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P-Berwald manifolds

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Dedicated to Professor Lajos Tamássy on the occasion of his 85th birthday

Abstract. We introduce a new class of special Finsler manifolds, the class of p-Berwald manifolds. P-Berwald manifolds are defined as Finsler manifolds for which the projected Berwald curvature vanishes. We show that an at least 3-dimensional Finsler manifold is a p-Berwald manifold if and only if it is a weakly Berwald Douglas manifold. 2-dimensional p-Berwald manifolds are characterized by means of a differential equation concerning the main scalar. We prove that a p-Berwald manifold is R-quadratic if and only if its stretch tensor vanishes.

1. Introduction

By a p-Berwald manifold we mean a Finsler manifold whose projected Berwald curvature vanishes. The concept of a "projected Finsler tensor" was first systematically investigated by M. Matsumoto under the quite strange term "indicatorizaion", using the arsenal of classical tensor calculus [8]. An index-free description of Matsumoto's indicatorization was presented by Sz. Vattamány [18], working on TTM and using the Frölicher–Nijenhuis calculus of vector-valued forms. It seems to us that the pull-back bundle $\mathring{\tau}^* TM$ is a more economical framework for these constructions, and the Berwald derivative arising naturally from a Finsler structure is an adequate tool for calculations in this setting. For the

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readers' convenience, we briefly summarize these basic technicalities in Section 2, and, partly, in Section 3. We follow the notation and conventions of reference [17] and, with some modifications, [5]. These papers also give some links to the classical approach. In Section 4 we discuss basic curvature relations in a Finsler manifold. The most interesting is formulated in Proposition 4.2; it has a "converse" (see (25)) in p-Berwald manifolds. In Section 5 it turns out that in $n > 2$ dimensions p-Berwald manifolds form the intersection of the class of Douglas manifolds and the class of weakly Berwald manifolds – of two classes of special Finsler manifolds which have been investigated extensively [1]–[4], [6].

2. Preliminaries

Throughout the paper M will be an n-dimensional ($n \geq 1$), second countable, Hausdorff, smooth manifold. $C^\infty(M)$ is the ring of real-valued smooth functions on M; the $C^\infty(M)$-module of smooth vector fields on M is denoted by $\mathfrak{X}(M)$. d is the operator of exterior derivative, i_X is the substitution operator induced by $X \in \mathfrak{X}(M)$.

If TM is the $2n$-dimensional manifold of all tangent vectors to M, and $\tau : TM \to M$ is the natural projection, the "foot map", then τ is said to be the tangent bundle of M, TM is the total space of the tangent bundle. The complete lift of a function $f \in C^\infty(M)$ is

$$f^{\mathsf{c}} : v \in TM \longmapsto f^{\mathsf{c}}(v) := v(f).$$

The complete lift of a vector field $X \in \mathfrak{X}(M)$ is the unique vector field $X^{\mathsf{c}} \in \mathfrak{X}(TM)$ such that

$$X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}}, \quad f \in C^\infty(M).$$

Let $\widetilde{TM} \subset TM$ be an open subset satisfying $\tau(\widetilde{TM}) = M$, and let $\widetilde{\tau} := \tau \restriction \widetilde{TM}$. If

$$\widetilde{\tau}^* TM =: \widetilde{TM} \times_M TM := \left\{ (u,v) \in \widetilde{TM} \times TM \mid \widetilde{\tau}(u) = \tau(v) \right\}$$

and $\widetilde{\pi}(u,v) := u$ for $(u,v) \in \widetilde{\tau}^* TM$, then $\widetilde{\pi}$ is a vector bundle of rank n, the *pull-back of* τ *over* $\widetilde{\tau}$. The most important special cases arise when $\widetilde{TM} := TM$, $\widetilde{\tau} := \tau$ and $\widetilde{TM} := \mathring{T}M := TM \backslash o(M)$ ($o \in \mathfrak{X}(M)$ is the zero vector field), $\widetilde{\tau} := \mathring{\tau} := \tau \restriction \mathring{T}M$. Then we get the pull-back bundles $\pi : TM \times_M TM \to TM$ and $\mathring{\pi} : \mathring{T}M \times_M TM \to \mathring{T}M$.

We denote by $\Gamma(\widetilde{\pi})$ the $C^\infty(\widetilde{TM})$-module of smooth sections of $\widetilde{\pi}$. A typical element of $\Gamma(\widetilde{\pi})$ is of the form

$$\widetilde{X} : v \in \widetilde{TM} \longmapsto \widetilde{X}(v) = (v, \underline{X}(v)) \in \widetilde{TM} \times_M TM,$$

where $\underline{X} : \widetilde{TM} \to TM$ is a smooth map such that $\tau \circ \underline{X} = \widetilde{\tau}$. Any vector field X on M yields a section

$$\widehat{X} : v \in \widetilde{TM} \longmapsto \widehat{X}(v) = (v, X \circ \widetilde{\tau}(v)) \in \widetilde{TM} \times_M TM,$$

of $\widetilde{\pi}$, called a *basic vector field*. Basic vector fields generate the $C^\infty(\widetilde{TM})$-module $\Gamma(\widetilde{\pi})$. The *canonical section* δ of $\widetilde{\pi}$ sends $v \in \widetilde{TM}$ to $(v, v) \in \widetilde{\tau}^* TM$.

