Publ. Math. Debrecen
73/3-4 (2008), 497–500

On the volume of the convex hull of $d+1$ segments in $\mathbb{R}^d$

By GENNADIY AVERKOV (Magdeburg) and HORST MARTINI (Chemnitz)

Abstract. Let $d \geq 2$, $m \geq d$, and $u_1, \dots, u_m$ be non-zero vectors linearly spanning $\mathbb{R}^d$. The note is devoted to the problem of minimizing the volume of the polytopes $P := \operatorname{conv}(I_1 \cup \dots \cup I_m)$, where, for $j = 1, \dots, m$, I_j is a translate of $\operatorname{conv}\{o, u_j\}$. The solution of this problem for the case $m = d$ was previously known. For the case $m = d+1$ the minimal volume is evaluated and the class of minimizing polytopes P is studied.

1. Introduction

Let $d \geq 2$. The origin in $\mathbb{R}^d$ is denoted by o. The abbreviations conv and vol stand for convex hull and volume, respectively. Let $d \geq 2$, $m \geq d$, and $u_1, \dots, u_m$ be non-zero vectors linearly spanning $\mathbb{R}^d$. We consider the class $\mathcal{P}(u_1, \dots, u_m)$ of convex polytopes P in $\mathbb{R}^d$ such that $P = \operatorname{conv}(I_1 \cup \dots \cup I_m)$, where, for $j = 1, \dots, m$, the set I_j is a translate of the segment $\operatorname{conv}\{o, u_j\}$. By $v(u_1, \dots, u_m)$ we denote the minimum among the volumes of all polytopes from $\mathcal{P}(u_1, \dots, u_d)$, and by $\mathcal{P}_0(u_1, \dots, u_k)$ the subclass of $\mathcal{P}(u_1, \dots, u_m)$ minimizing the volume. It is known that $v(u_1, \dots, u_d) = \frac{1}{d!} \det(u_1, \dots, u_d)$. In this note we evaluate $v(u_1, \dots, u_{d+1})$ and study the properties of $\mathcal{P}_0(u_1, \dots, u_{d+1})$. For the case $m = d$ the class $\mathcal{P}_0(u_1, \dots, u_m)$ and the corresponding quantity $v(u_1, \dots, u_m)$ were studied in [7] and [5]. In particular, in the above paper it was shown that

$$v(u_1, \dots, u_d) = \frac{1}{d!} |\det(u_1, \dots, u_d)|, \tag{1}$$

where $\det(u_1, \dots, u_d)$ denotes the determinant of the matrix with columns $u_1, \dots, u_d$. We perform an analogous study for the case $m = d+1$.

Mathematics Subject Classification: Primary: 52B11; Secondary: 52A40.
Key words and phrases: geometric inequality, minimum width, Minkowski space, polytope, reduced body, simplex, volume.

Theorem. *Let $d \geq 2$, and $u_1, \ldots, u_{d+1}$ be non-zero vectors linearly spanning $\mathbb{R}^d$. Then the following statements hold true.*

I. *The quantity $v(u_1, \ldots, u_{d+1})$ is equal to the maximum of $v(u'_1, \ldots, u'_d)$ over all possible subsets $\{u'_1, \ldots, u'_d\}$ of $\{u_1, \ldots, u_{d+1}\}$.*

II. *The class $\mathcal{P}_0(u_1, \ldots, u_{d+1})$ necessarily contains simplices.*

III. *Every polytope from $\mathcal{P}_0(u_1, \ldots, u_{d+1})$ has at most $2d$ vertices.*

The statement of Theorem can be reformulated in terms of *Minkowski geometry* (that is, the geometry of finite dimensional normed spaces); see [8] and [6]. In fact, if $\{\pm u_1, \ldots, \pm u_m\}$ is a set of vertices of some o-symmetric d-dimensional convex polytope B and $\mathcal{M}^d$ is the normed space whose unit ball is B, then the class $\mathcal{P}_0(u_1, \ldots, u_m)$ is precisely the class of convex bodies whose *minimum width*, measured with respect to $\mathcal{M}^d$, is equal to one and whose volume is minimal. It should also be mentioned that the elements of $\mathcal{P}(u_1, \ldots, u_m)$ contain all $\mathcal{M}^d$-*reduced bodies*, which were introduced in [4]. For more information on minimum width and reduced bodies in Minkowski spaces see [4], [3], and [2].

