

Minimal solution of a Riccati type differential equation

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Abstract. We consider a Riccati type differential equation which appears in the oscillation theory of half-linear differential equations. We establish the existence of the so-called *minimal solution* of this equation and we investigate basic properties of this solution. In particular, we prove a Sturmian type theorem for minimal solutions of a pair of considered equations.

1. Introduction

The Riccati type differential equation which we investigate in this paper comes from the oscillation theory of half-linear differential equations. Recall that the half-linear differential equation is an equation of the form

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1, \quad (1)$$

and that oscillation theory of this equation attracted considerable attention in the recent years, let us mention at least the books [1, 6] and the references given therein. It was shown that oscillatory properties of (1) are essentially the same as those of the linear Sturm–Liouville equation (which is the special case $p = 2$ in (1))

$$(r(t)x')' + c(t)x = 0 \quad (2)$$

Mathematics Subject Classification: 34C10.

Key words and phrases: Riccati type differential equation, minimal solution, half-linear differential equation, Sturmian comparison theorem, principal solution.

The first author is supported by the Grant OTKA K61620 and the second author is supported by the Research Project MSM0021622409 of the Ministry of Education of the Czech Republic and the Grant 201/08/0469 of the Grant Agency of the Czech Republic.

and hence equation (1) can be classified as oscillatory or nonoscillatory similarly as for (2). As pioneers of the half-linear oscillation theory are usually regarded ELBERT and MIRZOV with their papers [7] and [14], even if elements of the theory of half-linear equations had already appeared in BIHARI's papers [2], [3], [4].

In the classical oscillation theory of (1), this equation is regarded as a perturbation of the (nonoscillatory) *one-term equation*

$$(r(t)\Phi(x'))' = 0 \quad (3)$$

and an important role is played there by the Riccati type differential equation (related to (1) by the substitution $w = r\Phi(x'/x)$)

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q := \frac{p}{p-1}. \quad (4)$$

Comparing this equation with the classical Riccati equation (associated with (2) by the substitution $w = rx'/x$)

$$w' + c(t) + \frac{w^2}{r(t)} = 0, \quad (5)$$

the power q in (4) makes no essential difference with respect to the power 2 in (5), so (non)oscillation criteria for (1) derived in this way are similar to those for (2). Another reason for this similarity is the fact that the solution space of (3) is actually *linear*.

Recently, a more general approach (sometimes called the *perturbation principle*) to half-linear oscillation theory has been introduced. There, equation (1) is viewed as a perturbation of (nonoscillatory) half-linear equation *of the same form* (i.e., linearity of its solution space is lost)

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0. \quad (6)$$

More precisely, let h be an eventually positive solution of (6) and let $w_h := r\Phi(h'/h)$ be the solution of the Riccati equation associated with (6). If w is a solution of (4) and $v = h^p(w - w_h)$, then v solves the first order equation

$$v' + (c(t) - \tilde{c}(t))h^p(t) + (p-1)r^{1-q}(t)h^{-q}(t)H(t, v) = 0, \quad (7)$$

where

$$H(t, v) := |v + G(t)|^q - q\Phi^{-1}(G(t))v - |G(t)|^q, \quad G(t) := r(t)h(t)\Phi(h'(t)),$$

$\Phi^{-1}(s) = |s|^{q-2}s$ being the inverse function of Φ . Of course, if $\tilde{c}(t) \equiv 0$ and $h(t) \equiv 1$, then (7) reduces to (4). We refer to [5] and to [6, Section 5.6] for a brief summary of basic ideas of this “perturbation” approach.

Motivated by the above mentioned facts, the main objective of our paper is the Riccati type differential equation of the form

$$v' + c(t) + r^{-1}(t)H(g(t), v) = 0, \quad (8)$$

where c, r, g are continuous functions, $r(t) > 0$, and

$$H(g, v) = |v + g(t)|^q - q\Phi^{-1}(g(t))v - |g(t)|^q.$$

We will show that similarly to (4), among all proper solutions of (8) (see the next section for the definition of this concept) there exists the so-called *minimal solution*, and we establish Sturmian type comparison theorem for minimal solutions of two equations of the form (8).

