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CD-independent subsets in distributive lattices

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Abstract. A subset X of a lattice L with 0 is called *CD-independent* if for any $x, y \in X$, either $x \leq y$ or $y \leq x$ or $x \wedge y = 0$. In other words, if any two elements of X are either comparable or "disjoint". Maximal CD-independent subsets are called *CD-bases*.

The main result says that any two CD-bases of a finite distributive lattice L have the same number of elements. It is also shown that distributivity cannot be replaced by a weaker lattice identity. However, weaker assumptions on L are still relevant: semimodularity implies that no CD-basis can have fewer elements than a maximal chain, while lower semimodularity yields that each maximal chain together with all atoms forms a CD-basis.

Let L be a lattice with 0. A subset X of L will be called *CD-independent* if for any $x, y \in X$, either $x \leq y$ or $y \leq x$ or $x \wedge y = 0$. In other words, if any two elements of X either form a <u>c</u>hain (i.e., they are <u>c</u>omparable) or they are "<u>d</u>isjoint"; the initials explain our terminology. As one might expect, maximal CD-independent subsets are called *CD-bases* of L.

The classical notion of independent subsets of (semimodular or modular) lattices has many applications ranging from von Neumann's coordinatization theory to combinatorial applications via matroid theory. Some other notions of independence were introduced in [1] and [2], and there was a decade witnessing an

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intensive study of weak independence, cf. LENGVÁRSZKY's [10] and his other papers. Recently, the result of [1] has been successfully applied to combinatorial problems, cf. [3], PLUHÁR [13] and HORVÁTH, NÉMETH and PLUHÁR [8].

The present research started with the (easy) observation that many subsets occurring in [3], [8] and [13] are, in fact, CD-independent. At the time of the final revision of this paper, we add that so are the subsets in LENGVÁRSZKY [11] and [12], and E. K. HORVÁTH, G. HORVÁTH, NÉMETH and SZABÓ [9].

As a general reference to (the rudiments of) lattice theory the reader is referred to GRÄTZER [6]. For $b \in L$, $\downarrow b$ will stand for the principal ideal $\{u \in L : u \le b\}$. The *length*, that is the supremum of $\{|C| - 1 : C \text{ is a chain in } L\}$, of L is denoted by $\ell(L)$. For $u \in L$, let $h(u) = \ell(\downarrow u)$ denote the *height* of u. If for all $a, b, c \in L$, $a \preceq b$ implies $a \vee c \preceq b \vee c$ then L is called *semimodular*. Lattices satisfying the dual property are called *lower semimodular*. It is well-known that any two maximal chains of a semimodular lattice L of finite length have the same number of elements, and for any $u \le v \in L$ the length $\ell([u, v])$ of the interval $[u, v] = \{x \in L : u \le x \le v\}$ equals $h(v) - h(u)$.

Facts and notation. For a lattice L of finite length and a CD-basis X of L,

- $0, 1 \in X$;
- $\max(X)$ denotes the set of maximal elements of $X \setminus \{1\}$,
- the set of all CD-bases of L will be denoted by $\mathfrak{B}(L)$;
- for $b \in X$, we define $X(b) = (X \cap \downarrow b) \setminus \{0\}$, and we have

$$X(b) \cup \{0\} \in \mathfrak{B}(\downarrow b); \tag{1}$$

- if $\max(X)$ consists of k elements, say $\max(X) = \{a_1, \dots, a_k\}$, then

$$X = \{0, 1\} \,\dot\cup\, X(a_1) \,\dot\cup \cdots \dot\cup\, X(a_k) \text{ and } a_i \wedge a_j = 0 \quad \text{for all } i \ne j, \tag{2}$$

where $\dot\cup$ stands for (pairwise) disjoint union, and

$$\text{either } k = 1 \text{ and } a_1 \text{ is a coatom or } a_1 \vee \cdots \vee a_k = 1. \tag{3}$$

Facts (1) and (2) are trivial, while (3) is straightforward from the assumption that X is a *maximal* CD-independent subset.

