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Approximation ratio of the digits in Oppenheim series expansion

By BAO-WEI WANG (Wuhan) and JUN WU (Wuhan)

Abstract. This paper is concerned with the Hausdorff dimensions of some sets determined by the approximation ratio of the digits in Oppenheim series expansion. We give a general characterization on the Hausdorff dimensions of such sets. As it's corollaries, we answer the questions posed by Galambos.

1. Introduction

Oppenheim series expansion is a generalized tool to the representation of real numbers by infinite series, including LÜROTH [12], ENGEL, SYLVESTER expansions [3] and Cantor product [14] as special cases, which is given by the following algorithm. For any $x \in (0,1]$, set

$$x = x_1,\ d_n = [1/x_n] + 1, \quad x_n = 1/d_n + \gamma_n \cdot x_{n+1}, \tag{1}$$

where $\gamma_n = \gamma_n(d_1, \dots, d_n)$ is some positive rational valued function and $[y]$ denotes the integer part of y. Then it leads to an infinite series expansion for each $x \in (0,1]$ with the form

$$x \sim \frac{1}{d_1} + \gamma_1 \frac{1}{d_2} + \dots + \gamma_1\gamma_2 \dots \gamma_n \frac{1}{d_{n+1}} + \dots, \tag{2}$$

which is called the Oppenheim expansion of x, see [13]. The expansion (2) is termed as the restricted Oppenheim expansion of x if γ_n depends on the last denominator d_n only and the function

$$h_n(d) = \gamma_n(d) \cdot d(d-1) \tag{3}$$

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is integer-valued for all $d \geq 2$ and $n \geq 1$. In this restricted case, a sufficient and necessary condition for the series on the right hand side in (2) to be the expansion of its sum by the algorithm (1) is, see [13],

$$d_{n+1} \geq \gamma_n d_n(d_n - 1) + 1 \quad \text{for all } n \geq 1. \tag{4}$$

In the present paper, we deal with the restricted Oppenheim expansion only.

The representation (2) under (1) was first studied by Oppenheim [13], where he established the arithmetical properties, including the question of rationality of this expansion. The foundations of the metric theory were laid down by Galambos [6], [7], [8], [10], [11], see also the monographs of Galambos [9], Schweiger [15], Vervaat [16], Dajani and Kraaikamp [2]. From [9], Chapter 6, it can be seen that the integer approximations $T_n(x)$ to the ratios $d_n(x)/h_{n-1}(d_{n-1}(x))$ given by

$$T_n(x) < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq T_n(x) + 1, \quad n \geq 1, \tag{5}$$

where $h_0 \equiv 1$, plays an important role in the metric theory of Oppenheim expansion. They are stochastically independent and are distributed as the denominators in the Lüroth expansion. Galambos ([9], Chapter 6) showed that

$$\frac{1}{n \log n}(T_1(x) + \cdots + T_n(x)) \to 1, \quad \text{as } n \to \infty \tag{6}$$

in probability but it has Lebesgue measure zero where the above convergence actually occurs. Let

$$\mathbb{B}_m = \{x \in (0,1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\}, \qquad m \geq 2,$$
$$\mathbb{D} = \Big\{x \in (0,1] : \frac{1}{n \log n}(T_1(x) + \cdots + T_n(x)) \to 1, \text{ as } n \to \infty\Big\},$$

and for any $k > 0$, let

$$\mathbb{D}_k = \{x \in (0,1] : T_1(x) + \cdots + T_n(x) \leq kn \log n, \text{ for sufficiently large } n\}.$$

Galambos, see [9], page 132–133, posed the questions to calculate the Hausdorff dimension of the sets $\mathbb{B}_m$, $\mathbb{D}$ and $\mathbb{D}_k$ above.

In this paper, we give a general characterization on the Hausdorff dimension of the set determined by the approximation ratio $\frac{d_{n+1}(x)}{h_n(d_n(x))}$. More precisely, denote

$$C = \Big\{x \in (0,1] : L_n < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq M_n \text{ for all } n \geq 1\Big\},$$

where $\{L_n : n \geq 1\}$ and $\{M_n : n \geq 1\}$ are two given integer-valued sequences, then what is the Hausdorff dimension of C? We get the point under some natural restrictions on L_n and M_n. By (5), it is easy to check that

$$C = \{x \in (0,1] : L_n \leq T_n(x) \leq M_n - 1 \text{ for all } n \geq 1\}.$$

So, as corollaries, we answer the questions posed by Galambos.

Remark 1.1. Since $T_j(x) \geq 1$ for all $j \geq 1$, we have $T_1(x)+\cdots+T_n(x) \geq n$ for all $n \geq 1$. Thus we should modify GALAMBOS' question to calculate the Hausdorff dimension of E_k (see [9], page 133) to calculate the Hausdorff dimension of $\mathbb{D}_k$, where E_k is defined as the set of points with the inequality in $\mathbb{D}_k$ holding for all $n \geq 1$.

The Hausdorff dimension of $\mathbb{B}_m$ has been considered in [17]. Some other exceptional sets associated with the Oppenheim series expansion were discussed in [18], [20], [21].

