

A note on the diagonal mapping in spaces of analytic functions in the unit polydisc

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Abstract. We define two spaces $K^{p,q,\alpha,\beta}$ and $M^{p,\alpha}$ of analytic functions in the unit polydisc U^n of C^n , closely related to the mixed norm and the Bergman spaces on U^n , and for any holomorphic function F in $K^{p,q,\alpha,\beta}$ or in $M^{p,\alpha}$ we consider its restriction to the diagonal, i.e., the function in the unit disc U of C defined by $DF(z) = F(z, \dots, z)$, and prove that the diagonal mapping D maps $K^{p,q,\alpha,\beta}$ onto the mixed-norm space $H^{p,q,\beta+\frac{q}{p}(|\alpha|+2n-1)}(U)$ and the space $M^{p,\alpha}$ onto the Bergman space $A^{p,|\alpha|+2n-1}(U)$.

Introduction

Let U^n be the unit polydisc in C^n and T^n be its Shilov boundary (see [R]) ($U^1 = U$ and $T^1 = T$). Denote by dm_n the normalized volume measure in U^n , and by $d\sigma_n$ the normalized surface measure on T^n .

For any Lebesgue measurable function f in U^n , we define

$$M_p(r, f) = \left(\int_{T^n} |f(r\xi)|^p d\sigma_n(\xi) \right)^{1/p},$$

where $0 < p < \infty$ and $r\xi = (r_1\xi_1, \dots, r_n\xi_n)$.

If $0 < p < \infty$, $0 < q < \infty$, and $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$, let

$$\|f\|_{p,q,\alpha}^q = \int_{I^n} \left(\prod_{j=1}^n (1 - r_j^2)^{\alpha_j} M_p(r, f)^q \right) dr,$$

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where $I^n = [0, 1]^n$ and $dr = dr_1 \dots dr_n$. The mixed norm space $H^{p,q,\alpha} = H^{p,q,\alpha}(U^n)$ is then defined to be the space of functions f holomorphic in U^n , ($f \in H(U^n)$) such that $\|f\|_{p,q,\alpha} < \infty$.

First we give a new characterization of the mixed norm spaces $H^{p,q,\alpha}(U)$.

Theorem 1. *Let $0 < p, q < \infty$ and $-1 < \beta, \gamma < \infty$. A function $f \in H(U)$ belongs to $H^{p,q,\beta+q/p(\gamma+1)}(U)$ if and only if*

$$\int_0^1 \left(\int_{|z|<r} |f(z)|^p (1-|z|)^\gamma dm_1(z) \right)^{q/p} (1-r)^\beta dr < \infty.$$

Motivated by this characterization we define the spaces $K^{p,q,\alpha,\beta}(U^n)$ as follows.

The space $K^{p,q,\alpha,\beta} = K^{p,q,\alpha,\beta}(U^n)$ consists of all $f \in H(U^n)$ such that

$$\|f\|_{p,q,\alpha,\beta}^q = \int_0^1 \left(\int_{|z_1|<r} \dots \int_{|z_n|<r} |f(z)|^p \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dm_n(z) \right)^{q/p} (1-r)^\beta dr < \infty,$$

where, $0 < p, q < \infty$, $-1 < \beta < \infty$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$.

We note that the spaces $K^{p,q,\alpha,\beta}(U)$ were studied by [AJ] and [J].

To each $F \in H(U^n)$, we associate a function DF , defined on the unit disc U of C , by

$$DF(z) = F(z, \dots, z), \quad z \in U.$$

The problem of description of diagonal of subspaces of $H(U^n)$ was studied by many authors (see [DS], [MR], [HO], [RS], [Sh1], [Sh2], [S].)

Now we are ready to state the main result of this paper.