We denote by $\mathcal{T}^k_l(\widetilde{\pi})$ the $C^\infty(\widetilde{TM})$-module of all tensors of type (k, l) over $\Gamma(\widetilde{\pi})$ ($(k, l) \in \mathbb{N} \times \mathbb{N}$; $\mathcal{T}^0_0(\widetilde{\pi}) := C^\infty(\widetilde{TM})$). Elements of $\mathcal{T}^1_l(\widetilde{\pi})$ may naturally be interpreted as $\Gamma(\widetilde{\pi})$-valued $C^\infty(\widetilde{TM})$-multilinear maps. The unit tensor in $\mathcal{T}^1_1(\widetilde{\pi})$ will simply be denoted by $\mathbf{1}$. We note that $\mathcal{T}^k_l(\pi)$ may (and will) be considered as a submodule of $\mathcal{T}^k_l(\mathring{\pi})$.

$\mathbf{i}$ denotes the canonical bundle injection $\widetilde{TM} \times_M TM \to T\widetilde{TM}$, $\mathbf{j}$ is the canonical bundle surjection of $T\widetilde{TM}$ onto $\widetilde{TM} \times_M TM$. Then $\mathbf{j} \circ \mathbf{i} = 0$, while $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is another canonical bundle map, the *vertical endomorphism* of $T\widetilde{TM}$. $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{J}$ induce the $C^\infty(\widetilde{TM})$-homomorhpisms

$$\begin{aligned}
\Gamma(\widetilde{\pi}) &\longrightarrow \mathfrak{X}(TM), & \widetilde{X} &\longmapsto \mathbf{i}\widetilde{X} := \mathbf{i} \circ \widetilde{X},\\
\mathfrak{X}(\widetilde{TM}) &\longrightarrow \Gamma(\pi), & \xi &\longmapsto \mathbf{j}\xi := \mathbf{j} \circ \xi,\\
\mathfrak{X}(\widetilde{TM}) &\longrightarrow \mathfrak{X}(TM), & \xi &\longmapsto \mathbf{J}\xi := \mathbf{J} \circ \xi.
\end{aligned}$$

Then

$$\mathfrak{X}^{\mathsf{v}}(\widetilde{TM}) := \mathbf{i}(\Gamma(\widetilde{\pi})) = \mathrm{Im}(\mathbf{J}) = Ker(\mathbf{J})$$

is the $C^\infty(\widetilde{TM})$-module of *vertical vector fields* on $\widetilde{TM}$, $X^{\mathsf{v}} := \mathbf{i}\widehat{X}$ is the *vertical lift* of $X \in \mathfrak{X}(M)$. $C := \mathbf{i}\delta$ is a canonical vertical vector field on $\widetilde{TM}$, the *Liouville vector field*. For any vector field X on M we have

$$[C, X^{\mathsf{v}}] = -X^{\mathsf{v}}, \quad [C, X^{\mathsf{c}}] = 0. \tag{1}$$

We define the *vertical differential* $\nabla^{\mathsf{v}} F \in \mathcal{T}^0_1(\widetilde{\pi})$ of a function $F \in C^\infty(\widetilde{TM})$ by

$$\nabla^{\mathsf{v}} F(\widetilde{X}) := (\mathbf{i}\widetilde{X})F, \quad \widetilde{X} \in \Gamma(\widetilde{\pi}). \tag{2}$$

The vertical differential of a section $\widetilde{Y} \in \Gamma(\widetilde{\pi})$ is the $(1, 1)$ tensor $\nabla^{\mathsf{v}}\widetilde{Y} \in \mathcal{T}^1_1(\widetilde{\pi})$ given by

$$\nabla^{\mathsf{v}}\widetilde{Y}(\widetilde{X}) =: \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} := \mathbf{j}\big[\mathbf{i}\widetilde{X}, \eta\big], \quad \widetilde{X} \in \Gamma(\widetilde{\pi}), \tag{3}$$

where $\eta \in \mathfrak{X}(\widetilde{TM})$ is such that $\mathbf{j}\eta = \widetilde{Y}$. (It is easy to check that the result does not depend on the choice of η.) Using the Leibnizian product rule as a guiding principle, the operators $\nabla^{\mathsf{v}}_{\widetilde{X}}$ may uniquely be extended to a tensor derivation of the tensor algebra of $\Gamma(\widetilde{\pi})$. Forming the vertical differential of a tensor over $\Gamma(\widetilde{\pi})$, we use the following convention: if, e.g., $\mathbf{A} \in \mathcal{T}^1_2(\widetilde{\pi})$, then $\nabla^{\mathsf{v}}(\mathbf{A}) \in \mathcal{T}^1_3(\widetilde{\pi})$, given by

$$\nabla^{\mathsf{v}}\mathbf{A}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) := (\nabla^{\mathsf{v}}_{\widetilde{X}}\mathbf{A})(\widetilde{Y}, \widetilde{Z}) = \nabla^{\mathsf{v}}_{\widetilde{X}}\mathbf{A}(\widetilde{Y}, \widetilde{Z}) - \mathbf{A}(\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) - \mathbf{A}(\widetilde{Y}, \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Z}).$$

3. Finsler functions and associated objects

Let m_λ, where λ is a real number, denote the map $v \in TM \mapsto \lambda v \in TM$. By a *Finsler function* we mean a function $F : TM \to \mathbb{R}$ satisfying:

(F1) F is smooth on $\overset{\circ}{T}M$.

(F2) $F \circ m_\lambda = \lambda F$ for all real numbers $\lambda \geq 0$.

(F3) $F \geq 0$ and equals 0 only on $o(M)$.

(F4) The $(0, 2)$ tensor $g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2 \in \mathcal{T}^0_2(\overset{\circ}{\pi})$ is (fibrewise) positive definite.