We will use $|\cdot|$ to denote the diameter of a set, $\dim_{\mathrm{H}}$ to denote the Hausdorff dimension and 'cl' the closure of a subset of $(0,1]$ respectively.

2. Main results

In this section, we collect some elementary properties on Oppenheim series expansion and state our main results.

Definition 2.1. Let $d_1, d_2, \ldots, d_n$ be an admissible sequence, i.e., $d_1 \geq 2$ and $d_{j+1} \geq h_j(d_j) + 1$ for all $1 \leq j < n$. We call the set

$$I(d_1, \ldots, d_n) := \{x \in (0,1] : d_1(x) = d_1, \ldots, d_n(x) = d_n\}$$

an n-th order admissible interval.

Proposition 2.2 ([9]). *Let $d_1, d_2, \ldots, d_n$ be an admissible sequence. Then the n-th order admissible interval $I(d_1, \ldots, d_n)$ is the interval with two endpoints*

$$\frac{1}{d_1} + \gamma_1(d_1)\frac{1}{d_2} + \cdots + \gamma_1(d_1)\ldots\gamma_{n-1}(d_{n-1})\frac{1}{d_n},$$

and

$$\frac{1}{d_1} + \gamma_1(d_1)\frac{1}{d_2} + \cdots + \gamma_1(d_1)\ldots\gamma_{n-1}(d_{n-1})\frac{1}{d_n} + \gamma_1(d_1)\ldots\gamma_{n-1}(d_{n-1})\frac{1}{d_n(d_n-1)}.$$

Thus

$$|I(d_1,\dots,d_n)|=\gamma_1(d_1)\dots\gamma_{n-1}(d_{n-1})\frac{1}{d_n(d_n-1)}=\prod_{j=0}^{n-1}\frac{h_j(d_j)}{d_{j+1}(d_{j+1}-1)}.$$

Definition 2.3. We call $\{h_n, n\geq 1\}$ is of order t, if there exist two constants $0<c_1\leq c_2$ such that

$$c_1d^t\leq h_n(d)\leq c_2d^t$$

for all $d\geq 2$ and $n\geq 1$.

Let $\{L_n, n\geq 1\}$ and $\{M_n, n\geq 1\}$ be two positive integer sequences which are non-decreasing and satisfy

$$L_n<M_n,\quad M_n\geq 3,\quad \text{and}\quad \sup_{n\geq 1}\frac{M_n}{L_n}:=\alpha<\infty. \tag{7}$$

Recall that

$$\begin{aligned}C&=\left\{x\in(0,1]:L_n<\frac{d_n(x)}{h_{n-1}(d_{n-1}(x))}\leq M_n \text{ for all } n\geq 1\right\}\\&=\{x\in(0,1]:L_n\leq T_n(x)\leq M_n-1 \text{ for all } n\geq 1\}.\end{aligned}$$

Theorem 2.4. *Assume $h_n(d)\geq d-1$ for all $d\geq 2$ and $n\geq 1$.*

(1) *When $\{M_n, n\geq 1\}$ is bounded, we have* $\dim_{\mathrm{H}} C=1$.

(2) *When $M_n\to\infty$ as $n\to\infty$, we have*

   (i) *If* $\lim_{n\to\infty}\frac{\log M_{n+1}}{\log M_n}=1$, *then* $\dim_{\mathrm{H}} C=1$.

   (ii) *If* $\lim_{n\to\infty}\frac{\log M_{n+1}}{\log M_n}=b>1$, *$\{h_n, n\geq 1\}$ is of order t and* $\lim_{n\to\infty}\frac{\log(M_n-L_n)}{\log M_n}=\beta$, *then* $\dim_{\mathrm{H}} C=\frac{\beta(b-t)+t}{(2b-\beta b+\beta)(b-t)+t}$ *if $b>t$ and* $\dim_{\mathrm{H}} C=1$ *if $b\leq t$.*

Remark 2.5. It would become clear from the details of the proof that the assumptions of monotonicity and $\sup_{n\geq 1}\frac{M_n}{L_n}<\infty$ in (7) is actually not necessary. But, from the dimensional number of C at the case $b>t$, the extra assumption $\lim_{n\to\infty}\frac{\log(M_n-L_n)}{\log M_n}=\beta$ is not superfluous.

Corollary 2.6 ([17])**.** *If $h_n(d)\geq d-1$ for all $d\geq 2$ and $n\geq 1$, then for any $m\geq 2$,*

$$\dim_{\mathrm{H}}\{x\in(0,1]:1\leq T_n(x)\leq m \textit{ for all } n\geq 1\}=1.$$

Proof. This is a direct consequence of Theorem 2.4. □

Corollary 2.7. *Suppose $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. Then we have*

$$\dim_{\mathrm{H}}\{x \in (0,1] : \lim_{n\to\infty} \frac{T_1(x)+\cdots+T_n(x)}{n\log n} = 1\} = 1.$$