Theorem 2. *Let $0 < p, q < \infty$, $-1 < \beta < \infty$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$. Then*

$$DK^{p,q,\alpha,\beta}(U^n) = H^{p,q,\beta+q/p(|\alpha|+2n-1)}(U),$$

where $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Note that in [RS] it was shown that

$$DH^{p,q,\alpha}(U^n) = H^{p,q,|\alpha|+(q/p+1)(n-1)}(U).$$

A special case of mixed norm spaces $H^{p,q,\alpha}(U^n)$ are the Bergman spaces $A^{p,\alpha}(U^n)$, $0 < p < \infty$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$,

$$A^{p,\alpha} = A^{p,\alpha}(U^n) = H^{p,p,\alpha}(U^n).$$

J. SHAPIRO [S] and F. SHAMOYAN [Sh2] proved that

$$DA^{p,\alpha}(U^n) = A^{p,|\alpha|+2n-2}(U).$$

For example, if $\alpha' = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \dots, \alpha_n)$, then

$$DA^{p,\alpha'}(U^n) = A^{p,|\alpha|+2n-1}(U).$$

In this paper we define the space $M^{p,\alpha}(U^n)$, $0 < p < \infty$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$, that contains $A^{p,\alpha'}(U^n)$, for $\alpha' = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \dots, \alpha_n)$ and show that $DM^{p,\alpha}(U^n) = A^{p,|\alpha|+2n-1}(U)$.

Let $0 < p < \infty$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$. Define $M^{p,\alpha} = M^{p,\alpha}(U^n)$ as the space of all $f \in H(U^n)$ such that

$$\|f\|_{p,\alpha}^p = \int_T \int_{\Gamma_t(\xi)} \dots \int_{\Gamma_t(\xi)} |f(z)|^p \prod_{k=1}^n (1 - |z_k|^2)^{\alpha_k} dm_n(z) d\sigma_1(\xi) < \infty,$$

where $\Gamma_t(\xi) = \{z \in U : |z - \xi| < t(1 - |z|)\}$, $t > 1$, $\xi \in T$ is the Stolz angle with vertex ξ .

Theorem 3. *Let $0 < p < \infty$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$. Then*

$$DM^{p,\alpha}(U^n) = A^{p,|\alpha|+2n-1}(U),$$

where $|\alpha| = \alpha_1 + \dots + \alpha_n$.

1. Preliminaries

In this Section we gather several partially well-known lemmas that will be used in the proofs of our results.

Lemma 1. *Let $b > a > 0$, $c > 0$ and let $g : [0, 1] \rightarrow [0, \infty)$ be measurable. Assume either $0 < k < 1$ and g is increasing or $1 \leq k < \infty$. Then*

$$\int_0^1 (1-r)^{ka-1} \left(\int_0^1 \frac{g(\rho)(1-\rho)^{c-1} d\rho}{(1-r\rho)^b} \right)^k dr \leq C \int_0^1 (1-r)^{k(a+c-b)-1} g^k(r) dr.$$

See [RS, Proposition 3.1]. See also [AJ] and [M].

Lemma 2. *Let $0 < p < \infty$ and $\beta > 0$. Let (a_n) be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} 2^{-n\beta} a_n^p < \infty$. Then there is a constant $C > 0$, depending only on p and β , such that*

$$\sum_{n=1}^{\infty} 2^{-n\beta} a_n^p \leq C \left(a_0^p + \sum_{n=1}^{\infty} 2^{-n\beta} |a_n - a_{n-1}|^p \right).$$

See [MP, Lemma HL].

Lemma 3. *Let $0 < p < 1$, $\beta > -1$ and $\beta p + 2p > 1$. If $f \in H^{p,p,\beta p+2p-2}(U)$, then $f \in H^{1,1,\beta}(U)$. Moreover, there is a constant $C > 0$, depending only on p and β , such that*

$$\|f\|_{1,1,\beta} \leq C \|f\|_{p,p,\beta p+2p-2}.$$

PROOF. By using the standard estimate

$$M_1(r^2, f) \leq C(1-r)^{1-(1/p)} M_p(r, f),$$

(see [D]), we find that

$$\int_U |f(z)|(1-|z|)^\beta dm_1(z) \leq C \int_0^1 M_p(r, f)(1-r)^{\beta-(1/p)+1} dr.$$

Let $r_n = 1 - 2^{-n}$, $n = 0, 1, 2, \dots$. Then we have

$$\begin{aligned} \|f\|_{1,1,\beta}^p &\leq C \left(\sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} M_p(r, f)(1-r)^{\beta-(1/p)+1} dr \right)^p \\ &\leq C \sum_{n=1}^{\infty} 2^{-np(\beta-(1/p)+2)} M_p^p(r_n, f) \\ &\leq C \sum_{n=0}^{\infty} \int_{r_n}^{r_{n+1}} M_p^p(r, f)(1-r)^{p(\beta-(1/p)+2)-1} dr \\ &\leq C \int_U |f(z)|^p (1-|z|)^{\beta p+2p-2} dm_1(z). \end{aligned} \quad \square$$

An integration in polar coordinates shows that the following is true.