Proposition 1. *Let X be a CD-basis of a finite semimodular lattice L. Then X has at least $\ell(L) + 1$ elements.*

PROOF. We prove the statement by induction on the length of L. The case $\ell(L) \le 1$ is evident, so we assume that $\ell(L) > 1$. If $|\max(X)| = 1$, then (1), (2), (3) and the induction hypothesis give

$$|X| = |(\{0\} \mathbin{\dot\cup} X(a_1)) \mathbin{\dot\cup} \{1\}| \ge \ell(\downarrow a_1) + 1 + 1 = \ell(L) + 1.$$

Hence we may assume that $\max(X) = \{a_1, \ldots, a_k\}$ consists of at least two elements. For $i \in \{1, \ldots, k\}$, denote $X(a_1) \cup \cdots \cup X(a_i)$ by X_i and $a_1 \vee \cdots \vee a_i$ by b_i. Then $X_k = X \setminus \{0, 1\}$ by (2) and $h(b_k) = h(1) = \ell(L)$, whence it suffices to show that

$$|X_i| \ge h(b_i) \tag{4}$$

for $i = 1, \ldots, k$. For $i = 1$ this is clear from the induction hypothesis on the length of the lattice. Now, let us assume the validity of (4) for $i < k$. Since finite semimodular lattices satisfy the well-known *"dimension inequality"*

$$h(x) + h(y) \ge h(x \wedge y) + h(x \vee y) \tag{5}$$

for any $x, y \in L$ (cf. GRÄTZER [6], Theorem IV.2.2), we have

$$\ell(\downarrow a_{i+1}) \ge h(a_{i+1}) - h(b_i \wedge a_{i+1}) \ge h(b_i \vee a_{i+1}) - h(b_i) = \ell([b_i, b_{i+1}]). \tag{6}$$

Since $\ell(\downarrow a_{i+1}) = h(a_{i+1}) < h(1) = \ell(L)$, the induction hypothesis (on the length) gives $|X(a_{i+1})| \ge \ell(\downarrow a_{i+1})$. Hence it follows from (6) and the induction hypothesis (on i) that $|X_{i+1}| = |X_i| + |X(a_{i+1})| \ge h(b_i) + \ell([b_i, b_{i+1}]) = h(b_{i+1})$, showing (4). □

The black-filled elements of the lattice A, cf. Figure 1, form a CD-basis with less than $\ell(A)+1$ elements. This indicates that semimodularity cannot be dropped from Proposition 1.

Proposition 2. *Let C be a maximal chain in a finite lower semimodular lattice L, and let $A(L)$ denote the set of atoms in L. Then $A(L) \cup C$ is a CD-basis of L.*

PROOF. Let $C = \{0 = c_0 \prec c_1 \prec c_2 \prec \cdots \prec c_n = 1\}$. It is clear, even without assuming lower semimodularity, that $C \cup A(L)$ is a CD-independent set. Let $y \in L \setminus C$ such that $C \cup \{y\}$ is CD-independent; we need to show that $y \in A(L)$. Let c_i be the smallest member of C such that $y \le c_i$. Then $i > 0$, $c_i = c_{i-1} \vee y$ and c_{i-1} is incomparable with y. The CD-independence of $C \cup \{y\}$ gives $y \wedge c_{i-1} = 0$. Hence lower semimodularity yields $0 \prec y$, i.e., $y \in A(L)$. □

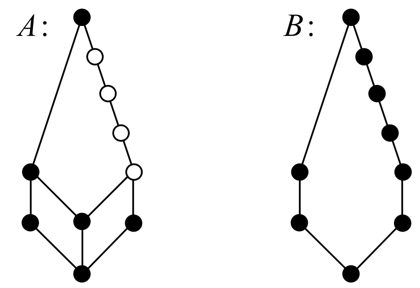

Figure 1. Lattices A and B

Note that B, cf. Figure 1, is a CD-basis of itself. Hence no maximal chain plus the atoms of B form a CD-basis. This indicates that lower semimodularity cannot be dropped from Proposition 2.