Proof. Choose

$$L_n = [\log(n+6)], \quad M_n = \left[\log(n+6) + (\log(n+6))^{1-\frac{1}{\sqrt{\log\log(n+6)}}}\right] + 2$$

for all $n \geq 1$. Applying Theorem 2.4, we get the desired result immediately. □

Corollary 2.8. *Suppose $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. Then for any $k > 0$,*

$$\dim_{\mathrm{H}}\{x \in (0,1] : T_1(x)+\cdots+T_n(x) \leq kn\log n, \text{ for sufficiently large } n\} = 1.$$

Proof. Choose

$$L_n = \left[\frac{k}{2}\log n\right], \quad M_n = \left[\frac{k}{2}\left(\log n + (\log n)^{1-\frac{1}{\sqrt{\log\log n}}}\right)\right]$$

when n is large enough. Then the desired result is an easy consequence of Theorem 2.4. □

At the end of this section, we state the Billingsley theorem (see [1], [4], [5], [19]), which will be used to obtain the lower bound of the Hausdorff dimension of a fractal set.

Lemma 2.9. *Let $E \subset (0,1]$ be a Borel set and μ be a measure with $\mu(E) > 0$. If for any $x \in E$,*

$$\liminf_{r\to 0} \frac{\log\mu(B(x,r))}{\log r} \geq s,$$

where $B(x,r)$ denotes the open ball with center at x and radius r, then $\dim_{\mathrm{H}} E \geq s$.

3. Proof of Theorem 2.4

We fix some notation at first.

For any admissible sequence $d_1, d_2, \ldots, d_n$, let

$$J(d_1,\ldots,d_n) = \bigcup_{L_{n+1}h_n(d_n) < d_{n+1} \leq M_{n+1}h_n(d_n)} \mathrm{cl}\, I(d_1,\ldots,d_n,d_{n+1}),$$

and call $J(d_1,\dots,d_n)$ an n-th order basic interval. By Proposition 2.2, we have

$$|J(d_1,\dots,d_n)| = \left(\frac{1}{L_{n+1}} - \frac{1}{M_{n+1}}\right)|I(d_1,\dots,d_n)|. \tag{8}$$

Fix $k \geq 1$ and $\bar{d}_1,\dots,\bar{d}_k$ an admissible sequence. For any $n \geq k$, let

$$\begin{aligned} D_n(\bar{d}_1,\dots,\bar{d}_k) =& \Big\{(d_1,\dots,d_n) \in \mathbb{N}^n : d_j = \bar{d}_j, 1 \leq j \leq k, \\ & L_{j+1} < \frac{d_{j+1}}{h_j(d_j)} \leq M_{j+1}, k \leq j < n\Big\}. \end{aligned}$$

In the following, we shall give a bound estimation of the gap $G^l(d_1,\dots,d_n)$ which is the gap between $J(d_1,\dots,d_n)$ and the closest n-th order basic interval which lies on the left hand side of $J(d_1,\dots,d_n)$, and give a bound estimation of the gap $G^r(d_1,\dots,d_n)$ which is the gap between $J(d_1,\dots,d_n)$ and the closest n-th order basic interval, (if it exsits, otherwise we set $G^r(d_1,\dots,d_n) = \infty$), which lies on the right hand side of $J(d_1,\dots,d_n)$. For $G^l(d_1,\dots,d_n)$, it is clear that $G^l(d_1,\dots,d_n)$ is not less than the distance between the left endpoint of $J(d_1,\dots,d_n)$ and the left endpoint of $I(d_1,\dots,d_n)$. Thus by Proposition 2.2,

$$G^l(d_1,\dots,d_n) \geq \frac{1}{M_{n+1}}|I(d_1,\dots,d_n)|.$$

For $G^r(d_1,\dots,d_n)$, let $J(\tilde{d}_1,\dots,\tilde{d}_n)$ be the n-th order basic interval which lies on the right hand side of $J(d_1,\dots,d_n)$ and closest to it. Let $j_0 = \min\{j : d_j \neq \tilde{d}_j\}$. Then $d_j = \tilde{d}_j$ for $1 \leq j < j_0$ and $d_{j_0} > \tilde{d}_{j_0}$. Moreover, it is clear that $G^r(d_1,\dots,d_n)$ is not less than the distance between the left endpoint of $J(\tilde{d}_1,\dots,\tilde{d}_{j_0})$ and the left endpoint of $I(\tilde{d}_1,\dots,\tilde{d}_{j_0})$. Thus, by Proposition 2.2, we have

$$\begin{aligned} G^r(d_1,\dots,d_n) &\geq \frac{1}{M_{j_0+1}} I(\tilde{d}_1,\dots,\tilde{d}_{j_0}) \geq \frac{1}{M_{j_0+1}} I(d_1,\dots,d_{j_0}) \\ &\geq \frac{1}{M_{j_0+1}} I(d_1,\dots,d_n) \geq \frac{1}{M_{n+1}} I(d_1,\dots,d_n). \end{aligned} \tag{9}$$

Write

$$G(d_1,\dots,d_n) = \frac{1}{M_{n+1}} I(d_1,\dots,d_n). \tag{10}$$

Now we are in the position to show Theorem 2.4. We divide the proof into three propositions.