Lemma 4. *Let $\beta > -1$ and $\gamma - \beta > 2$. Then there is a constant C , depending only on β and γ , such that*

$$\int_{|z|<r} \frac{(1-|z|)^\beta}{|1-\bar{w}z|^\gamma} dm_1(z) \leq \frac{C}{(1-r|w|)^{\gamma-\beta-2}}, \quad |w| < 1, \quad 0 < r \leq 1.$$

Lemma 5. *Let $\beta > -1$, $\gamma > \beta + 2$ and $t > 1$. Then*

$$\int_{\Gamma_t(\xi)} \frac{(1 - |z|)^\beta}{|1 - \bar{z}w|^\gamma} dm_1(z) \leq \frac{C}{|1 - \xi\bar{w}|^{\gamma-\beta-2}}, \quad \xi \in T, |w| < 1,$$

where C is a constant depending only on β , γ and t .

Lemma 6. *Let $\beta > 1$. Then*

$$\int_T \frac{d\sigma_1(\xi)}{|1 - \xi z|^\beta} \leq \frac{C}{(1 - |z|)^{\beta-1}}, \quad |z| < 1.$$

See [D].

2. On characterization of the mixed norm spaces in the unit polydisc U

In this section we prove Theorem 1.

PROOF OF THEOREM 1. First, assume that $\|f\|_{p,q,\beta+q/p(\gamma+1)} < \infty$. For $0 < t < 1$ define $f_t(z) = f(tz)$, $z \in U$. Let $r_n = 1 - 2^{-n}$, $n = 0, 1, \dots$. Then

$$\begin{aligned} & \int_0^1 (1-r)^\beta \left(\int_{|z|<r} |f_t(z)|^p (1-|z|)^\gamma dm_1(z) \right)^{q/p} dr \\ &= \sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} (1-r)^\beta \left(\int_{|z|<r} |f_t(z)|^p (1-|z|)^\gamma dm_1(z) \right)^{q/p} dr \leq C \sum_{n=1}^{\infty} 2^{-n(\beta+1)} A_n^{q/p}, \end{aligned}$$

where

$$A_n = \int_{|z|<r_n} |f_t(z)|^p (1-|z|)^\gamma dm_1(z).$$

Now by using Lemma 2 we find that

$$\begin{aligned} & \int_0^1 (1-r)^\beta \left(\int_{|z|<r} |f_t(z)|^p (1-|z|)^\gamma dm_1(z) \right)^{q/p} dr \\ & \leq C \sum_{n=1}^{\infty} 2^{-n(\beta+1)} \left(\int_{r_{n-1} \leq |z| < r_n} |f_t(z)|^p (1-|z|)^\gamma dm_1(z) \right)^{q/p} \\ & \leq C \sum_{n=1}^{\infty} 2^{-n(\beta+1)} M_p(r_n, f_t)^q 2^{-n(\gamma+1)q/p} \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{n=1}^{\infty} \int_{r_n}^{r_{n+1}} M_p(r, f_t)^q (1-r)^{\beta+q/p(\gamma+1)} dr \\
&\leq C \int_0^1 (1-r)^{\beta+q/p(\gamma+1)} M_p(r, f_t)^q dr.
\end{aligned}$$

Letting $t \rightarrow 1$, we get

$$\int_0^1 (1-r)^{\beta} \left(\int_{|z|<r} |f(z)|^p (1-|z|)^{\gamma} dm_1(z) \right)^{q/p} dr \leq C \|f\|_{p,q,\beta+q/p(\gamma+1)}^q.$$