2. Half-linear and Riccati type differential equations

First of all, observe that if $p = 2$ in (7), then this equation takes the form

$$v' + (c(t) - \tilde{c}(t))h^2(t) + \frac{v^2}{r(t)h^2(t)} = 0$$

which is the equation of the same form as (5) and this is the Riccati equation associated with the second order Sturm–Liouville equation resulting from (2) upon the transformation $x = h(t)u$, where h is a solution of (6) with $p = 2$. It is known (see, e.g., [6, Section 1.3]) that the linear transformation theory does not extend to (1) (since the function Φ is not additive), so from this point of view equation (7) can be regarded as an attempt to overcome this difficulty.

If $\tilde{c}(t) \equiv 0$, $\int_t^\infty r^{1-q}(s) ds < \infty$, and we take $h(t) = \int_t^\infty r^{1-q}(s) ds$, then $G(t) = -h(t)$ and (7) reads as

$$v' + c(t)h^p(t) + (p-1)h^{-q}(t)r^{1-q}(t) \{ |v - h(t)|^q + q\Phi^{-1}(h(t))v - |h(t)|^q \} = 0,$$

and this Riccati type differential equation played an important role in the paper [13]. If $r(t) \equiv 1$ and $\tilde{c}(t) = \gamma_p t^{-p}$, $\gamma_p := \left(\frac{p-1}{p}\right)^p$, i.e., (6) reduces to the half-linear Euler equation with the critical coefficient

$$(\Phi(x'))' + \frac{\gamma_p}{t^p} \Phi(x) = 0 \quad (9)$$

and with the solution $h(t) = t^{\frac{p-1}{p}}$. Then (7) takes the form

$$v' + (c(t) - \gamma_p t^{-p})t^{p-1} + \frac{p-1}{t} \left[\left| v + \left(\frac{p-1}{p} \right)^{p-1} \right|^q - v - \left(\frac{p-1}{p} \right)^p \right] = 0,$$

and this equation was an important tool in proving the main results of [11], [16].

In the next part of this section we recall the main results of the papers [8], [9], where the equation

$$(r(t)x')' + c(t)f(x, r(t)x') = 0 \quad (10)$$

is considered under the assumptions on f :

- (i) The function f is continuous on $\Omega = \mathbb{R} \times \mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$;
- (ii) It holds $xf(x, y) > 0$ if $xy \neq 0$;
- (iii) The function f is homogeneous, i.e., $f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda \in \mathbb{R}_0$ and $(x, y) \in \Omega$;
- (iv) The function f is sufficiently smooth in order to ensure the continuous dependence and the uniqueness of solutions of the initial value problem $x(t_1) = x_0$, $x'(t_1) = x_1$ at some $(x_0, x_1) \in \Omega$;
- (v) Let

$$F(t) = tf(t, 1), \quad (11)$$

then

$$\int_{-\infty}^{\infty} \frac{dt}{1 + F(t)} < \infty \quad \text{and} \quad \lim_{|t| \rightarrow \infty} F(t) = \infty.$$

Let g be the differentiable function given by the formula

$$g(u) = \begin{cases} \int_{1/u}^{\infty} \frac{ds}{F(s)} & \text{if } u > 0, \\ -\int_{-\infty}^{1/u} \frac{ds}{F(s)} & \text{if } u < 0, \end{cases} \quad (12)$$

and $g(0) = 0$. Then g is strictly increasing and $\lim_{u \rightarrow \pm\infty} g(u) = \pm\infty$. If x is a solution of (10) such that $x(t) \neq 0$, then the function $u = g(rx'/x)$ solves the Riccati type differential equation

$$u' + c(t) + r^{-1}(t)H(u) = 0, \quad (13)$$

where the function H is given by

$$\int_{g(u)}^{\infty} \frac{ds}{H(s)} = \frac{1}{u}, \quad \text{if } u > 0, \quad \int_{-\infty}^{g(u)} \frac{ds}{H(s)} = -\frac{1}{u}, \quad \text{if } u < 0$$