Proposition 3.1. *Suppose $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. If $\{M_n, n \geq 1\}$ is bounded, we have $\dim_{\mathrm{H}} C = 1$.*

Proposition 3.2. *Suppose $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. If $M_n \to \infty$, as $n \to \infty$, and $\lim_{n\to\infty} \frac{\log M_{n+1}}{\log M_n} = 1$, we have $\dim_{\mathrm{H}} C = 1$.*

Proposition 3.3. *Assume $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. Suppose $M_n \to \infty$, as $n \to \infty$, $\lim_{n\to\infty} \frac{\log M_{n+1}}{\log M_n} = b > 1$, $\{h_n, n \geq 1\}$ is of order t and $\lim_{n\to\infty} \frac{\log(M_n - L_n)}{\log M_n} = \beta$, then $\dim_{\mathrm{H}} C = \frac{\beta(b-t)+t}{(2b-\beta b+\beta)(b-t)+t}$ if $b > t$ and $\dim_{\mathrm{H}} C = 1$ if $b \leq t$.*

The proof of Proposition 3.1 is the same as that in Proposition 3.2, except some minor modifications. Also it can be done with the ideas given in [17]. So, we show Proposition 3.2 and 3.3 in details only.

In the sequel, the following Stolz's formula is used several times, so we state it as a lemma here.

Lemma 3.4. *Let $\{a_n, b_n, n \geq 1\}$ be two real sequences. If a_n tends to infinity increasingly as $n \to \infty$ and there exists $-\infty \leq \alpha \leq \infty$ such that*

$$\lim_{n\to\infty} \frac{b_n - b_{n-1}}{a_n - a_{n-1}} = \alpha,$$

Then $\lim_{n\to\infty} \frac{b_n}{a_n} = \alpha$.

Proof of Proposition 3.2. By (7), the assumptions on M_n and Stolz's formula, we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n-1} \log\left[\left(\frac{1}{2\alpha}\right)^j \prod_{i=1}^{j} M_i\right]}{\sum_{j=1}^{n+1} 2\log M_j + \log M_{n+1}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \log M_i - n\log 2\alpha}{3\log M_{n+2} - \log M_{n+1}} = \infty. \tag{11}$$

For any $0 < \epsilon < 1$, let $\epsilon' = \frac{\epsilon}{1-\epsilon}$. By (11), there exists $k_1 \in \mathbb{N}$ such that for any $n \geq k_1$, we have

$$\prod_{j=1}^{n-1} \left(\frac{1}{(2\alpha)^j} \prod_{i=1}^{j} M_i\right)^{\epsilon'} \geq \prod_{j=1}^{n+1} M_j^2 \cdot M_{n+1}. \tag{12}$$

Set

$$E(M_1) = \left\{x \in (0,1] : d_1(x) = M_1,\ L_{n+1} < \frac{d_{n+1}(x)}{h_n(d_n(x))} \leq M_{n+1} \text{ for all } n \geq 1\right\}.$$

It is easy to see $E(M_1) \subset C$. From the definition of $D_n(\bar{d_1}, \dots, \bar{d_k})$ and $J(d_1, \dots, d_n)$, we have

$$E(M_1) = \bigcap_{n=1}^{\infty} \bigcup_{(d_1,\dots,d_n)\in D_n(M_1)} J(d_1, \dots, d_n).$$

Set $\mu(J(M_1)) = 1$. For any $n \geq 2$ and $J(d_1, \dots, d_n) \in D_n(M_1)$, set

$$\mu(J(d_1, \dots, d_n)) = \prod_{j=1}^{n-1} \left(\frac{1}{M_{j+1} - L_{j+1}} \cdot \frac{1}{h_j(d_j)} \right). \tag{13}$$

Then μ is a probability mass distribution supported on $E(M_1)$, because

$$\begin{aligned} &\sum_{L_{n+1}h_n(d_n)<d_{n+1}\leq M_{n+1}h_n(d_n)} \mu(J(d_1, \dots, d_n, d_{n+1})) \\ &= \mu(J(d_1, \dots, d_n)) \times \sum_{L_{n+1}h_n(d_n)<d_{n+1}\leq M_{n+1}h_n(d_n)} \frac{1}{M_{n+1} - L_{n+1}} \cdot \frac{1}{h_n(d_n)} \\ &= \mu(J(d_1, \dots, d_n)). \end{aligned}$$

In order to apply Lemma 2.9 to give a lower bound estimation on $\dim_H E(M_1)$, we will estimate the measure of arbitrary balls.