Main Theorem. *Any two CD-bases of a finite distributive lattice have the same number of elements.*

Proof. The notations from the previous two proofs will be in effect. Let L be a finite distributive lattice. Clearly, we can assume that $|L| \geq 3$. In virtue of Proposition 2, it suffices to show that, for every CD-basis X of L, we have

$$|X| = \ell(L) + |A(L)|. \tag{7}$$

Since $X \cup A(L)$ is CD-independent, the maximality of X implies that

$$A(L) \subseteq X. \tag{8}$$

We prove (7) by induction on $|L|$. Notice that, in formulas (2) and (3), $k = |\max(X)|$ must be 1 or 2. Indeed, if $k \geq 3$ then, for $i \geq 3$, $a_i \wedge (a_1 \vee a_2) = (a_i \wedge a_1) \vee (a_i \wedge a_2) = 0$. Since $a_i \neq 0$, we conclude $a_1 \vee a_2 \neq 1$, which means that $X \,\dot\cup\, \{a_1 \vee a_2\}$ is CD-independent, a contradiction.

First we consider the case k=1. Then a_1, the unique element of $\max(X)$, is a coatom by (3). Hence we conclude by (8) that $A(\downarrow a_1) = A(L)$. Now $X = \{1\} \,\dot\cup\, (\{0\} \,\dot\cup\, X(a_1))$ and, by (1), we know that $\{0\} \,\dot\cup\, X(a_1) \in \mathfrak{B}(\downarrow a_1)$. Hence we can apply the induction hypothesis to the distributive lattice $\downarrow a_1$:

$$|X| = 1 + |\{0\} \,\dot\cup\, X(a_1)| = 1 + \ell(\downarrow a_1) + |A(\downarrow a_1)| = \ell(L) + |A(L)|,$$

as desired.

Secondly, let $k = 2$. Then, by (2) and (3), a_2 is a complement of a_1. Hence $\ell(L) = h(1) = h(a_1) + h(a_2)$ by distributivity. Now it is well-known that L is (isomorphic to) the direct product of $L_1 = {\downarrow}a_1$ and $L_2 = {\downarrow}a_2$. Let $X_i = X \cap L_i = X(a_i) \cup \{0\} \in \mathfrak{B}({\downarrow}a_i)$, and let $A_i = A(L) \cap L_i$. Clearly, $X = X_1 \dot\cup \big((X_2 \cup \{1\}) \setminus \{0\}\big)$, $A(L_i) = A_i$, and $\ell(L) = h(a_1) + h(a_2) = \ell(L_1) + \ell(L_2)$. Hence the induction gives $|X| = |A(L)| + \ell(L)$ easily. $\square$

Now, by giving an unusual characterization of the variety of all distributive lattices, we point out that "distributivity" in the Main Theorem cannot be replaced by a weaker lattice identity. A lattice variety is called *nontrivial* if it is distinct from the class of all one-element lattices.

Corollary 3. *For every nontrivial variety $\mathcal{V}$ of lattices, the following two conditions are equivalent.*

(1) *any two CD-bases of each finite member of $\mathcal{V}$ have the same number of elements;*

(2) *$\mathcal{V}$ is the variety of all distributive lattices.*

Proof. Let $C_n = \{0 = d_0 \prec d_1 \prec \cdots \prec d_n = 1\}$ denote the chain of length n. (Although it would suffice to consider $n = 2$ in the present proof, the needs of a forthcoming proof makes it reasonable that we allow $n \geq 2$ here.) Given a lattice K, let $K[C_n] = \{(x_1, x_2, \ldots, x_n) \in K^n : x_1 \leq x_2 \leq \cdots \leq x_n\}$. Then $K[C_n]$ is a sublattice of the n-th direct power of K. (In fact, the constant n-tuples show that $K[C_n]$ is a subdirect power of K.) For $k \geq 3$, let $M_k = \{0, a_1, \ldots, a_k, 1\}$ denote the modular lattice of length 2 with exactly k atoms, and let N_5 be the five element nonmodular lattice with elements 0, 1, a, b, c such that $a < c$.