Conversely,

$$\begin{aligned}
\|f\|_{p,q,\beta+q/p(\gamma+1)}^q &= \int_0^1 (1-r)^{\beta+q/p(\gamma+1)} M_p(r, f)^q dr \\
&= \sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} (1-r)^{\beta+q/p(\gamma+1)} M_p(r, f)^q dr \\
&\leq C \sum_{n=1}^{\infty} 2^{-n(\beta+q/p(\gamma+1)+1)} M_p(r_n, f)^q \\
&\leq C \sum_{n=1}^{\infty} \left(\int_{r_n < |z| < r_{n+1}} |f(z)|^p (1-|z|)^{\gamma} dm_1(z) \right)^{q/p} 2^{-n(\beta+1)} \\
&\leq C \sum_{n=1}^{\infty} \left(\int_{r_n < |z| < r_{n+1}} |f(z)|^p (1-|z|)^{\gamma} dm_1(z) \right)^{q/p} \int_{r_{n+1}}^{r_{n+2}} (1-r)^{\beta} dr \\
&\leq C \sum_{n=1}^{\infty} \int_{r_{n+1}}^{r_{n+2}} (1-r)^{\beta} \left(\int_{|z|<r} |f(z)|^p (1-|z|)^{\gamma} dm_1(z) \right)^{q/p} dr \\
&\leq C \int_0^1 (1-r)^{\beta} \left(\int_{|z|<r} |f(z)|^p (1-|z|)^{\gamma} dm_1(z) \right)^{q/p} dr.
\end{aligned}$$

3. On the diagonal of $K^{p,q,\alpha,\beta}(U^n)$

In this section we prove Theorem 2.

PROOF OF THEOREM 2. For $k = 0, 1, \dots$, let

$$U_k = \{U_{k,j} : j = 0, 1, \dots, 2^{k+1} - 1\},$$

where

$$U_{k,j} = \left\{ z \in U : 1 - \frac{1}{2^k} \leq |z| < 1 - \frac{1}{2^{k+1}}, \frac{2\pi j}{2^{k+1}} \leq \arg z < \frac{2\pi(j+1)}{2^{k+1}} \right\},$$

and let

$$V_k = \{V_{k,j} : j = 0, 1, \dots, 2^{k+1} - 1\},$$

where

$$V_{k,j} = \left\{ z \in U : 1 - \frac{1}{2^{k-1}} \leq |z| < 1 - \frac{1}{2^{k+2}}, \frac{2\pi(j-1/2)}{2^{k+1}} \leq \arg z < \frac{2\pi(j+3/2)}{2^{k+1}} \right\}.$$

To prove Theorem 2, we will use the dyadic decomposition of the polydisc (see [DjS]):

$$U_{k,l_1,\dots,l_n} = U_{k,l_1} \times \dots \times U_{k,l_n},$$

where $k = 0, 1, \dots$ and $U_{k,l_j} \in U_k$, $j = 1, \dots, n$.

Now, let $r_m = 1 - \frac{1}{2^m}$, $m = 0, 1, \dots$. Then by using Theorem 1 we find that

$$\begin{aligned} & \|Df\|_{p,q,\beta+q/p(|\alpha|+2n-1)}^q \\ & \leq C \int_0^1 (1-r)^\beta \left(\int_{|z|<r} |Df(z)|^p (1-|z|^2)^{|\alpha|+2n-2} dm_1(z) \right)^{q/p} dr \\ & = C \sum_{m=0}^{\infty} \int_{r_m}^{r_{m+1}} (1-r)^\beta \left(\int_{|z|<r} |Df(z)|^p (1-|z|^2)^{|\alpha|+2n-2} dm_1(z) \right)^{q/p} dr \\ & \leq C \sum_{m=0}^{\infty} 2^{-m(\beta+1)} \left(\int_{|z|<r_{m+1}} |Df(z)|^p (1-|z|^2)^{|\alpha|+2n-2} dm_1(z) \right)^{q/p} \\ & \leq C \sum_{m=0}^{\infty} 2^{-m(\beta+1)} \left(\sum_{k \leq m} \sum_{l_1, \dots, l_n} \left(\max_{z \in U_{k,l_1, \dots, l_n}} |f(z)|^p \right) 2^{-k(|\alpha|+2n-2)} 2^{-2k} \right)^{q/p}. \end{aligned} \quad (1)$$

From the n -subharmonicity of $|f(z)|^p$, $0 < p < \infty$, it follows that

$$\max_{z \in U_{k,l_1, \dots, l_n}} |f(z)|^p \leq C 2^{2kn} \int_{V_{k,l_1, \dots, l_n}} |f(w_1, \dots, w_n)|^p dm_n(w). \quad (2)$$

See [DjS].