We claim first that, for any $(d_1, \dots, d_n) \in D_n(M_1)$,

$$|J(d_1, \dots, d_n)| \geq M_{n+1} \big(\mu(J(d_1, \dots, d_n))\big)^{1+\epsilon'}, \tag{14}$$

which, in fact, is the essential point in getting the desired result. Note that for any $(d_1, \dots, d_n) \in D_n(M_1)$,

$$h_n(d_n) \geq \frac{1}{2} d_n \geq \frac{1}{2} L_n h_{n-1}(d_{n-1}) \geq \cdots \geq \frac{1}{2^n} \prod_{j=1}^{n} L_j \geq \frac{1}{(2\alpha)^n} \prod_{j=1}^{n} M_j. \tag{15}$$

Combine (12) and (15), we have,

$$\Big| J(d_1, \dots, d_n) \Big| \geq \frac{1}{M_{n+1}^2} \prod_{j=1}^{n} \frac{1}{M_j^2} \prod_{j=1}^{n-1} \frac{1}{h_j(d_j)} \geq M_{n+1} \left(\prod_{j=1}^{n-1} \frac{1}{h_j(d_j)} \right)^{1+\epsilon'}. \tag{16}$$

So, we get the claims.

Let $r_0 = \min\{G(d_1, \dots, d_{k_1}), (d_1, \dots, d_{k_1}) \in D_{k_1}(M_1)\}$. Now we estimate $\mu(B(x, r))$ for any $x \in E(M_1)$ and $0 < r < r_0$. For any $x \in E(M_1)$, there exists a sequence $d_1, d_2, \ldots$ such that $(d_1, \ldots, d_n) \in D_n(M_1)$ and $x \in J(d_1, \ldots, d_n)$ for all $n \geq 1$. Choose $n \geq k_1$ such that

$$G(d_1, \ldots, d_{n+1}) \leq r < G(d_1, \ldots, d_n).$$

By the definition of $G(d_1, \ldots, d_n)$, we know that $B(x, r)$ can intersect only one n-th order basic interval which is $J(d_1, \ldots, d_n)$. For the number of $(n+1)$-th basic intervals that $B(x, r)$ can intersect, we distinguish two cases.

Case I. $G(d_1, \ldots, d_{n+1}) \leq r < |I(d_1, \ldots, d_{n+1})|$.
In this case, $B(x, r)$ can intersect at most six $(n+1)$-th order admissible intervals $I(d_1, \ldots, d_n, d_{n+1} + i)$, $-1 \leq i \leq 4$. This is because

$$r \leq \min\Big\{\big|I(d_1, \ldots, d_{n+1} - 1)\big|, \ \sum_{i=1}^{4} \big|I(d_1, \ldots, d_{n+1} + i)\big|\Big\}.$$

for $d_{n+1} \geq h_n(d_n) + 1 \geq d_n \geq M_1 \geq 3$. Thus, by (14), we get

$$\begin{aligned} \mu(B(x, r)) &\leq 6\mu(J(d_1, \ldots, d_{n+1})) \leq 6\left(\frac{1}{M_{n+2}}|J(d_1, \ldots, d_{n+1})|\right)^{1-\epsilon} \\ &\leq 6\left(\frac{1}{M_{n+2}}|I(d_1, \ldots, d_{n+1})|\right)^{1-\epsilon} = 6|G(d_1, \ldots, d_{n+1})|^{1-\epsilon} \leq 6r^{1-\epsilon}. \end{aligned} \tag{17}$$

Case II. $|I(d_1, \ldots, d_{n+1})| \leq r < G(d_1, \ldots, d_n)$.
By Proposition 2.2, we have for any $(d_1, \ldots, d_n, d'_{n+1}) \in D_{n+1}(M_1)$,

$$\begin{aligned} |I(d_1, \ldots, d_n, d'_{n+1})| &= |I(d_1, \ldots, d_n)| \cdot \frac{h_n(d_n)}{d'_{n+1}(d'_{n+1} - 1)} \\ &\geq |I(d_1, \ldots, d_n)| \cdot \frac{1}{M_{n+1}^2 h_n(d_n)}, \end{aligned}$$

thus $B(x, r)$ can intersect at most

$$\ell := 4rM_{n+1}^2 h_n(d_n) \cdot |I(d_1, \ldots, d_n)|^{-1}$$

$(n+1)$-th order basic intervals. Therefore

$$\mu(B(x, r)) \leq \min\Big\{\mu(J(d_1, \ldots, d_n)), \ \sum_{i} \mu(J(d_1, \ldots, d_n, i))\Big\},$$

where the sum is over all i such that $\max\{d_{n+1} - \ell, \ h_n(d_n) + 1\} \leq i \leq d_{n+1} + \ell$. By (13) and (14), we have

$$\mu(B(x, r)) \leq \mu(J(d_1, \ldots, d_n))$$

$$
\begin{aligned}
&\cdot \min\left\{1,\ 8rM_{n+1}^2 h_n(d_n)|I(d_1,\ldots,d_n)|^{-1}\frac{1}{M_{n+1}-L_{n+1}}\frac{1}{h_n(d_n)}\right\}\\
&\le 8\left(\frac{1}{M_{n+1}}|J(d_1,\ldots,d_n)|\right)^{1-\epsilon}\\
&\cdot 1^{\epsilon}(rM_{n+1}^2)^{1-\epsilon}|I(d_1,\ldots,d_n)|^{-(1-\epsilon)}\left(\frac{1}{M_{n+1}-L_{n+1}}\right)^{1-\epsilon}\\
&= 8\left(\frac{1}{L_{n+1}}\right)^{1-\epsilon} r^{1-\epsilon} \le 8r^{1-\epsilon}. \qquad (18)
\end{aligned}
$$