A *Finsler manifold* is a pair (M, F) consisting of a manifold M and a Finsler function on TM. By Euler's theorem on homogeneous functions, condition (F2) may equivalently be written in the form $CF = F$. $E := \frac{1}{2}F^2$ is the *energy function* of the Finsler manifold. It is positive-homogeneous of degree 2, i.e., $CE = 2E$, smooth on $\overset{\circ}{T}M$ and identically zero on $o(M)$. It may be shown (see e.g. [19]) that, actually, E is C^1 on TM and is C^2, if and only if, E is the norm associated with a Riemannian structure on M in which case E is smooth on TM. $g = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}E$ is said to be the *metric tensor* of (M, F). For any vector fields X, Y on M we have

$$g(\widehat{X}, \widehat{Y}) = X^{\mathsf{v}}(Y^{\mathsf{v}}E). \tag{4}$$

Since $[X^{\mathsf{v}}, Y^{\mathsf{v}}] = 0$, this implies that g is symmetric. It would have been sufficient to assume only the (fibrewise) non-singularity of this tensor for positive definiteness is then a consequence of the other conditions on F.

Now we list some basic data arising immediately from a Finsler function.

(i) $\delta_\flat : \widetilde{X} \in \Gamma(\overset{\circ}{\pi}) \longmapsto \delta_\flat(\widetilde{X}) := g(\widetilde{X}, \delta)$ - the *canonical 1-form* of (M, F),

(ii) $\ell := \frac{1}{F}\delta \in \Gamma(\overset{\circ}{\pi})$ - the *normalized support element field*,

(iii) $\ell_\flat := \frac{1}{F}\delta_\flat \in \mathcal{T}^0_1(\overset{\circ}{\pi})$ - the dual form of ℓ,

(iv) $\eta := g - \ell_\flat \otimes \ell_\flat$ - the *angular metric tensor*.

We have the following relation:

$$\delta_\flat = F\nabla^{\mathsf{v}}F = \nabla^{\mathsf{v}}E. \tag{5}$$

Indeed for any vector field X on M, $\delta_\flat(\widehat{X}) := g(\widehat{X},\delta) = g(\delta,\widehat{X}) = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}E(\delta,\widehat{X}) = \nabla^{\mathsf{v}}_\delta\nabla^{\mathsf{v}}E(\widehat{X}) = C(X^{\mathsf{v}}E) - \nabla^{\mathsf{v}}E(\nabla^{\mathsf{v}}_\delta\widehat{X}) \overset{(3)}{=} [C,X^{\mathsf{v}}]E + X^{\mathsf{v}}(CE) - \nabla^{\mathsf{v}}E(\mathbf{j}[C,X^{\mathsf{c}}]) \overset{(1)}{=} -X^{\mathsf{v}}E + 2X^{\mathsf{v}}E = \frac{1}{2}X^{\mathsf{v}}F^2 = F(X^{\mathsf{v}}F) \overset{(3)}{=} F\nabla^{\mathsf{v}}F(\widehat{X})$, which proves the formula.

From this observation relations

$$g(\delta,\delta) = \delta_\flat(\delta) = F^2, \quad \ell_\flat(\ell) = g(\ell,\ell) = 1, \tag{6}$$

$$\eta = g - \nabla^{\mathsf{v}}F \otimes \nabla^{\mathsf{v}}F \tag{7}$$

are immediately deduced.

If (M,F) is a Finsler manifold, then there is a unique vector field S on TM defined to be zero on $o(M)$, and defined on $\mathring{T}M$ to be the unique vector field such that

$$i_S d(\nabla^{\mathsf{v}}F^2 \circ \mathbf{j}) = -dF^2.$$

Then S is C^1 on TM, smooth on $\mathring{T}M$ and has the properties

$$JS = C, \quad [C,S] = S, \tag{8}$$

therefore S is a spray, called the *canonical spray* of the Finsler manifold. It is less known, but a proof of this really fundamental fact may also be found in Warner's above cited paper [19]. The canonical spray induces an Ehresmann connection $\mathcal{H} : \mathring{T}M \times_M TM \longrightarrow T\mathring{T}M$ such that for any vector field X on M,

$$X^{\mathsf{h}} := \mathcal{H}\widehat{X} := \mathcal{H}\circ\widehat{X} := \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}},S]). \tag{9}$$

$\mathcal{H}$ is said to be the *Barthel connection* of (M,F), X^{h} is the *horizontal lift* of X. $\mathcal{H}$ is *homogeneous* in the sense that

$$\left[C,X^{\mathsf{h}}\right] = 0, \quad X \in \mathfrak{X}(M). \tag{10}$$

Indeed, $2[C,X^{\mathsf{h}}] = [C,X^{\mathsf{c}}] + [C,[X^{\mathsf{v}},S]] \overset{(1)}{=} [C,[X^{\mathsf{v}},S]] = -[X^{\mathsf{v}},[S,C]] - [S,[C,X^{\mathsf{v}}]] \overset{(1)),((8)}{=} [X^{\mathsf{v}},S] + [S,X^{\mathsf{v}}] = 0$.

An important property of the Barthel connection is that the Finsler function is a first integral for the horizontal lifts, i.e.,

$$X^{\mathsf{h}}F = 0, \quad X \in \mathfrak{X}(M). \tag{11}$$

Equivalently, $dF \circ \mathcal{H} = 0$. For a recent, simple proof of this fact we refer to [16].

To the Barthel connection (as to any Ehresmann connection) we associate

(i) the *horizontal projector* $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$,

(ii) the *vertical projector* $\mathbf{v} := 1_{T\overset{\circ}{T}M} - \mathbf{h}$,

(iii) the *vertical map* $\mathcal{V} : T\overset{\circ}{T}M \to \overset{\circ}{T}M \times_M TM$ such that $\mathbf{i} \circ \mathcal{V} = \mathbf{v}$.

