Publ. Math. Debrecen
74/3-4 (2009), 453–455

A note on the exponential diophantine equation $(2^n - 1)(b^n - 1) = x^2$

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Abstract. Let b be a fixed positive integer with $b > 2$. In this paper, using some elementary methods, we prove that if $3 \mid b$, then the equation $(2^n - 1)(b^n - 1) = x^2$ has no positive integer solution (n, x).

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let a and b be fixed positive integers with $1 < a < b$. Recently, there were many works concerned the equation

$$(a^n - 1)(b^n - 1) = x^2, \quad n, x \in \mathbb{N} \tag{1}$$

(see [1], [2], [3], [4], [6]). In this paper we consider the case that $a = 2$. Then, equation (1) can be written as

$$(2^n - 1)(b^n - 1) = x^2, \quad n, x \in \mathbb{N}. \tag{2}$$

In this respect, L. Szalay [6] proved that if $b = 3$, then (2) has no solution (n, x). L. Hajdu and L. Szalay [3] proved that if $b = 6$, then (2) has no solution (n, x). In this paper we prove a general result as follows.

Theorem. *If* $3 \mid b$, *then* (2) *has no solution* (n, x).

Mathematics Subject Classification: 11D61.
Key words and phrases: exponential diophantine equation, Pell's equation.
Supported by the National Natural Science Foundation of China (No. 10771186) and the Guangdong Provincial Natural Science Foundation (No. 06029035).

In addition, we notice that (2) has solutions for infinitely many b. In fact, (1) has solutions if b satisfies one of the following conditions.

(i) If $b = 1 + c^2$, where c is a positive integer with $c > 1$, then (2) has the solution $(n, x) = (1, c)$.

(ii) Let $\alpha = 2+\sqrt{3}$ and $\beta = 2-\sqrt{3}$. For any positive integer k, if $b = (\alpha^k+\beta^k)/2$, then (2) has the solution $(n, x) = (2, 3(\alpha^k - \beta^k)/2\sqrt{2}\,)$.

(iii) If $b = 4$ or 22, then (2) has the solutions $(n, x) = (3, 21)$ and $(n, x) = (3, 273)$, respectively.

By the above mentioned observations, we propose the following conjecture.

Conjecture. *Excepting the above cases* (i), (ii) *and* (iii), (2) *has no solution* (n, x).

2. Proof of Theorem

Let d be a positive integer which is not a square. It is a well known fact that the Pell equation

$$u^2 - dv^2 = 1, \quad u, v \in \mathbb{N} \tag{3}$$

has solution (u, v).

Lemma ([5, Lemma 3])**.** *Let* $(u, v) = (u_1, v_1)$ *denote the least solution of* (3)*. Then we have*

(i) *For any solution* (u, v) *of* (3)*, we have* $v_1 \mid v$.

(ii) *If* $(u, v) = (u', v')$ *is a solution of* (3) *such that* u' *is a power of* 2*, then* (u', v') *is the least solution of* (3).

Proof of Theorem. Let b be a positive integer with $3 \mid b$. If (2) has a solution (n, x), then we have

$$2^n - 1 = dy^2, \tag{4}$$

and

$$b^n - 1 = dz^2, \tag{5}$$

where d, y and z are positive integers satisfying $dyz = x$, and d is square free. Since $3 \mid b$, we see from (5) that $dz^2 \equiv -1 \equiv 2 \pmod 3$. It implies that $3 \nmid d$, $3 \nmid z$, $z^2 \equiv 1 \pmod 3$ and $d \equiv 2 \pmod 3$.

If $3 \nmid y$, then $y^2 \equiv 1 \pmod 3$. Further, since $d \equiv 2 \pmod 3$, we get $dy^2+1 \equiv d + 1 \equiv 0 \pmod 3$. But, by (4), it is impossible. Therefore, we have

$$3 \mid y \tag{6}$$

Then, by (4), we get $2^n \equiv 1 \pmod 3$. It implies that n must be even.

Since $2 \mid n$, we see from (4) that the Pell equation (3) has the solution $(u,v) = (2^{n/2}, y)$. Therefore, by (ii) of Lemma, $(u_1, v_1) = (2^{n/2}, y)$ is the least solution of (3).

On the other hand, we find from (5) that (3) has an other solution $(u,v) = (b^{n/2}, z)$. By (i) of Lemma, we get $y \mid z$. Further, by (6), we obtain $3 \mid z$. But, since $3 \mid b$, it is impossible by (5). Thus, if $3 \mid b$, then (2) has no solution (n,x). The theorem is proved. □

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(Received October 31, 2008)