is not an absolute extensor for  $X$ . Then there exists a compact subset  $A$  of  $X$  and a map  $f : A \rightarrow K$  that does not extend to a map of  $X$  to  $K$ .*

PROOF. There exists a closed subspace  $B$  of  $X$  and a map  $g : B \rightarrow K$  that does not extend to a map of  $X$  to  $K$ . Write  $X = \bigcup \{X_i \mid i \in \mathbb{N}\}$  where for each  $i \in \mathbb{N}$ ,  $X_i$  is a compact Hausdorff space.

If  $g \mid (B \cap X_1) : B \cap X_1 \rightarrow K$  does not extend to a map of  $X_1 \rightarrow K$ , then define  $A = B \cap X_1$  and  $f = g \mid A : A \rightarrow K$ . Otherwise, choose a map  $h_1 : B \cup X_1 \rightarrow K$  that extends  $g$ . We may as well assume that the domain of  $h_1$  is a closed neighborhood  $N_1$  of  $B \cup X_1$ . Suppose that  $k \in \mathbb{N}$  and we have found closed subsets  $N_1 \subset \cdots \subset N_k$ , of  $X$ , and maps  $h_i : N_i \rightarrow K$ ,  $1 \leq i \leq k$ , such that for  $1 \leq i \leq j \leq k$ ,

- (i)  $h_j \mid N_i = h_i$ ,
- (ii)  $X_i \subset \text{int}_X N_i$ , and
- (iii)  $h_i \mid B = g$ .

If  $h_k \mid (N_k \cap X_{k+1})$  does not extend to a map of  $X_{k+1}$  to  $K$ , then choose  $A = N_k \cap X_{k+1}$  and  $f = h_k \mid A : A \rightarrow K$ . If it does extend, there exists a closed neighborhood  $N_{k+1}$  of  $N_k \cup X_{k+1}$  and a map  $h_{k+1} : N_{k+1} \rightarrow K$  such that  $h_{k+1} \mid N_k = h_k : N_k \rightarrow K$ .

If this recursive process ends after finitely many steps, then our proof is complete. If it does not end, then put  $G = \bigcup \{h_i \mid i \in \mathbb{N}\} : X \rightarrow K$ . Then  $G$  is a map that extends  $g$ , and we have reached a contradiction.  $\square$

We have the following statement in case  $K$  is a CW-complex,  $Y$  a space, and  $K \notin \text{AE}(Y)$ .

**Lemma 5.7.** *Let  $K$  be a CW-complex,  $Y$  a space,  $A$  a closed subspace of  $Y$ ,  $L$  a subcomplex of  $K$ , and  $f : A \rightarrow L$  a map that does not extend to a map of  $Y$  to  $K$ . Then for any subcomplex  $M$  of  $K$  with  $L \subset M$ , the map  $f : A \rightarrow M$  does not extend to a map of  $Y$  to  $M$ .  $\square$*

This motivates us to define the notion of an “anti-basis” and show how this is related to Problem 5.4.

*Definition 5.8.* Let  $\mathcal{K}^*$  be a class of spaces,  $K$  be a simplicial complex, and  $\mathcal{F}$  a collection of subcomplexes of  $K$  having the property that whenever  $Y \in \mathcal{K}^*$  and  $|K|$  is not an absolute extensor for  $Y$ , then there exist a closed subspace  $A$  of  $Y$ ,  $F \in \mathcal{F}$ , and map  $f : A \rightarrow |F|$  that does not extend to a map of  $Y$  into  $|K|$ . Then we shall call  $\mathcal{F}$  an *anti-basis* for  $K$  relative to  $\mathcal{K}^*$ .

An application of Lemma 5.6 shows the following.

*Example 5.9.* Let  $\mathcal{K}^*$  be a class of Hausdorff  $\sigma$ -compacta and  $K$  a simplicial complex. Then  $\mathcal{F}_K$  is an anti-basis for  $K$  relative to  $\mathcal{K}^*$ .

Now we have the following theorem.

**Theorem 5.10.** *Let  $\mathcal{K}^*$  be a class of Hausdorff  $\sigma$ -compacta,  $X \in \mathcal{K}^*$ , and  $K$  a simplicial complex. Suppose that  $\text{extdim}_{(\mathcal{K}^*, \mathcal{T}_{\text{CW}})} X$  exists and equals  $[|K|]_{(\mathcal{K}^*, \mathcal{T}_{\text{CW}})}$ . If  $K$  has a countable anti-basis  $\mathcal{F}$  relative to  $\mathcal{K}^*$  such that  $\mathcal{F}$  consists of finite subcomplexes of  $K$ , then there is a countable representative of  $\text{extdim}_{(\mathcal{K}^*, \mathcal{T}_{\text{CW}})} X$ . Indeed,  $M = \Psi^\infty(\bigcup \mathcal{F})$  is a countable subcomplex of  $K$  and  $|M|$  represents  $\text{extdim}_{(\mathcal{K}^*, \mathcal{T}_{\text{CW}})} X$ .*

PROOF. Put  $L = \bigcup \mathcal{F}$ . Then  $L$  is a countable subcomplex of  $K$ . Let  $M = \Psi^\infty(L)$ . By Lemma 3.2(3),  $M$  is a countable subcomplex of  $K$ . Moreover since  $X \tau |K|$ , by Corollary 4.5,  $X \tau |M|$ . We know that  $|K| \leq_{(\mathcal{K}^*, \mathcal{T}_{\text{CW}})} |M|$ . It remains to prove the opposite inequality.

Suppose that  $Y \in \mathcal{K}^*$ ,  $Y \tau |M|$ , and  $Y \not\tau |K|$  is false. By Definition 5.8, there is an element  $L$  of  $\mathcal{F}$ , a closed subspace  $A$  of  $Y$ , and a map  $f : A \rightarrow |L|$  that does not extend to a map of  $Y$  to  $K$ . But  $|L| \subset |M|$  and  $Y \tau |M|$ , so  $f$  extends to a map of  $Y$  to  $|M|$ . This contradicts Lemma 5.7.  $\square$

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