We define the *h-Berwald differentials* $\nabla^{\mathsf{h}} F \in \mathcal{T}^0_1(\overset{\circ}{\pi})$ $(F \in C^\infty(\overset{\circ}{T}M))$ and $\nabla^{\mathsf{h}} \widetilde{Y} \in \mathcal{T}^1_1(\overset{\circ}{\pi})$ $(\widetilde{Y} \in \Gamma(\overset{\circ}{\pi}))$ by the following rules:

$$\nabla^{\mathsf{h}} F(\widetilde{X}) := (\mathcal{H}\widetilde{X})F, \quad \widetilde{X} \in \Gamma(\overset{\circ}{\pi}); \tag{12}$$

$$\nabla^{\mathsf{h}} \widetilde{Y}(\widetilde{X}) := \nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} := \mathcal{V}\big[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}\big], \quad \widetilde{X} \in \Gamma(\overset{\circ}{\pi}). \tag{13}$$

Then the operators $\nabla^{\mathsf{h}}_{\widetilde{X}}$ $(\widetilde{X} \in \Gamma(\overset{\circ}{\pi}))$ may uniquely be extended to the whole tensor algebra of $\Gamma(\overset{\circ}{\pi})$ as tensor derivations. Forming the h-Berwald differential of an arbitrary tensor, we adopt the same convention as in the vertical case. We note that the homogeneity of the Barthel connection implies

$$\nabla^{\mathsf{h}} \delta = 0. \tag{14}$$

From the operators ∇^{v} and ∇^{h} we build the *Berwald derivative*

$$\nabla : (\xi, \widetilde{Y}) \in \mathfrak{X}(\overset{\circ}{T}M) \times \Gamma(\overset{\circ}{\pi}) \longmapsto \nabla_\xi \widetilde{Y} := \nabla^{\mathsf{v}}_{\mathcal{V}\xi} \widetilde{Y} + \nabla^{\mathsf{h}}_{\mathbf{j}\xi} \widetilde{Y} \in \Gamma(\overset{\circ}{\pi}).$$

Then, by (3) and (13),

$$\nabla_\xi \widetilde{Y} = \mathbf{j}\big[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}\big] + \mathcal{V}\big[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}\big].$$

In particular,

$$\begin{aligned} &\nabla_{\mathbf{i}\widetilde{X}} \widetilde{Y} = \nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y}, \quad \nabla_{\mathcal{H}\widetilde{X}} \widetilde{Y} = \nabla^{\mathcal{H}}_{\widetilde{X}} \widetilde{Y}; \qquad && \widetilde{X}, \widetilde{Y} \in \Gamma(\overset{\circ}{\pi}); \\ &\nabla_{X^{\mathsf{v}}} \widehat{Y} = 0, \qquad\quad \nabla_{X^{\mathsf{h}}} \widehat{Y} = \mathcal{V}\left[X^{\mathsf{h}}, Y^{\mathsf{v}}\right]; && X, Y \in \mathfrak{X}(M). \end{aligned} \tag{15}$$

4. Curvature properties

We assume for the remainder of the paper that (M, F) is a fixed n-dimensional Finsler manifold. To introduce some curvature data in (M, F), we start from the classical curvature tensor R^∇ of the Berwald derivative on M given by

$$R^\nabla(\xi, \eta)\widetilde{Z} := \nabla_\xi \nabla_\eta \widetilde{Z} - \nabla_\eta \nabla_\xi \widetilde{Z} - \nabla_{[\xi,\eta]} \widetilde{Z}, \quad (\xi, \eta \in \mathfrak{X}(\overset{\circ}{T}M),\ \widetilde{Z} \in \Gamma(\overset{\circ}{\pi})).$$

By the *affine curvature tensor* of (M, F) we mean the tensor $\mathbf{H} \in \mathcal{T}_3^1(\mathring{\pi})$ given by

$$\mathbf{H}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := R^{\nabla}(\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y})\widetilde{Z}; \quad \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(\mathring{\pi}).$$

Here we followed L. Berwald's terminology. According to Z. Shen's usage, we say that (M, F) is *R-quadratic* if $\nabla^{\mathsf{v}}\mathbf{H} = 0$, i.e., the affine curvature "depends only on the position".

The type $(1, 3)$ tensor $\mathbf{B}$ given by

$$\mathbf{B}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := R^{\nabla}(\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y})\widetilde{Z}; \quad \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(\mathring{\pi})$$

is said to be the *Berwald curvature* of (M, F). Evaluating on basic vector fields, we find that

$$\mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} = \mathcal{V}\left[X^{\mathsf{v}}, \left[Y^{\mathsf{h}}, Z^{\mathsf{v}}\right]\right] \quad \text{or} \quad \mathbf{iB}(\widehat{X}, \widehat{Y})\widehat{Z} = \left[X^{\mathsf{v}}, \left[Y^{\mathsf{h}}, Z^{\mathsf{v}}\right]\right].$$

It is then a straightforward matter to check that $\mathbf{B}$ is totally symmetric. We also have:

$$\delta \in \left\{\widetilde{X}, \widetilde{Y}, \widetilde{Z}\right\} \Rightarrow \mathbf{B}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = 0. \tag{16}$$

A Finsler manifold is said to be a *Berwald manifold* if its Berwald curvature vanishes. (M, F) is a *weakly Berwald manifold* provided $\operatorname{tr}\mathbf{B} = 0$, where tr denotes the trace of the $C^{\infty}(\mathring{T}M)$-linear map $\widetilde{X} \mapsto \mathbf{B}(\widetilde{X}, \widetilde{Y})\widetilde{Z}$.