with $H(0) = 0$. Conversely, having a function $H(u) > 0$ for $u \neq 0$, with $H(0) = 0$, such that

$$\int_{-\infty}^{\infty} \frac{ds}{H(s)} < \infty, \quad \int_{-\infty}^{\infty} \frac{ds}{H(s)} < \infty$$

one can associate with (13) equation (10) with f satisfying (i)–(v). More precisely, the function g is given as the solution of the differential equation

$$g'(u) = \frac{1}{u^2} H(g(u)), \quad g(0) = 0, \quad (14)$$

and the function $f : \mathbb{R} \times \mathbb{R}_0 \rightarrow \mathbb{R}$ is given by the formula

$$f(1, u) := \frac{1}{g'(u)}, \quad f(t, s) := \begin{cases} tf(1, s/t), & t \neq 0 \\ 0 & t = 0 \end{cases} \quad (15)$$

We finish this section with presenting some properties of the function $H(g, v)$ in (8). Directly one can verify that the derivative $H_v(g, v) = 0$ if and only if $v = 0$ and that H is strictly convex with respect to the second variable. Also, $H(g, v)$ is Lipschitzian in v , hence the initial value problem for (8) has a unique solution. This means that the graphs of solutions of (8) cannot intersect.

3. Minimal solution

Suppose that (1) is nonoscillatory, i.e., there exists a solution of this equation which is eventually of one sign. Then the associated Riccati equation (4) possesses the so-called *proper solution*, i.e., a solution which is defined on some interval $[t_0, \infty)$. MIRZOV [15] and independently ELBERT and KUSANO [10] showed that among all proper solutions of (4) there exists the so-called *minimal solution* $\tilde{w}$, which is the proper solution with the property that any other proper solution w of (4) satisfies $w(t) > \tilde{w}(t)$ on the interval of its existence. The minimal solution of (4) defines then the so-called *principal solution* of (1) via the formula

$$\tilde{x}(t) = C \exp \left\{ \int^t \Phi^{-1}(\tilde{w}(s)/r(s)) ds \right\}, \quad 0 \neq C \in \mathbb{R},$$

Note that the principal solution of (1) plays an important role in the oscillation theory of these equations, see [6, Section 4.2].

To establish the existence of the minimal solution of (8), we need the following auxiliary result.

Lemma 1. *Consider equation (8) in an interval $[t_0, t_0 + T]$, $T > 0$ arbitrary. There exists $v_0 < 0$ such that any solution of (8) with $v(t_0) < v_0$ satisfies*

$$\lim_{t \rightarrow t_1^-} v(t) = -\infty$$

for some $t_1 \in [t_0, t_0 + T]$.

PROOF. Denote

$$\hat{c} = \min_{t_0, t_0+T} c(t), \quad \hat{r} = \max_{t_0, t_0+T} r(t).$$

Since the function g is continuous, it attains for $t \in [t_0, t_0 + T]$ values in some bounded closed interval, denote it $[A, B]$, and for this interval let

$$\hat{H}(v) := \min_{\tau \in [A, B]} H(\tau, v).$$

Consequently, for $t \in [t_0, t_0 + T]$ and $v \in \mathbb{R}$ we have $\hat{H}(v) \leq H(g(t), v)$. Together with (8), consider the equation

$$u' + \hat{c} + \hat{r}^{-1} \hat{H}(u) = 0. \quad (16)$$

Then by the standard theorem for differential inequalities (see, e.g. [12]), if $v(t_0) < u(t_0)$, then $v(t) < u(t)$ for $t > t_0$ for which $v(t)$ exists.