We ask about possible extensions of Theorem for the case of a larger number of segments. More precisely, we consider $d \geq 2$, $m \geq d$, and vectors $u_1, \ldots, u_m$ linearly spanning $\mathbb{R}^d$, and we pose the following problems.

1. Describe m and d such that for all $u_1, \ldots, u_m$ the quantity $v(u_1, \ldots, u_m)$ is the maximum of $v(u'_1, \ldots, u'_d)$ over all $\{u'_1, \ldots, u'_d\} \subseteq \{u_1, \ldots, u_m\}$.
2. Describe m and d such that for all $u_1, \ldots, u_m \in \mathbb{R}^d$ the class $\mathcal{P}_0(u_1, \ldots, u_m)$ necessarily contains simplices.
3. Describe m and d such that for all $u_1, \ldots, u_m$ the class $\mathcal{P}_0(u_1, \ldots, u_m)$ consists of polytopes with at most $2d$ vertices.

The solutions for the case $m \leq d+1$ are provided by Theorem and the results from [7] and [5]. Problems 2 and 3 can be solved for $d = 2$. Indeed, from the main result of [1] it follows that for $d = 2$ the class $\mathcal{P}_0(u_1, \ldots, u_m)$ necessarily contains triangles and, furthermore, the polygons from $\mathcal{P}_0(u_1, \ldots, u_m)$ distinct from triangles are necessarily quadrilaterals. To the best of our knowledge, all the remaining cases of Problems 1–3 are open.

2. The proof

Proof of Theorem. For every subset $\{u'_1, \dots, u'_d\}$ of $\{u_1, \dots, u_{d+1}\}$ the inequality $v(u'_1, \dots, u'_d) \le v(u_1, \dots, u_{d+1}\}$ is obvious. Let us show that for some $u'_1, \dots, u'_d$ as above the reverse inequality is fulfilled. Without loss of generality we assume that the maximum of $v(u'_1, \dots, u'_d)$ over $\{u'_1, \dots, u'_d\} \subseteq \{u_1, \dots, u_{d+1}\}$ is attained when $u'_i = u_i$ for $i = 1, \dots, d$.

For $i = 1, \dots, d$ we replace u_i by $-u_i$ (or equivalently, translate the original segment $\operatorname{conv}\{o, u_i\}$ by $-u_i$) arriving at the representation $u_{d+1} = \sum_{i=1}^{d} \beta_i u_i$ with each $\beta_i \ge 0$. Moreover, we assume that $u_1, \dots, u_d$ are ordered so that $\beta_1 \le \cdots \le \beta_d$. We have $\beta_d \le 1$, since otherwise $v(u_1, \dots, u_{d-1}, u_{d+1})$ would be larger than $v(u_1, \dots, u_d)$. We define the points $p_1, p_2, \dots, p_d$ by

$$p_i := \sum_{j=i}^{d} u_j.$$

Let us show that the simplex $T := \operatorname{conv}\{o, p_1, \dots, p_d\}$ belongs to $\mathcal{P}_0(u_1, \dots, u_{d+1})$. We introduce the values $\alpha_1, \dots, \alpha_d$ by the formulas

$$\alpha_1 := \beta_1, \qquad \alpha_i := \beta_i - \beta_{i-1} \qquad (2 \le i \le d).$$

It is easy to verify that for every $j = 1, \dots, d$ we have $0 \le \alpha_j \le 1$ and $\sum_{i=1}^{j} \alpha_i = \beta_j \le 1$. Now let us show that $u_{d+1} \in T$. We have

$$\begin{aligned}\left(1 - \sum_{i=1}^{d} \alpha_i\right) o + \sum_{i=1}^{d} \alpha_i p_i &= \sum_{i=1}^{d} \alpha_i p_i = \sum_{i=1}^{d} \alpha_i \sum_{j=i}^{d} u_j \\ &= \sum_{1 \le i \le j \le d} \alpha_i u_j = \sum_{j=1}^{d} \left(\sum_{i=1}^{j} \alpha_i\right) u_j = \sum_{j=1}^{d} \beta_j u_j = u_{d+1}.\end{aligned}$$