We shall need the following Bianchi identity:

$$\nabla^{\mathsf{v}}\mathbf{H}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) + \nabla^{\mathsf{h}}\mathbf{B}(\widetilde{Y}, \widetilde{Z}, \widetilde{X}, \widetilde{U}) - \nabla^{\mathsf{h}}\mathbf{B}(\widetilde{Z}, \widetilde{Y}, \widetilde{X}, \widetilde{U}) = 0 \tag{17}$$

$(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U} \in \Gamma(\mathring{\pi}))$; see [14], p. 1331.

The *Landsberg tensor* of (M, F) is

$$\mathbf{P} := -\frac{1}{2}\nabla^{\mathsf{h}}g. \tag{18}$$

As a special case of 2.50, Lemma 5 in [14], we obtain

Lemma 4.1. *The Berwald curvature and the Landsberg tensor of a Finsler manifold are related by*

$$\nabla^{\mathsf{v}}E \circ \mathbf{B} = -2\mathbf{P}, \tag{19}$$

where E is the energy function.

Notice that relation (19) implies immediately that Berwald manifolds have vanishing Landsberg tensor.

By the *stretch tensor* of (M,F) we mean the tensor $\mathbf{\Sigma}\in\mathcal{T}^0_4(\overset{\circ}{\pi})$ given by

$$\frac{1}{2}\mathbf{\Sigma}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) := \nabla^{\mathsf{h}}\mathbf{P}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) - \nabla^{\mathsf{h}}\mathbf{P}(\widetilde{Y},\widetilde{X},\widetilde{Z},\widetilde{U}). \tag{20}$$

The next important observation gives an index-free reformulation of relation (3.3.2.5) in [10]. For completeness we present an immediate (and also index-free) proof, which differs essentially from Matsumoto's argument based on classical tensor calculus.

Proposition 4.2. *For any sections $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$, $\widetilde{U}$ in $\Gamma(\overset{\circ}{\pi})$,*

$$\nabla^{\mathsf{v}}E\circ\nabla^{\mathsf{v}}\mathbf{H}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) = \mathbf{\Sigma}(\widetilde{Y},\widetilde{X},\widetilde{Z},\widetilde{U}). \tag{21}$$

Proof. It is enough to check the relation for basic vector fields $\widehat{X}$, $\widehat{Y}$, $\widehat{Z}$, $\widehat{U}$.

$$\begin{aligned}\nabla^{\mathsf{v}}E(\nabla^{\mathsf{v}}\mathbf{H}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U})) &\overset{(2)}{=} (\mathbf{i}\nabla^{\mathsf{v}}\mathbf{H}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U}))E\\ &\overset{(17)}{=} \mathbf{i}(-\nabla^{\mathsf{h}}\mathbf{B}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U})+\nabla^{\mathsf{h}}\mathbf{B}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U}))E.\end{aligned}$$

Here

$$\begin{aligned}\nabla^{\mathsf{h}}\mathbf{B}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U}) &= (\nabla_{Y^{\mathsf{h}}}\mathbf{B})(\widehat{Z},\widehat{X},\widehat{U}) = \nabla_{Y^{\mathsf{h}}}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U}\\ &\quad - \mathbf{B}(\nabla_{Y^{\mathsf{h}}}\widehat{Z},\widehat{X})\widehat{U} - \mathbf{B}(\widehat{Z},\nabla_{Y^{\mathsf{h}}}\widehat{X})\widehat{U} - \mathbf{B}(\widehat{Z},\widehat{X})\nabla_{Y^{\mathsf{h}}}\widehat{U},\end{aligned}$$

and by (15)

$$\nabla_{Y^{\mathsf{h}}}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U} = \mathcal{V}\big[Y^{\mathsf{h}},\mathbf{i}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U}\big].$$

Therefore, applying (19) we get

$$\begin{aligned}\mathbf{i}\nabla^{\mathsf{h}}\mathbf{B}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U})E &= \big[Y^{\mathsf{h}},\mathbf{i}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U}\big]E + 2\mathbf{P}(\nabla_{Y^{\mathsf{h}}}\widehat{Z},\widehat{X},\widehat{U})\\ &\quad + 2\mathbf{P}(\widehat{Z},\nabla_{Y^{\mathsf{h}}}\widehat{X},\widehat{U}) + 2\mathbf{P}(\widehat{Z},\widehat{X},\nabla_{Y^{\mathsf{h}}}\widehat{U}).\end{aligned}$$

Since $Y^{\mathsf{h}}E=0$ by (11), at the right-hand side the first term is

$$Y^{\mathsf{h}}((\mathbf{i}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U})E) \overset{(19)}{=} -2Y^{\mathsf{h}}\mathbf{P}(\widehat{Z},\widehat{X},\widehat{U}),$$

therefore the right-hand side is just $-2\nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U})$. In the same way we find that

$$\mathbf{i}\nabla^{\mathsf{h}}\mathbf{B}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U})E = -2\nabla^{\mathsf{h}}\mathbf{P}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U}).$$

Hence

$$\begin{aligned}\nabla^{\mathsf{v}}E\big(\nabla^{\mathsf{v}}\mathbf{H}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U})\big) &= 2\big(\nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U}) - \nabla^{\mathsf{h}}\mathbf{P}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U})\big)\\ &\overset{(20)}{=} \mathbf{\Sigma}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U}),\end{aligned}$$

as was to be proved. □

Corollary 4.3. *R-quadratic Finsler manifolds have vanishing stretch tensor.*

5. P-Berwald manifolds

Lemma 5.1. *If*
$$\mathbf{p} := \mathbf{1} - \frac{1}{2E}\nabla^{\mathsf{v}}E \otimes \delta, \tag{22}$$
then $\mathbf{p}(\delta) = 0$*, and* $\mathbf{p}$ *is a projection operator on* $\Gamma(\overset{\circ}{\pi})$*, i.e.,* $\mathbf{p}^2 = \mathbf{p}$.