The family of enlarged sets $V_{k,j}$, $k = 0, 1, \dots$, $j = 0, 1, \dots, 2^{k+1} - 1$, is a finite covering of U^n . Thus,

$$\begin{aligned} & \sum_{k \leq m} \sum_{l_1, \dots, l_n} \int_{V_{k,l_1, \dots, l_n}} |f(w_1, \dots, w_n)|^p \prod_{j=1}^n (1-|w_j|^2)^{\alpha_j} dm_n(w) \\ & \leq C \int_{|w_1| < r_{m+2}} \dots \int_{|w_n| < r_{m+2}} |f(w_1, \dots, w_n)|^p \prod_{j=1}^n (1-|w_j|^2)^{\alpha_j} dm_n(w). \end{aligned} \quad (3)$$

Combining (1), (2) and (3) we find that

$$\begin{aligned}
& \|Df\|_{\beta+q/p(\alpha+2n-1)}^q \\
& \leq C \sum_{m=0}^{\infty} 2^{-m(\beta+1)} \int_{|w_1|<r_{m+2}} \cdots \int_{|w_n|<r_{m+2}} |f(w_1, \dots, w_n)|^p \\
& \quad \times \prod_{j=1}^n (1 - |w_j|^2)^{\alpha_j} dm_n(w) \\
& \leq C \sum_{m=0}^{\infty} \int_{r_{m+2}}^{r_{m+3}} (1-r)^{\beta} \left(\int_{|w_1|<r} \cdots \int_{|w_n|<r} |f(w_1, \dots, w_n)|^p \right. \\
& \quad \times \left. \prod_{j=1}^n (1 - |w_j|^2)^{\alpha_j} dm_n(w) \right)^{q/p} dr \\
& \leq C \int_0^1 (1-r)^{\beta} \left(\int_{|w_1|<r} \cdots \int_{|w_n|<r} |f(w)|^p \prod_{j=1}^n (1 - |w_j|^2)^{\alpha_j} dm_n(w) \right)^{q/p} dr.
\end{aligned}$$

It remains to show that

$$D : K^{p,q,\alpha,\beta}(U^n) \rightarrow H^{p,q,\beta+q/p(\alpha+2n-1)}(U)$$

is onto.

Let $f \in H^{p,q,\beta+q/p(\alpha+2n-1)}(U)$. Define a function F on U^n by

$$F(z_1, \dots, z_n) = s \int_U \frac{f(w)(1 - |w|^2)^{s-1}}{\prod_{j=1}^n (1 - z_j \bar{w})^{(s+1)/n}} dm_1(w),$$

where s is a positive real number. Then F is holomorphic in U^n and it is easy to see that

$$DF(z) = F(z, \dots, z) = f(z), \quad z \in U.$$

Let $0 < p \leq 1$. We may assume that $p(s+1) - n \max_{1 \leq j \leq n} \alpha_j - 2n > (\beta+1)p/q$. Then using Lemma 3 and Lemma 4 we find that

$$\begin{aligned}
\|F\|_{p,q,\alpha,\beta}^q &= \int_0^1 \left(\int_{|z_1|<r} \cdots \int_{|z_n|<r} |F(z)|^p \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j} dm_n(z) \right)^{q/p} (1-r)^{\beta} dr \\
&\leq C \int_0^1 \left(\int_{|z_1|<r} \cdots \int_{|z_n|<r} \left(\int_U \frac{|f(w)|^p (1 - |w|)^{p(s-1)+2p-2} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{(s+1)p/n}} \right) \right. \\
&\quad \times \left. \prod_{j=1}^n (1 - |z_j|)^{\alpha_j} dm_n(z) \right)^{q/p} (1-r)^{\beta} dr
\end{aligned}$$

$$\leq C \int_0^1 \left(\int_0^1 M_p^p(\rho, f) \frac{(1-\rho)^{p(s-1)+2p-2}}{(1-r\rho)^{p(s+1)-|\alpha|-2n}} d\rho \right)^{q/p} (1-r)^\beta dr.$$

An application of Lemma 1 gives

$$\begin{aligned} \|F\|_{p,q,\alpha,\beta}^q &\leq C \int_0^1 (1-r)^{\beta+q/p(|\alpha|+2n-1)} M_p(r, f)^q dr \\ &= C \|f\|_{p,q,\beta+q/p(|\alpha|+2n-1)}^q. \end{aligned}$$

Thus, $F \in K^{p,q,\alpha,\beta}(U^n)$ and $DF = f$.