By (17), (18) and Lemma 2.9, we have

$$\dim_{\mathrm{H}} E(M_1) \ge 1-\epsilon.$$

Since $\epsilon > 0$ is arbitrary and $E(M_1) \subset C$, we have

$$\dim_{\mathrm{H}} C = 1. \qquad \square$$

Proof of Proposition 3.3. For any $n \ge 1$ and $\bar{d_1} \ge 2$, let

$$
\begin{aligned}
H_n(\bar{d_1}) &= \left[\left(\frac{c_1}{\alpha}\right)^{1+t+\cdots+t^{n-1}} M_n^t M_{n-1}^{t^2}\ldots M_2^{t^{n-1}} \bar{d_1}^{\,t^n}\right],\\
G_n(\bar{d_1}) &= \left[c_2^{1+t\cdots+t^{n-1}} M_n^t M_{n-1}^{t^2}\ldots M_2^{t^{n-1}} \bar{d_1}^{\,t^n}\right] + 1.
\end{aligned}
$$

We divide the proof into two parts.

Part I. $b > t$. Write $s_0 = \frac{\beta(b-t)+t}{(2b-\beta b+\beta)(b-t)+t}$. For this case, we know

$$\lim_{n\to\infty} \frac{\sum_{j=2}^n \log(M_j - L_j) + \sum_{j=2}^n \log G_{j-1}(\bar{d_1})}{2\sum_{j=2}^n \log M_{j+1} + \sum_{j=2}^n \log G_{j-1}(\bar{d_1}) - \log(M_{n+1}-L_{n+1})} = s_0, \qquad (19)$$

which gives, for any $s > s_0$ and n large enough,

$$\prod_{j=2}^n \Big((M_j - L_j)G_{j-1}(\bar{d_1})\Big)^{1-s} \alpha^{2ns} \prod_{j=2}^n \Big(\frac{M_{j+1}-L_{j+1}}{M_{j+1}^2}\Big)^s \le 1. \qquad (20)$$

For any $\bar{d_1} \ge 2$, let

$$E(\bar{d_1}) = \Big\{x \in (0,1] : d_1(x) = \bar{d_1},\ L_{n+1} < \frac{d_{n+1}(x)}{h_n(d_n(x))} \le M_{n+1},\ \text{for all } n \ge 1\Big\}.$$

Then

$$C = \bigcup_{\bar{d_1}=2}^{\infty} E(\bar{d_1}).$$

For any $\bar{d_1} \geq 2$, $x \in E(\bar{d_1})$ and $n \geq 1$, since $\{h_n, n \geq 1\}$ is of order t, then

$$h_n(d_n(x)) \leq c_2 d_n^t(x) \leq c_2 M_n^t h_{n-1}^t(d_{n-1}(x)).$$

By iteration, we have for any $x \in E(\bar{d_1})$ and $n \geq 1$,

$$H_n(\bar{d_1}) < h_n(d_n(x)) \leq G_n(\bar{d_1}).$$

Note that

$$E(\bar{d_1}) = \bigcap_{n=1}^{\infty} \bigcup_{(d_1,\ldots,d_n)\in D_n(\bar{d_1})} J(d_1, \ldots, d_n),$$

which follows that

$$\begin{aligned}
\mathbf{H}^s(E(\bar{d_1})) &\leq \liminf_{n\to\infty} \sum_{(d_1,\ldots,d_{n-1},d_n)\in D_n(\bar{d_1})} \big|J(d_1, \ldots, d_{n-1}, d_n)\big|^s \\
&= \liminf_{n\to\infty} \sum_{(d_1,\ldots,d_{n-1})\in D_{n-1}(\bar{d_1})} \big|J(d_1, \ldots, d_{n-1})\big|^s \\
&\quad \cdot \sum_{L_n < \frac{d_n}{h_{n-1}(d_{n-1})} \leq M_n} \left(\frac{M_{n+1} - L_{n+1}}{M_{n+1}L_{n+1}} \, \frac{M_n L_n}{M_n - L_n} \, \frac{h_{n-1}(d_{n-1})}{d_n(d_n - 1)} \right)^s \\
&\leq \liminf_{n\to\infty} \sum_{(d_1,\ldots,d_{n-1})\in D_{n-1}(\bar{d_1})} \big|J(d_1, \ldots, d_{n-1})\big|^s \\
&\quad \cdot \big((M_n - L_n)h_{n-1}(d_{n-1})\big)^{1-s} \alpha^{2s} \left(\frac{M_{n+1} - L_{n+1}}{M_{n+1}^2} \right)^s \leq \cdots \\
&\leq \liminf_{n\to\infty} \prod_{j=2}^{n} \big((M_j - L_j)h_{j-1}(d_{j-1})\big)^{1-s} \alpha^{2ns} \prod_{j=2}^{n} \left(\frac{M_{j+1} - L_{j+1}}{M_{j+1}^2} \right)^s \\
&\leq \liminf_{n\to\infty} \prod_{j=2}^{n} \big((M_j - L_j)G_{j-1}(\bar{d_1})\big)^{1-s} \alpha^{2ns} \prod_{j=2}^{n} \left(\frac{M_{j+1} - L_{j+1}}{M_{j+1}^2} \right)^s \\
&\leq 1. \qquad\qquad \text{(by (20))}
\end{aligned}$$