Proof. Since the energy function is positive-homogeneous of degree 2,
$$\mathbf{p}(\delta) := \delta - \frac{1}{2E}\nabla^{\mathsf{v}}E(\delta)\delta = \delta - \frac{1}{2E}(CE)\delta = \delta - \delta = 0.$$
Using this observation, for any section $\widetilde{X}$ in $\Gamma(\overset{\circ}{\pi})$,
$$\mathbf{p}^2(\widetilde{X}) = \mathbf{p}(\widetilde{X} - \frac{1}{2E}(\mathbf{i}\widetilde{X})E\delta) = \mathbf{p}(\widetilde{X}),$$
thus proving the claim. □

By the *projected tensor* of a tensor $\mathbf{K} \in \mathcal{T}^0_k(\overset{\circ}{\pi})$ or $\mathbf{L} \in \mathcal{T}^1_k(\overset{\circ}{\pi})$ we mean the tensors $\mathbf{pK}$ and $\mathbf{pL}$ given by
$$\mathbf{pK}(\widetilde{X}_1, \dots, \widetilde{X}_k) := \mathbf{K}(\mathbf{p}\widetilde{X}_1, \dots, \mathbf{p}\widetilde{X}_k)$$
and
$$\mathbf{pL}(\widetilde{X}_1, \dots, \widetilde{X}_k) := \mathbf{p}(\mathbf{L}(\mathbf{p}\widetilde{X}_1, \dots, \mathbf{p}\widetilde{X}_k)).$$

Corollary 5.2. *Let* $\mathbf{K} \in \mathcal{T}^0_k(\overset{\circ}{\pi})$, $\mathbf{L} \in \mathcal{T}^1_k(\overset{\circ}{\pi})$ *. If*
$$\delta \in \left\{\widetilde{X}_1, \dots, \widetilde{X}_k\right\} \Rightarrow \mathbf{K}(\widetilde{X}_1, \dots, \widetilde{X}_k) = 0, \quad \mathbf{L}(\widetilde{X}_1, \dots, \widetilde{X}_k) = 0,$$
then $\mathbf{pK} = \mathbf{K}$, $\mathbf{pL} = \mathbf{p} \circ \mathbf{L}$.

Example. The projected tensor of the metric tensor g is the angular metric tensor η. Indeed, for any vector fields X, Y on M,
$$\begin{aligned}
\mathbf{p}g(\widehat{X}, \widehat{Y}) &:= g\big(\mathbf{p}(\widehat{X}), \mathbf{p}(\widehat{Y})\big) = g\left(\widehat{X} - \frac{1}{2E}(X^{\mathsf{v}}E)\delta, \widehat{Y} - \frac{1}{2E}(Y^{\mathsf{v}}E)\delta\right) \\
&= g(\widehat{X}, \widehat{Y}) - \frac{1}{2E}(X^{\mathsf{v}}E)g(\delta, \widehat{Y}) - \frac{1}{2E}(Y^{\mathsf{v}}E)g(\widehat{X}, \delta) \\
&\quad + \frac{1}{4E^2}(X^{\mathsf{v}}E)(Y^{\mathsf{v}}E)g(\delta, \delta) \overset{(5),(6)}{=} g(\widehat{X}, \widehat{Y}) - \frac{1}{F^2}(X^{\mathsf{v}}E)\nabla^{\mathsf{v}}E(\widehat{Y}) \\
&\quad - \frac{1}{F^2}(Y^{\mathsf{v}}E)\nabla^{\mathsf{v}}E(\widehat{X}) + \frac{1}{F^2}(X^{\mathsf{v}}E)(Y^{\mathsf{v}}E) \\
&= \left(g - \frac{1}{F^2}\nabla^{\mathsf{v}}E \otimes \nabla^{\mathsf{v}}E\right)(\widehat{X}, \widehat{Y}) = (g - \nabla^{\mathsf{v}}F \otimes \nabla^{\mathsf{v}}F)(\widehat{X}, \widehat{Y}) = \eta(\widehat{X}, \widehat{Y}).
\end{aligned}$$

Lemma 5.3. *The projected tensor of the Berwald curvature of a Finsler manifold is*

$$\mathbf{pB} = \mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta. \tag{23}$$

Proof. By (16) and Corollary 5.2, $\mathbf{pB} = \mathbf{p} \circ \mathbf{B}$. Now, for any vector fields X, Y, Z on M,

$$\begin{aligned}(\mathbf{pB})(\widehat{X}, \widehat{Y}, \widehat{Z}) &= \mathbf{p}(\mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z}) \overset{(22)}{=} \mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} - \frac{1}{2E}(\mathbf{iB}(\widehat{X}, \widehat{Y})\widehat{Z})E\delta \\ &\overset{(19)}{=} \mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} + \frac{1}{E}\mathbf{P}(\widehat{X}, \widehat{Y}, \widehat{Z})\delta = \left(\mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta\right)(\widehat{X}, \widehat{Y}, \widehat{Z}),\end{aligned}$$

hence our statement. □

Definition. By a *p-Berwald manifold* we mean a Finsler manifold in which the projected Berwald curvature vanishes, i.e., which has the property

$$\mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta = 0. \tag{24}$$

Proposition 5.4. *Any p-Berwald manifold is a weakly Berwald manifold.*

Proof. We have to show that if (M, F) is a p-Berwald manifold, then $\operatorname{tr}\mathbf{B} = 0$. By (24) and Lemma 1 of [15], $\operatorname{tr}\mathbf{B} = -\frac{1}{E}\operatorname{tr}(\mathbf{P} \otimes \delta) = -\frac{1}{E}i_\delta\mathbf{P}$. Here $i_\delta\mathbf{P} = -\frac{1}{2}i_\delta\nabla^{\mathsf{h}}g = 0$; for an index-free proof of this well-known fact we refer to [14], 3.11 (p. 1381). □