It suffices to show that Condition (1) implies that neither M_3 nor N_5 belongs to $\mathcal{V}$, for the reverse implication is just the Main Theorem. Suppose that $\mathcal{V}$ satisfies Condition (1).

By way of contradiction, suppose first that $M_3 \in \mathcal{V}$. Then $M_3[C_n] \in \mathcal{V}$ as well. (For $n = 3$, it is depicted in Figure 2.) Consider the following principal ideals of $M_3[C_n]$:

$$\downarrow(a_i, \ldots, a_i) = \big\{(0, \ldots, 0, 0), (0, \ldots, 0, a_i), (0, \ldots, a_i, a_i), \ldots, (a_i, \ldots, a_i, a_i)\big\},$$

for $i = 1, 2, 3$. They are chains of length n. Using modularity and the fact that the constant n-tuples in $M_3[C_n]$ form a sublattice isomorphic to M_3, we obtain that $\ell(M_3[C_n]) = 2n$. Notice that $M_3[C_n]$ has exactly three atoms: the $(0, \ldots, 0, a_i)$ for $i = 1, 2, 3$. Therefore, in virtue of Proposition 2, $M_3[C_n]$ has a CD-basis G of size $2n + 3$, cf. the cross-filled elements in the figure. By similar argument,

$$M_k[C_n] \text{ has a CD-basis of size } 2n + k; \tag{9}$$

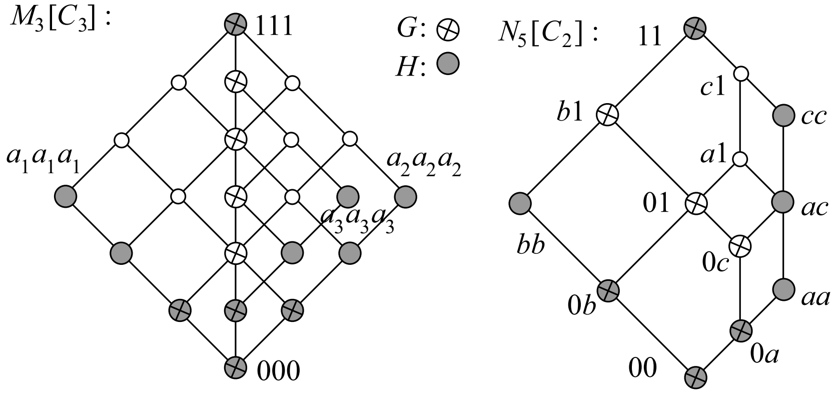

Figure 2. $M_3[C_3]$ and $N_5[C_2]$

we have noticed this for later reference. On the other hand, let

$$H = \{(1,\dots,1)\} \cup \downarrow(a_1,\dots,a_1) \cup \downarrow(a_2,\dots,a_2) \cup \downarrow(a_3,\dots,a_3).$$

Then H is a CD-independent subset and $|H| = 3n+2$, cf. the grey-filled elements in the figure. (It is not hard to see that H is a CD-basis, but we do not need this fact.) Similarly,

$$M_k[C_n] \text{ has a CD-independent subset of size at least } kn+2. \tag{10}$$

Now $3n+2 > 2n+3$ for $n \geq 2$, contradicting Condition (1).

Secondly, suppose that $N_5 \in \mathcal{V}$. Then $N_5[C_2] \in \mathcal{V}$ as well; cf. Figure 2, which is quoted from [16]. The cross-filled elements form a CD-basis G while the gray-filled elements form a CD-basis H. So $|G| = 7 \neq 8 = |H|$ contradicts Condition (1). □

Remark 4. Let $\mathcal{V}$ be a lattice variety containing a non-distributive member. Then, for each $t \in \mathbf{N}$, there are a finite lattice $L \in \mathcal{V}$ and CD-bases X and Y of L such that $|X| - |Y| > t$.

Proof. If Z is a CD-basis of L then it is straightforward to see that $Z' = (Z \times \{0\}) \cup (\{0\} \times Z) \cup \{(1,1)\}$ is a CD-basis of L^2 with $|Z'| = 2 \cdot |Z|$. This together with Corollary 3 implies the above remark. □

Remark 5. For each $t \in \mathbf{N}$, there are a finite modular lattice L and CD-bases X and Y of L such that $|X| > t \cdot |Y|$.