Now consider equation (16). We have

$$\int_{u(t_0)}^{u(t)} \frac{ds}{-\hat{c} - \hat{r}^{-1} \hat{H}(s)} = t - t_0.$$

Since $\hat{H}(u) = H(g(t_0), u)$ for some $t_0 \in [a, b]$, we have $\hat{H}(u) \rightarrow \infty$ as $u \rightarrow -\infty$ and there exists $\tilde{u}$ such that $-\hat{c} - \hat{r}^{-1} \hat{H}(s) < 0$ for $u < \tilde{u}$, i.e., $u(t)$ is decreasing and

$$\int_{u(t)}^{u(t_0)} \frac{ds}{\hat{c} + \hat{r}^{-1} \hat{H}(s)} = t - t_0$$

if $u(t_0) < \tilde{u}$. Hence

$$\infty > \int_{-\infty}^{u(t_0)} \frac{ds}{\hat{c} + \hat{r}^{-1} \hat{H}(s)} > \int_{u(t)}^{u(t_0)} \frac{ds}{\hat{c} + \hat{r}^{-1} \hat{H}(s)} = t - t_0.$$

Now, if $u(t_0) \rightarrow -\infty$, the first integral in the previous formula tends to 0, which means that $t \rightarrow t_0$, i.e., $t - t_0 < T$ for $u(t_0)$ sufficiently negative. Hence $u(t)$ has to blow down to $-\infty$ inside of the interval $[t_0, t_0 + T]$ and inequality for solutions of (8) and (16) implies that a solution v of (8) starting with sufficiently negative initial value $v(t_0)$ has the same property. $\square$

In the remaining part of this section we assume that there exists $t_0 \in T$ such that

$$(8) \text{ possesses a solution defined on } [t_0, \infty). \quad (17)$$

Similarly to (4), such a solution we will call *the proper* solution of (8).

Definition 1. A proper solution $\tilde{v}$ of (8) is said to be *minimal*, if any other proper solution v of (8) satisfies $v(t) > \tilde{v}(t)$ on the interval of existence of v .

Denote

$$\mathcal{V} = \{v \in \mathbb{R}, \text{ the solution of (8) given by } v(t_0) = v \text{ is proper}\}.$$

By our assumption $\mathcal{V} \neq \emptyset$ and by Lemma 1 the set $\mathcal{V}$ is bounded below. Let

$$v_0 = \inf \mathcal{V}. \quad (18)$$

Theorem 1. Suppose that (17) holds and let $\tilde{v}$ be the solution of (8) given by the initial condition $\tilde{v}(t_0) = v_0$, where v_0 is given by (18). Then $\tilde{v}$ is a proper solution, i.e., it exists on $[t_0, \infty)$ and it is the minimal solution of (8).

PROOF. By contradiction, suppose that $\tilde{v}$ is not proper, i.e., $\tilde{v}(T_1-) = -\infty$ for some $T_1 > t_0$. Let $T_2 > T_1$ be arbitrary. For $t \in [t_0, T_2]$ the function g attains the values in an interval $[A, B]$. Denote

$$\begin{aligned} \hat{H}(u) &= \min_{\tau \in [A, B]} \{ |u + \tau|^q - q\Phi^{-1}(\tau)u - |\tau|^q \}, \\ \tilde{H}(u) &= \max_{\tau \in [A, B]} \{ |u + \tau|^q - q\Phi^{-1}(\tau)u - |\tau|^q \}. \end{aligned} \quad (19)$$

Then we have for $t \in [t_0, T_2]$

$$\hat{H}(u) \leq H(g(t), u) \leq \tilde{H}(u). \quad (20)$$

Consider the Riccati type equations

$$u' + c(t) + r^{-1}(t)\hat{H}(u) = 0, \quad (21)$$

$$u' + c(t) + r^{-1}(t)\tilde{H}(u) = 0. \quad (22)$$

These equations are of the same form as (13), so one can associate with them the second order differential equations

$$(r(t)z')' + c(t)\hat{f}(z, r(t)z') = 0, \quad (23)$$

$$(r(t)z')' + c(t)\tilde{f}(z, r(t)z') = 0, \quad (24)$$

the functions $\hat{f}$, $\tilde{f}$ are related to $\hat{H}$, $\tilde{H}$ as described in Section 2 by relations (14) and (15).