Assume now that $1 < p < \infty$ and $(s+1) > n \max_{1 \leq k \leq n} (\alpha_k + 2)$. Let γ_1 and γ_2 be positive real numbers such that $\gamma_1 + \gamma_2 = (s+1)/n$, and

$$\frac{s+1}{pn} - \frac{\min_{1 \leq k \leq n} \alpha_k + 1}{p} < \gamma_1 < \frac{s+1}{pn}.$$

Let q denote the conjugate of p , i.e. $1/p + 1/q = 1$. Then we have

$$\begin{aligned} \|F\|_{p,q,\alpha,\beta}^q &\leq C \int_0^1 \left(\int_{|z_1| < r} \cdots \int_{|z_n| < r} \left(\int_U \frac{|f(w)|(1-|w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{(s+1)/n}} \right)^p \right. \\ &\quad \times \left. \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j} dm_n(z) \right)^{q/p} (1-r)^\beta dr. \end{aligned} \quad (4)$$

By using Hölder's inequality we find that

$$\begin{aligned} \left(\int_U \frac{|f(w)|(1-|w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{(s+1)/n}} \right)^p &\leq \int_U \frac{|f(w)|^p (1-|w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_1 p}} \\ &\quad \times \left(\int_U \frac{(1-|w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_2 q}} \right)^{p/q}. \end{aligned} \quad (5)$$

Using the Hölder's inequality and Lemma 4 we see that

$$\begin{aligned} \left(\int_U \frac{(1-|w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_2 q}} \right)^{p/q} &\leq \prod_{j=1}^n \left(\int_U \frac{(1-|w|^2)^{s-1} dm_1(w)}{|1 - z_j \bar{w}|^{n\gamma_2 q}} \right)^{p/(qn)} \\ &\leq \prod_{j=1}^n \frac{1}{(1 - |z_j|)^{(s+1)/n - p\gamma_1}}. \end{aligned} \quad (6)$$

Thus, from (4), (5) and (6) it follows that

$$\|F\|_{p,q,\alpha,\beta}^q \leq C \int_0^1 \left(\int_{|z_1| < r} \cdots \int_{|z_n| < r} \int_U \frac{|f(w)|^p (1-|w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_1 p}} \right)$$

$$\begin{aligned}
& \times \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j - (s+1)/n + \gamma_1 p} dm_n(z) \Big)^{q/p} (1-r)^\beta dr \\
& \leq C \int_0^1 \left(\int_U |f(w)|^p (1 - |w|^2)^{s-1} \right. \\
& \quad \times \left. \left(\prod_{j=1}^n \int_{|z_j| < r} \frac{(1 - |z_j|)^{\alpha_j - (s+1)/n + \gamma_1 p} dm_1(z_j)}{|1 - z_j \bar{w}|^{\gamma_1 p}} \right) dm_1(w) \right)^{q/p} (1-r)^\beta dr.
\end{aligned}$$

Now by using Lemma 4 and Lemma 1 we get

$$\begin{aligned}
\|F\|_{p,q,\alpha,\beta}^q & \leq C \int_0^1 \left(\int_U \frac{|f(w)|^p (1 - |w|^2)^{s-1} dm_1(w)}{(1 - r|w|)^{s+1-|\alpha|-2n}} \right)^{q/p} (1-r)^\beta dr \\
& \leq C \int_0^1 M_p(r, f)^q (1-r)^{\beta+q/p(|\alpha|+2n-1)} dr \\
& = C \|f\|_{p,q,\beta+q/p(|\alpha|+2n-1)}^q.
\end{aligned}$$

This finishes the proof of Theorem 2.

4. On the diagonal of $M^{p,\alpha}(U^n)$

PROOF OF THEOREM 3. Let $f \in H^{p,\alpha}(U^n)$. To show that

$$\|f\|_{p,\alpha}^p \geq C \int_U |Df(z)|^p (1 - |z|^2)^{|\alpha|+2n-1} dm_1(z) \quad (7)$$