Thus, u_{d+1} is a convex combination of points $o, p_1, \dots, p_d$, and hence $u_{d+1} \in T$. Obviously, T can be represented by

$$T = \operatorname{conv}\left(\bigcup_{i=1}^{d-1} \operatorname{conv}\{p_i, p_{i+1}\} \cup \operatorname{conv}\{o, p_d\}\right).$$

But $\operatorname{conv}\{o, p_d\} = \operatorname{conv}\{o, u_d\}$, while, for every $i = 1, \dots, d-1$, the segment $\operatorname{conv}\{p_i, p_{i+1}\}$ is a translate of $\operatorname{conv}\{o, u_i\}$. Since $\operatorname{conv}\{o, u_{d+1}\} \subseteq T$, we see that $T \in \mathcal{P}(u_1, \dots, u_d)$. By construction,

$$\operatorname{vol}(T) = \frac{1}{d!} |\det(p_1, \dots, p_d)| = \frac{1}{d!} |\det(u_1, \dots, u_d)| = v(u_1, \dots, u_d).$$

Hence $v(u_1, \ldots, u_d) \geq v(u_1, \ldots, u_{d+1})$. This shows Part I and (since T is a simplex) also Part II of the theorem.

We prove Part III by contradiction. Assume that P is a polytope from $\mathcal{P}_0(u_1, \ldots, u_{d+1})$ with at least $2d+1$ vertices. Let $J_1, \ldots, J_{d+1}$ be segments in $\mathbb{R}^d$ such that, for $i = 1, \ldots, d+1$, the segment J_i is a translate of $\operatorname{conv}\{o, u_i\}$ and $P = \operatorname{conv}(J_1 \cup \cdots \cup J_{d+1})$. Since P has at least $2d+1$ vertices, at least one of the endpoints of J_{d+1} is also a vertex of P. Hence, taking into account (1),

$$\operatorname{vol}(P) > \operatorname{vol}(\operatorname{conv}(J_1 \cup \cdots \cup J_d)) \geq \frac{1}{d!} |\det(u_1, \ldots, u_d)| = \operatorname{vol}(T),$$

and we arrive at the contradiction. This finishes the proof of Part III. $\square$

References

[1] G. Averkov, On planar convex bodies of given Minkowskian thickness and least possible area, *Arch. Math.* **84**, no. 2 (2005), 183–192.

[2] G. Averkov and H. Martini, On reduced polytopes and antipodality, *Adv. Geom.* **8** (2008), 617–628.

[3] M. Lassak, Characterizations of reduced polytopes in finite-dimensional normed spaces, *Beiträge Algebra Geom.* **47**, no. 2 (2006), 559–566.

[4] M. Lassak and H. Martini, Reduced bodies in Minkowski space, *Acta Math. Hungar.* **106**, no. 1–2 (2005), 17–26.

[5] H. Martini, Das Volumen spezieller konvexer Polytope, *Elem. Math.* **44**, no. 5 (1989), 113–115.

[6] H. Martini, K. J. Swanepoel and G. Weiss, The geometry of Minkowski spaces – a survey, Part I, *Expo. Math.* **19**, no. 2 (2001), 97–142.

[7] P. McMullen, The volume of certain convex sets, *Math. Proc. Cambridge Philos. Soc.* **91**, no. 1 (1982), 91–97.

[8] A. C. Thompson, Minkowski Geometry, Encyclopedia of Mathematics and its Applications, Vol. 63, *Cambridge University Press*, *Cambridge*, 1996.

GENNADIY AVERKOV
FACULTY OF MATHEMATICS
UNIVERSITY OF MAGDEBURG
39106 MAGDEBURG
GERMANY

E-mail: gennadiy.averkov@googlemail.com

HORST MARTINI
FACULTY OF MATHEMATICS
UNIVERSITY OF TECHNOLOGY
09107 CHEMNITZ
GERMANY

E-mail: horst.martini@mathematik.tu-chemnitz.de

(Received March 27, 2008; revised June 26, 2008)