Consider the solutions $\hat{z}$, $\tilde{z}$ of (23), (24), respectively, given by the initial condition

$$\hat{z}(T_2) = 0, \quad \hat{z}'(T_2) = -1, \quad \tilde{z}(T_2) = 0, \quad \tilde{z}'(T_2) = -1,$$

and let

$$\hat{u} = \hat{g}(r\hat{z}'/\hat{z}), \quad \tilde{u} = \tilde{g}(r\tilde{z}'/\tilde{z})$$

with $\hat{g}$, $\tilde{g}$ defined again via corresponding $\hat{f}$ and $\tilde{f}$ using formula (12). Then (since $\hat{g}(-\infty) = -\infty = \tilde{g}(-\infty)$)

$$\hat{u}(T_2-) = -\infty = \tilde{u}(T_2-)$$

and $\tilde{u}(T_1) \geq \hat{u}(T_1)$. Indeed, if, by contradiction, $\tilde{u}(T_1) < \hat{u}(T_1)$, then the solution $\bar{u}$ of (23) given by $\bar{u}(T_1) = \tilde{u}(T_1)$ satisfies $\bar{u}(t) \geq \tilde{u}(t)$, $t \in [T_1, T_2)$, so its graph either intersects that of $\hat{u}$ what is a contradiction with the unique solvability of (23), or $\bar{u}(T_2-) = -\infty$, again a contradiction, since the solutions of (23) satisfying $\hat{z}(T_2) = 0$ are determined up to a multiplicative factor (because of the homogeneity of the solution space), hence they determine the unique solution of (21).

Now consider the solution v of (8) with $v(T_1) \in [\hat{u}(T_1), \tilde{u}(T_1)]$. Since (20) holds, we have

$$\hat{u}(t) \leq v(t) \leq \tilde{u}(t) \quad \text{for } t \in [T_1, T_2),$$

i.e., $v(T_2-) = -\infty$. Moreover, the unique solvability of (8) implies that v exists on $[t_0, T_1]$ and $v(t_0) > v_0$. Indeed, if $v(t_1+) = \infty$ for some $t_1 \in [t_0, T_1)$ then the graph of v intersects the graph of any proper solution of (8) on $[t_0, \infty)$. Also, $v(t_0) \leq v_0$ implies the intersection of graphs of v and $\tilde{v}$ at some $t \in [t_0, T_1)$. Consequently, we have constructed a solution v of (8) starting with $v(t_0) > v_0$ which is not proper. This is contradiction with the definition of v_0 . $\square$

The next statement is a Sturmian type comparison theorem for minimal solutions of two equations of the form (8).

Theorem 2. *Together with (8) we consider the equation*

$$u' + C(t) + R^{-1}(t)H(g, u) = 0, \tag{25}$$

with $c(t) \leq C(t)$ and $0 < R(t) < r(t)$ for large t (i.e., (25) is a majorant of (8) in the classical Sturmian setting for $p = 2$). Suppose that (25) possessed a proper solution and let $\tilde{u}$ be its minimal solution which is defined for $t \geq t_0$. Then (8) possesses a proper solution as well and for its minimal solution $\tilde{v}$ we have $\tilde{v}(t) < \tilde{u}(t)$ for $t \geq t_0$.

PROOF. Let u be a proper solution of (25) and consider the solution v of (8) given by the initial condition $v(t_1) = u(t_1)$ for some (sufficiently large) t_1 .