we will use the following inequality

$$\begin{aligned}
& \int_U |Df(z)|^p (1 - |z|^2)^{|\alpha|+2n-1} dm_1(z) \\
& = \sum_{k,j} \int_{U_{k,j}} |Df(z)|^p (1 - |z|^2)^{|\alpha|+2n-1} dm_1(z) \\
& \leq C \sum_{k,j} 2^{-k(|\alpha|+2n)} \int_{2\pi j/2^{k+1}}^{2\pi(j+1)/2^{k+1}} \sup_{1-2^{-k} \leq \rho < 1-2^{-(k+1)}} |f(\rho e^{i\theta}, \dots, \rho e^{i\theta})|^p d\theta \\
& \leq C \sum_k 2^{-k(|\alpha|+2n)} \int_0^{2\pi} \sup_{1-2^{-k} \leq \rho < 1-2^{-(k+1)}} |f(\rho e^{i\theta}, \dots, \rho e^{i\theta})|^p d\theta \\
& \leq C \sum_{k_1, \dots, k_n} \int_{1-2^{-(k_1+1)}}^{1-2^{-(k_1+2)}} (1 - \rho_1^2)^{\alpha_1+1} d\rho_1 \cdots \int_{1-2^{-(k_n+1)}}^{1-2^{-(k_n+2)}} (1 - \rho_n^2)^{\alpha_n+1} d\rho_n
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^{2\pi} |f(\rho_1 e^{i\theta}, \dots, \rho_n e^{i\theta})|^p d\theta \\
& \leq C \int_T \left(\int_0^1 \cdots \int_0^1 |f(\rho_1 \xi, \dots, \rho_n \xi)|^p \prod_{j=1}^n (1 - \rho_j^2)^{\alpha_j+1} d\rho_1 \cdots d\rho_n \right) d\sigma_1(\xi). \quad (8)
\end{aligned}$$

If u is a function defined on U , and $0 < \alpha, p < \infty$, than as a special case of the following known asymptotic relation (see [C])

$$\int_U |g(z)|^p d\mu(z) \cong \int_T \left(\int_{\Gamma_t(\xi)} \frac{|g(z)|^p}{1 - |z|} d\mu(z) \right) d\sigma_1(\xi),$$

where $d\mu$ is a positive Borel measure on U , we have

$$\int_0^1 |u(re^{i\theta})|^p (1-r)^\alpha dr \leq C \int_{\Gamma_t(e^{i\theta})} |u(z)|^p (1-|z|^2)^{\alpha-1} dm_1(z).$$

Using this inequality, by each variable, we get the following

$$\begin{aligned}
& \int_0^1 \cdots \int_0^1 |f(\rho_1 e^{i\theta}, \dots, \rho_n e^{i\theta})|^p \prod_{j=1}^n (1 - \rho_j^2)^{\alpha_j+1} d\rho_1 \cdots d\rho_n \\
& \leq C \int_{\Gamma_t(e^{i\theta})} \cdots \int_{\Gamma_t(e^{i\theta})} |f(z)|^p \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j} dm_n(z).
\end{aligned}$$

Integrating both sides by T and using (8) we get (7). The argument given here is taken from [LS].

The inequality (7) shows that the diagonal mapping D maps $M^{p,\alpha}(U^n)$ into $A^{p,|\alpha|+2n-1}(U)$. That D is onto follows from the following two facts:

$$A^{p,\alpha'}(U^n) \subset M^{p,\alpha}(U^n) \quad \text{and} \quad DA^{p,\alpha'}(U^n) = A^{p,|\alpha|+2n-1}(U),$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\alpha' = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \dots, \alpha_n)$.

For a reader convenience, we show that an extension operator, the right inverse of the diagonal operator, is bounded from $A^{p,|\alpha|+2n-1}(U)$ into $M^{p,\alpha}$ and consequently $D : M^{p,\alpha}(U^n) \rightarrow A^{p,|\alpha|+2n-1}(U)$ is onto.

Let $f \in A^{p,|\alpha|+2n-1}(U)$. Define a function F on U^n by

$$F(z_1, \dots, z_n) = s \int_U \frac{f(w)(1-|w|^2)^{s-1}}{\prod_{j=1}^n (1 - z_j \bar{w})^{(s+1)/n}} dm_1(w),$$

where s is a positive real number. Obviously, $DF = f$, by the known Bergman representation formulas.