Therefore, $\dim_H E(\bar{d_1}) \leq s$. By the arbitrariness of $s > s_0$, we have

$$\dim_{\mathrm{H}} C \leq \sup_{\bar{d_1}\geq 2} \dim_{\mathrm{H}} E(\bar{d_1}) \leq \frac{\beta(b-t)+t}{(2b-\beta b+\beta)(b-t)+t}.$$

Now we prove the inverse inequality. Fix $\bar{d_1} \geq 2$. Let $\{t_n, n \geq 1\}$ be an integer sequence with $3 \leq t_n \leq (M_{n+1} - L_{n+1})G_n(\bar{d_1})$ for all $n \geq 1$. It should be noticed first that

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \log(M_{j+1} - L_{j+1}) + \sum_{j=1}^{n} \log H_j(\bar{d_1})}{2\sum_{j=1}^{n+1} \log M_j + \log M_{n+2} + \sum_{j=1}^{n} \log G_j(\bar{d_1})} = \frac{\beta(b-t)+t}{b(b-t)+b} \geq s_0.$$

Thus, for any $s' < s_0$, there exists $k_3 \in \mathbb{N}$ such that for any $n \geq k_3$ and $(d_1, \ldots, d_n) \in D_n(\bar{d_1})$, we have

$$\prod_{j=1}^{n} \frac{1}{M_{j+1} - L_{j+1}} \frac{1}{h_j(d_j)} \leq \left(\frac{1}{M_{n+2}} \prod_{j=0}^{n} \frac{h_j(d_j)}{d_{j+1}(d_{j+1}-1)} \right)^{s'}. \tag{21}$$

For any $n \geq 1$, denote

$$F_{\bar{d_1}}(t_n) = \frac{\sum_{j=1}^{n} \log(M_{j+1} - L_{j+1}) + \sum_{j=1}^{n} \log H_j(\bar{d_1}) - \log t_n}{2\sum_{j=1}^{n+1} \log M_j + \sum_{j=1}^{n} \log G_j(\bar{d_1}) - \log t_n + \log 12}.$$

Since $F_{\bar{d_1}}(\cdot)$ is monotonic decreasing with respect to t_n, we have

$$\liminf_{n\to\infty} F_{\bar{d_1}}(t_n) \geq \liminf_{n\to\infty} F_{\bar{d_1}}((M_{n+1} - L_{n+1})G_n(\bar{d_1})) = s_0.$$

Thus, for $s' < s_0$, there exists $k_4 \in \mathbb{N}$ such that for any $n \geq k_4$, $3 \leq t_n \leq (M_{n+1} - L_{n+1})G_n(\bar{d_1})$ and $(d_1, \ldots, d_n) \in D_n(\bar{d_1})$, we have

$$t_n \prod_{j=1}^{n} \frac{1}{M_{j+1} - L_{j+1}} \frac{1}{h_j(d_j)} \leq \left(\frac{t_n}{12} \prod_{j=1}^{n} \frac{1}{M_{j+1}^2 h_j(d_j)} \right)^{s'}. \tag{22}$$

We can do as the same way as in the proof of Proposition 3.2 to define a probability measure μ supported on $E(\bar{d_1})$, i.e., $\mu(J(\bar{d_1})) = 1$, and

$$\mu(J(d_1, \ldots, d_n)) = \prod_{j=1}^{n-1} \left(\frac{1}{M_{j+1} - L_{j+1}} \cdot \frac{1}{h_j(d_j)} \right),$$

for any $n \geq 2$ and $(d_1, \ldots, d_n) \in D_n(\bar{d_1})$.

In fact, by (19), s_0 is just the Billingsley dimension of $E(\bar{d_1})$ with respect to the measure μ defined above. In the following, what we will done is just to check that $\dim_H E(\bar{d_1})$ coincides with its Billingsley dimension.(see also [19] for cases when Hausdorff dimension coincides with its Billingsley dimension.)

Let $k_2 = \max\{k_3, k_4\}$. Then for any $n \geq k_2$ and $(d_1, \ldots, d_{n+1}) \in D_{n+1}(\bar{d_1})$, by (21), we have

$$\mu(J(d_1, \ldots, d_{n+1})) \leq \left(\frac{1}{M_{n+2}} \prod_{j=0}^{n} \frac{h_j(d_j)}{d_{j+1}(d_{j+1}-1)} \right)^{s'}. \tag{23}$$