Proof. Evident by (9) and (10). □

Historical remarks. The lattice $M_3[C_n]$ is just a particular case of the $M_3[D]$ construction for bounded distributive lattices D. While $M_3[D]$ was introduced in [15] in a very different way, by means of balanced triples, here we used the more general definition of $K[D]$ from [16]. For some other applications and generalizations of the $M_3[D]$ construction cf., e.g., [17], Farley [4] and [5], Grätzer and Wehrung [7], and Quackenbush [14].

Corollary 6. *Let L be a finite distributive lattice. Then L is Boolean if and only if $|X| = 2 \cdot \ell(L)$ holds for every (equivalently, some) CD-basis X of L.*

Proof. Let $J(L)$ denote the set of nonzero join-irreducible elements of L. Then $|J(L)| = \ell(L)$, and L is Boolean iff $J(L) = A(L)$, cf., e.g., Theorem II.1.9 and Corollary II.1.14 in Grätzer [6]. Hence the Main Theorem and Proposition 2 complete the proof. □

References

[1] G. Czédli, A. P. Huhn and E. T. Schmidt, Weakly independent subsets in lattices, *Algebra Universalis* **20** (1985), 194–196.

[2] G. Czédli and Zs. Lengvárszky, Two notes on independent subsets in lattices, *Acta Math. Hungarica* **53** (1–2) (1989), 169–171.

[3] G. Czédli, The number of rectangular islands by means of distributive lattices, *European Journal of Combinatorics* **30** (2009), 208–215.

[4] J. D. Farley, Priestley powers of lattices and their congruences: a problem of E. T. Schmidt, *Acta Sci. Math.* **62** (1996), 3–45.

[5] J. D. Farley, Tensor products of semilattices, semimodularity and supersolvability, a problem of E. T. Schmidt from 1974 and some conjectures of Quackenbush from 1985, *Algebra Universalis*, submitted.

[6] G. Grätzer, General Lattice Theory, *Birkhäuser Verlag, Basel – Stuttgart*, 1978, Second edition: Birkhäuser Verlag, 1998.

[7] G. Grätzer and F. Wehrung, Proper congruence-preserving extension of lattices, *Acta Math. Hungar.* **85** (1999), 175–185.

[8] E. K. Horváth, Z. Németh and G. Pluhár, The number of triangular islands on a triangular grid, *Periodica Math. Hungar.* (*to appear*).

[9] E. K. Horváth, G. Horváth, Z. Németh and Cs. Szabó, The number of square islands on a rectangular sea, submitted.

[10] Zs. Lengvárszky, Lower bound on the size of weak bases in lattices, *Algebra Universalis* **33** (1995), 207–208.

[11] Zs. Lengvárszky, The minimum cardinality of maximal systems of rectangular islands, *European Journal of Combinatorics* **30** (2009), 216–219.

[12] Zs. Lengvárszky, The size of maximal systems of square islands, *European Journal of Combinatorics*, in press (doi:10.1016/j.ejc.2008.07.023).

[13] G. Pluhár, The number of brick islands by means of distributive lattices, *Acta Sci. Math. (Szeged)* (*to appear*).

[14] R. W. Quackenbush, Non-modular varieties of semimodular lattices with a spanning M_3, *Discrete Math.* **53** (1985), 186–196.

[15] E. T. Schmidt, Zur Characterisierung der Kongruenzverbände, *Math. Časopis Sloven. Acad. Vied.* **18** (1968), 3–20.

[16] E. T. Schmidt, Lattices generated by partial lattices, Lattice theory (Proc. Colloq., Szeged, 1974), 343–353. Colloq. Math. Soc. Janos Bolyai, Vol. 14, North-Holland, Amsterdam, 1976.

[17] E. T. Schmidt, The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice, *Acta Sci. Math. (Szeged)* **43** (1981), 153–168.

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