First, assume that $0 < p \leq 1$. We may assume that $p(s+1) > n \max_{1 \leq j \leq n} \alpha_j + 2n + 1$. By using Lemma 3, Lemma 5 and Lemma 6 we obtain

$$\begin{aligned}
\|F\|_{p,\alpha}^p &\leq C \int_T \left(\int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} \int_U \frac{|f(w)|^p (1 - |w|^2)^{p(s-1)+2p-2} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{(s+1)p/n}} \right. \\
&\quad \left. \times \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j} dm_1(z_j) \right) d\sigma_1(\xi) \\
&\leq C \int_T \left(\int_U |f(w)|^p (1 - |w|^2)^{p(s-1)+2p-2} \right. \\
&\quad \left. \times \prod_{j=1}^n \int_{\Gamma_t(\xi)} \frac{(1 - |z_j|^2)^{\alpha_j} dm_1(z_j)}{|1 - z_j \bar{w}|^{(s+1)p/n}} dm_1(w) \right) d\sigma_1(\xi) \\
&\leq C \int_T \left(\int_U \frac{|f(w)|^p (1 - |w|^2)^{p(s-1)+2p-2}}{|1 - \xi \bar{w}|^{p(s+1)-|\alpha|-2n}} dm_1(w) \right) d\sigma_1(\xi) \\
&\leq C \int_U |f(w)|^p (1 - |w|^2)^{|\alpha|+2n-1} dm_1(w).
\end{aligned}$$

Case 1 $1 < p < \infty$. We may assume that $s > n \max_{1 \leq j \leq n} \alpha_j + 2n$.

Let q be the conjugate of p and let γ_1 and γ_2 be positive numbers such that $\gamma_1 + \gamma_2 = (s+1)/n$. and

$$\frac{s+1}{pn} - \frac{\alpha_j + 1}{p} < \gamma_1 < \frac{s+1}{pn}, \quad \text{for } j = 1, \dots, n.$$

By using Hölder's inequality we find that

$$\begin{aligned}
\|F\|_{p,\alpha}^p &\leq C \int_T \left(\int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} \left(\int_U \frac{|f(w)|^p (1 - |w|^2)^{s-1}}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_1 p}} dm_1(w) \right. \right. \\
&\quad \left. \left. \times \left(\int_U \frac{(1 - |w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_2 q}} \right)^{p/q} \right) \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j} dm_1(z_j) \right) d\sigma_1(\xi). \quad (9)
\end{aligned}$$

An application of Hölders inequality and Lemma 4 gives

$$\begin{aligned}
&\left(\int_U \frac{(1 - |w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_2 q}} \right)^{p/q} \\
&\leq \prod_{j=1}^n \left(\int_U \frac{(1 - |w|^2)^{s-1} dm_1(w)}{|1 - z_j \bar{w}|^{n \gamma_2 q}} \right)^{p/(qn)} \leq C \prod_{j=1}^n (1 - |z_j|)^{\gamma_1 p - (s+1)/n}. \quad (10)
\end{aligned}$$

By using (9), (10) and Lemma 6 we see that

$$\begin{aligned}
\|F\|_{p,\alpha}^p &\leq C \int_T \left(\int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} \int_U \frac{|f(w)|^p (1 - |w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_1 p}} \right. \\
&\quad \left. \times \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j + \gamma_1 p - (s+1)/n} dm_1(z_j) \right) d\sigma_1(\xi) \\
&= C \int_T \left(\int_U |f(w)|^p (1 - |w|^2)^{s-1} dm_1(w) \right. \\
&\quad \left. \times \prod_{j=1}^n \int_{\Gamma_t(\xi)} \frac{(1 - |z_j|^2)^{\alpha_j + \gamma_1 p - (s+1)/n}}{|1 - z_j \bar{w}|^{\gamma_1 p}} dm_1(z_j) \right) d\sigma_1(\xi) \\
&\leq C \int_T \left(\int_U \frac{|f(w)|^p (1 - |w|^2)^{s-1}}{|1 - \xi \bar{w}|^{s+1-|\alpha|-2n}} dm_1(w) \right) d\sigma_1(\xi) \\
&= C \int_U |f(w)|^p (1 - |w|^2)^{s-1} \left(\int_T \frac{d\sigma_1(\xi)}{|1 - \xi \bar{w}|^{s+1-|\alpha|-2n}} \right) dm_1(w) \\
&\leq C \int_U |f(w)|^p (1 - |w|^2)^{|\alpha|+2n-1} dm_1(w).
\end{aligned}$$

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