Theorem 5.5. *A p-Berwald manifold is R-quadratic, if and only if, its stretch tensor vanishes.*

Proof. The necessity of the condition is a consequence of Corollary 4.3. To prove the sufficiency, we show that in a p-Berwald manifold we have

$$\nabla^{\mathsf{v}}\mathbf{H}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) = \frac{1}{F^2}\mathbf{\Sigma}(\widetilde{Y}, \widetilde{Z}, \widetilde{X}, \widetilde{U}) \otimes \delta; \quad \widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U} \in \Gamma(\mathring{\pi}). \tag{25}$$

Observe first that

$$\nabla^{\mathsf{h}}\mathbf{B} \overset{(24)}{=} -\nabla^{\mathsf{h}}\left(\frac{1}{E}\mathbf{P} \otimes \delta\right) \overset{(11),(14)}{=} -\frac{1}{E}\nabla^{\mathsf{h}}\mathbf{P} \otimes \delta.$$

Now, applying Bianchi identity (17), we get

$$\begin{aligned}\nabla^{\mathsf{v}}\mathbf{H}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) &= \nabla^{\mathsf{h}}\mathbf{B}(\widetilde{Z}, \widetilde{Y}, \widetilde{X}, \widetilde{U}) - \nabla^{\mathsf{h}}\mathbf{B}(\widetilde{Y}, \widetilde{Z}, \widetilde{X}, \widetilde{U}) = -\frac{1}{E}(\nabla^{\mathsf{h}}\mathbf{P}(\widetilde{Z}, \widetilde{Y}, \widetilde{X}, \widetilde{U}) \\ &- \nabla^{\mathsf{h}}\mathbf{P}(\widetilde{Y}, \widetilde{Z}, \widetilde{X}, \widetilde{U})) \otimes \delta \overset{(20)}{=} \frac{1}{F^2}\mathbf{\Sigma}(\widetilde{Y}, \widetilde{Z}, \widetilde{X}, \widetilde{U}).\end{aligned}$$

This proves (25), whence the statement follows. □

To give a more precise characterization of p-Berwald manifolds, we need the concept of Douglas manifolds. By the *Douglas curvature* of a Finsler manifold we mean the tensor

$$\mathbf{D} := \mathbf{B} - \frac{1}{n+1}(\operatorname{tr}\mathbf{B} \odot \mathbf{1} + (\nabla^{\mathsf{v}} \operatorname{tr}\mathbf{B}) \otimes \delta), \tag{26}$$

where the symbol $\odot$ denotes symmetric product (without any extra numerical factor). An index-free representation of the Douglas curvature was first presented by J. Szilasi and Sz. Vattamány [13]; formula (26) is just a "pull back version" of formula (6.2b) of the cited paper. Finsler manifolds with vanishing Douglas curvature were baptized *Douglas manifolds* by S. Bácsó and M. Matsumoto, who devoted a series of papers to their thorough investigation [1]–[4]. Observe that in weakly Berwald manifolds, and hence in p-Berwald manifolds the Douglas and Berwald curvature coincide.

Lemma 5.6. *The projected tensor of the Douglas curvature is*

$$\mathbf{pD} = \mathbf{pB} - \frac{1}{n+1}\operatorname{tr}\mathbf{B} \odot \mathbf{p} = \mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta - \frac{1}{n+1}\operatorname{tr}\mathbf{B} \odot \mathbf{p}. \tag{27}$$

Proof. First we check that $\mathbf{D}$ satisfies the condition of Corollary 5.2, i.e., $\mathbf{D}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = 0$, if $\delta \in \{\widetilde{X}, \widetilde{Y}, \widetilde{Z}\}$. Let, for example, $\widetilde{X} := \delta$. Then

$$\begin{aligned}\mathbf{D}(\delta, \widetilde{Y}, \widetilde{Z}) &:= \mathbf{B}(\delta, \widetilde{Y}, \widetilde{Z}) - \frac{1}{n+1}(\operatorname{tr}\mathbf{B}(\delta, \widetilde{Y})\widetilde{Z} + \operatorname{tr}\mathbf{B}(\widetilde{Y}, \widetilde{Z})\delta + \operatorname{tr}\mathbf{B}(\widetilde{Z}, \delta)\widetilde{Y}) \\ &- \frac{1}{n+1}(\nabla_C \operatorname{tr}\mathbf{B})(\widetilde{Y}, \widetilde{Z})\delta \overset{(16)}{=} -\frac{1}{n+1}(\operatorname{tr}\mathbf{B}(\widetilde{Y}, \widetilde{Z})\delta + \nabla_C \operatorname{tr}\mathbf{B})(\widetilde{Y}, \widetilde{Z})\delta).\end{aligned}$$