Now we estimate $\mu(B(x,r))$ for any $x \in E(\bar{d_1})$ and $r > 0$ small enough. For any $x \in E(\bar{d_1})$, there exists a sequence $d_1, d_2, \ldots$ such that $(d_1, \ldots, d_n) \in D_n(\bar{d_1})$ and $x \in J(d_1, d_2, \ldots, d_n)$ for all $n \geq 1$. For any $0 < r < \min\{G(d_1, \ldots, d_{k_2}),$ $(d_1, \ldots, d_{k_2}) \in D_{k_2}(\bar{d_1})\}$, choose $n \geq k_2$ such that

$$G(d_1, \ldots, d_{n+1}) \leq r < G(d_1, \ldots, d_n).$$

Case I. $G(d_1, \ldots, d_{n+1}) \leq r < |I(d_1, \ldots, d_{n+1})|$.
In this case, by (23), we have

$$\mu(B(x,r)) \leq 6\mu(J(d_1, \ldots, d_{n+1})) \leq 6(G(d_1, \ldots, d_{n+1}))^{s'} \leq 6r^{s'}. \tag{24}$$

Case II. $|I(d_1, \ldots, d_{n+1})| \leq r < G(d_1, \ldots, d_n)$.
Denote by $t_n(r)$ the number of $(n+1)$-th order admissible intervals that the ball $B(x,r)$ can intersect. Then evidently that $1 \leq t_n(r) \leq (M_{n+1} - L_{n+1})h_n(d_n)$. If $t_n(r) \leq 5$, then by (24),

$$\mu(B(x,r)) \leq 5\mu(J(d_1, \ldots, d_{n+1})) \leq 5r^{s'}. \tag{25}$$

If $t_n(r) \geq 6$, then $B(x,r)$ contains at least $[\frac{t_n(r)}{3}]$ many $(n+1)$-th order admissible intervals, thus

$$r \geq \frac{t_n(r)}{12} \prod_{j=0}^{n-1} \frac{h_j(d_j)}{d_{j+1}(d_{j+1}-1)} \frac{1}{M_{n+1}^2 h_n(d_n)} \geq \frac{t_n(r)}{12} \prod_{j=1}^{n} \frac{1}{M_{j+1}^2 h_j(d_j)}.$$

By (22), we have

$$\mu(B(x,r)) \leq t_n(r) \prod_{j=1}^{n} \frac{1}{M_{j+1} - L_{j+1}} \frac{1}{h_j(d_j)} \leq r^{s'}. \tag{26}$$

Combine (24), (25) (26) and Lemma 2.9, we have $\dim_{\mathrm{H}} E(\bar{d_1}) \geq s'$. Since $E(\bar{d_1}) \subset C$ and $s' < s_0$ is arbitrary, we have $\dim_{\mathrm{H}} C \geq s_0$.

Part II. $b \le t$. Choose $\bar{d_1} \ge 2$ such that $\log \bar{d_1}^{t-1} > \log \frac{\alpha}{c_1}$, and let

$$E(\bar{d_1}) = \Big\{ x \in (0,1] : d_1(x) = \bar{d_1}, L_{n+1} < \frac{d_{n+1}(x)}{h_n(d_n(x))} \le M_{n+1} \text{ for all } n \ge 1 \Big\}.$$

By the choice of $\bar{d_1}$, we have

$$\liminf_{n\to\infty} \frac{\sum\limits_{j=1}^{n-1} \log H_j(\bar{d_1})}{\log M_{n+1} + 2\sum\limits_{j=1}^{n+1} \log M_j} \ge \liminf_{n\to\infty} \frac{\sum\limits_{j=1}^{n-1} \log (M_j^t \dots M_2^{t^{j-1}})}{\log M_{n+1} + 2\sum\limits_{j=1}^{n+1} \log M_j} = \infty. \tag{27}$$

For any $\epsilon > 0$, let $\epsilon' = \frac{\epsilon}{1-\epsilon}$. By (27), there exists $k_5 \in \mathbb{N}$ such that for any $n \ge k_5$,

$$M_{n+1} \prod_{j=1}^{n+1} M_j^2 \le \Big(\prod_{j=1}^{n-1} H_j(\bar{d_1}) \Big)^{\epsilon'}. \tag{28}$$

As a consequence, for any $n \ge k_5$ and $(d_1, \dots, d_n) \in D_n(\bar{d_1})$, we have

$$|J(d_1, \dots, d_n)| \ge \prod_{j=1}^{n+1} \frac{1}{M_j^2} \prod_{j=1}^{n-1} \frac{1}{h_j(d_j)} \ge M_{n+1} \Big(\prod_{j=1}^{n-1} \frac{1}{h_j(d_j)} \Big)^{1+\epsilon'}. \tag{29}$$

Combine this and formula (16) in Proposition 3.2, we can get $\dim_{\mathrm{H}} C = 1$ by following the proof in Proposition 3.2 step by step. The proof of Proposition 3.3 is finished. □

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BAO-WEI WANG
DEPARTMENT OF MATHEMATICS
HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY
WUHAN, HUBEI, 430074
P.R. CHINA

E-mail: bwei_wang@yahoo.com.cn

JUN WU
DEPARTMENT OF MATHEMATICS
HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY
WUHAN, HUBEI, 430074, P.R. CHINA

E-mail: wujunyu@public.wh.hb.cn

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