Then inequalities between c , C , r , and R imply that $v(t) \geq u(t)$ for $t \geq t_0$. Since $H(g, u) \geq 0$ for $u \in \mathbb{R}$, the solution v can not blow up to ∞ at some finite time t , we have that v is a proper solution of (8). By contradiction, suppose that the minimal solutions $\tilde{u}$, $\tilde{v}$ satisfy $\tilde{v}(t_2) > \tilde{u}(t_2)$ for some $t_2 > t_0$. Consider the solution v of (8) given by $v(t_2) = \tilde{u}(t_2)$. Then by the same argument as in the previous part of the proof we have $v(t) > \tilde{u}(t)$ for $t \geq t_2$. At the same time, since $v(t_1) < \tilde{v}(t_1)$, we have $v(t) < \tilde{v}(t)$. This means that we have found a proper solution v of (8) which is less than minimal solution of this equation. This leads to a contradiction. $\square$

Remark 1. The previous theorem is a comparison result with respect to c and r , while the function g is the same in (8), (25). The reason is that the behavior of H with respect to the first variable g is relatively complicated, since

$$\frac{\partial}{\partial g} H(g, u) = q \left[\Phi^{-1}(u + g) - (q - 1) |g|^{q-2} u - \Phi^{-1}(g) \right]$$

and it is difficult to compute explicitly the roots of the equation $\frac{\partial}{\partial g} H(g, u) = 0$.

The last statement deals with the case when the function g is bounded.

Theorem 3. Suppose that $\int_{t_0}^{\infty} r^{1-q}(t) dt = \infty$ for some $t_0 \in \mathbb{R}$, $c(t) \geq 0$, and g is bounded for $t \in [t_0, \infty)$. Then the minimal solution $\tilde{v}$ of (8) satisfies $\tilde{v}(t) \geq 0$ for $t \in [t_0, \infty)$.

PROOF. Suppose, by contradiction, that $\tilde{v}(T) < 0$ for some T . We proceed similarly as in the proof of Theorem 1. The function g is bounded, so it attains values in some bounded interval $[A, B]$. Consider equation (21) with the function $\hat{H}$ given by (19) and its solution satisfying $u(T) = \tilde{v}(T) < 0$. Then again $v(t) \leq u(t)$ for $t \geq T$. Since $c(t) \geq 0$, we have

$$u' + r^{-1}(t) \hat{H}(u) \leq 0. \quad (26)$$

The function u is decreasing (use that $\hat{H}(u) > 0$ for $u \neq 0$), hence from (26) for $t > T$

$$\int_{u(t)}^{u(T)} \frac{ds}{\hat{H}(s)} \geq \int_T^t r^{-1}(s) ds. \quad (27)$$

Letting $t \rightarrow \infty$, the integral on the left-hand side of (27) is convergent, while the integral on the right-hand side is divergent, which means that u cannot be a proper solution of (21) and hence $\tilde{v}$ is not a proper solution of (8) as well, a contradiction. $\square$

Remark 2. (i) The last statement gives an alternative proof of the fact that $h(t) = t^{\frac{p-1}{p}}$ is the principal solution of (9) (proved in [11] by a different method). Indeed, let x be a solution of (9) linearly independently of h , $w_x = r\Phi(x'/x)$, $w_h = r\Phi(h'/h)$, $w_x(t) \neq w_h(t)$, and $v = h^p(w_x - w_h)$. Then v satisfies the equation

$$v' + \frac{p-1}{t} \left[\left| v + \left(\frac{p-1}{p} \right)^{p-1} \right|^q - v + \left(\frac{p-1}{p} \right)^p \right] = 0.$$

Moreover, the unique solvability of Riccati type equation (4) associated with (9) implies that $w_x(t) \neq w_h(t)$. By Theorem 3 $v(t) \geq 0$ for $t \in [t_0, \infty)$ which means $w_x(t) > w_h(t)$ i.e., w_h is the minimal solution of the Riccati equation associated with (9) and hence h is the principal solution of (9).

(ii) Generally, *any* condition which guaranties that v is the minimal solution of (8) is a sufficient condition for $w = h^{-p}v + w_h$ to be the minimal solution of (4) and then

$$x(t) = C \exp \left\{ \int^t \Phi^{-1}(w(s)/r(s)) ds \right\}, \quad 0 \neq C \in \mathbb{R},$$

is the principal solution of (1), which, as we have already mentioned before, plays the important role in the oscillation theory of (1).

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(Received July 1, 2008)