It is known (see e.g. [13], Proposition 4.4) that $\mathbf{B}$ is homogeneous of degree -1, i.e., $\nabla_C \mathbf{B} = -\mathbf{B}$. Thus $\nabla_C \operatorname{tr}\mathbf{B} = \operatorname{tr}\nabla_C \mathbf{B} = -\operatorname{tr}\mathbf{B}$, and hence $\mathbf{D}(\delta, \widetilde{Y}, \widetilde{Z}) = 0$. The other two cases may be handled similarly. Now it follows that

$$\mathbf{pD} = \mathbf{p} \circ \mathbf{D} = \mathbf{pB} - \frac{1}{n+1}(\mathbf{p}(\operatorname{tr}\mathbf{B} \odot \mathbf{1}) + \mathbf{p}(\nabla^{\mathsf{v}} \operatorname{tr}\mathbf{B} \otimes \delta).$$

Here, for any vector fields X, Y, Z on M,

$$\begin{aligned}\mathbf{p}(\operatorname{tr}\mathbf{B} \odot \mathbf{1}(\widehat{X}, \widehat{Y}, \widehat{Z})) &:= \mathbf{p}(\operatorname{tr}\mathbf{B} \odot \mathbf{1}(\mathbf{p}\widehat{X}, \mathbf{p}\widehat{Y}, \mathbf{p}\widehat{Z})) \overset{(16),\mathrm{Cor.5.2}}{=} \mathbf{p}(\operatorname{tr}\mathbf{B}(\widehat{X}, \widehat{Y})\mathbf{p}(\widehat{Z}) \\ &+ \operatorname{tr}\mathbf{B}(\widehat{Y}, \widehat{Z})\mathbf{p}(\widehat{X}) + \operatorname{tr}\mathbf{B}(\widehat{Z}, \widehat{X})\mathbf{p}(\widehat{Y})) = \operatorname{tr}\mathbf{B}(\widehat{X}, \widehat{Y})\mathbf{p}(\widehat{Z}) \\ &+ \operatorname{tr}\mathbf{B}(\widehat{Y}, \widehat{Z})\mathbf{p}(\widehat{X}) + \operatorname{tr}\mathbf{B}(\widehat{Z}, \widehat{X})\mathbf{p}(\widehat{Y}) \\ &= (\operatorname{tr}\mathbf{B} \odot \mathbf{P})(\widehat{X}, \widehat{Y}, \widehat{Z}),\end{aligned}$$

while

$$\mathbf{p}(\nabla^{\mathsf{v}} \operatorname{tr} \mathbf{B} \otimes \delta)(\widehat{X}, \widehat{Y}, \widehat{Z}) = \mathbf{p}((\nabla^{\mathsf{v}}_{\mathbf{p}\widehat{X}} \operatorname{tr} \mathbf{B})(\mathbf{p}\widehat{Y}, \mathbf{p}\widehat{Z})\delta) = 0,$$

since $\mathbf{p}(\delta) = 0$.

This concludes the proof of (27). □

Theorem 5.7. *If (M, F) is a Finsler manifold of dimension $n > 2$, then (M, F) is a p-Berwald manifold, if and only if, it is a weakly Berwald Douglas manifold.*

Proof. If (M, F) is a p-Berwald manifold, then it is weakly Berwald by Proposition 5.4, therefore (27) reduces to $\mathbf{pD} = 0$. However, by a theorem of T. Sakaguchi [11] (see also [18]), $\mathbf{pD} = 0$ is equivalent to the vanishing of the Douglas tensor under the condition $n > 2$.

Conversely, if (M, F) is a weakly Berwald Douglas manifold, then $\mathbf{D} = \mathbf{pD} = 0$ and $\operatorname{tr} \mathbf{B} = 0$ imply by (27) that (M, F) is a p-Berwald manifold. □

Finally, we have a look at the "exceptional case" $\dim M = 2$. Then one can choose a section $m \in \Gamma(\mathring{\pi})$ such that

$$g(\ell, m) = 0, \quad g(m, m) = 1;$$

the pair (ℓ, m) is said to be a *Berwald frame* on (M, F). An immediate calculation shows that the only non vanishing component of the tensor $\nabla^{\mathsf{v}} g$ with respect to (ℓ, m) is the function

$$I := \frac{1}{2}\nabla^{\mathsf{v}} g(m, m, m),$$

it is called the *main scalar* of (M, F). For the Landsberg tensor of (M, F) we have the expression

$$2\mathbf{P} = \frac{SI}{I}\nabla^{\mathsf{v}} g, \tag{28}$$

where S is the canonical spray. By (16), the only surviving component of the Berwald curvature is $\mathbf{B}(m, m)m$. It may be shown that

$$\mathbf{B}(m, m)m = -\frac{2SI}{F}\ell + \big((\mathbf{i}m)(SI) + (\mathcal{H}m)I\big)m, \tag{29}$$

where $\mathcal{H}$ is the Barthel connection arising from S according to (9). By (28) and (29), condition $\mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta = 0$ takes the form

$$\mathbf{B}(m, m)m + \frac{1}{2E}\frac{SI}{I}\nabla^{\mathsf{v}} g(m, m, m)\delta = 0. \tag{30}$$

Since $\frac{1}{2E}\frac{SI}{I}\nabla^{\mathsf{v}} g(m, m, m)\delta = \frac{1}{E}(SI)\delta = \frac{2}{F}(SI)\ell$, (29) and (30) yield

$$(\mathbf{i}m)SI + (\mathcal{H}m)I = 0. \tag{31}$$

Thus we obtain:

Theorem 5.8. *A two-dimensional Finsler manifold is a p-Berwald manifold, if and only if, the main scalar satisfies relation* (31).

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