

Linear iterative equations of higher orders and random-valued functions

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Dedicated to Professor Zoltán Daróczy on his 70th birthday

Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$, a separable metric space X with the σ -algebra $\mathcal{B}$ of all its Borel subsets and a $\mathcal{B} \otimes \mathcal{A}$ -measurable $f : X \times \Omega \rightarrow X$ we consider the equation

$$(E) \quad \varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega)$$

and iterates $f^n, n \in \mathbb{N}$, of f defined on $X \times \Omega^{\mathbb{N}}$ by $f^1(x, \omega) = f(x, \omega_1)$ and $f^{n+1}(x, \omega) = f(f^n(x, \omega), \omega_{n+1})$. Assuming that for every $x \in X$ the sequence $(f^n(x, \cdot))_{n \in \mathbb{N}}$ converges in law and $\pi(x, \cdot)$ denotes the limit distribution we show that for every Borel and bounded $u : X \rightarrow \mathbb{R}$ the function $x \mapsto \int_X u(y) \pi(x, dy)$, $x \in X$, is a Borel solution of (E) and we study regularity of these solutions.

1. Throughout the paper $(\Omega, \mathcal{A}, P)$ is a probability space and (X, ϱ) is a separable metric space.

Let $\mathcal{B}$ denote the σ -algebra of all Borel subsets of X . We say that $f : X \times \Omega \rightarrow X$ is a *random-valued function* (shortly: an *rv-function*) if it is measurable with respect to the product σ -algebra $\mathcal{B} \otimes \mathcal{A}$. The iterates of such an rv-function are

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given by

$$\begin{aligned} f^1(x, \omega_1, \omega_2, \dots) &= f(x, \omega_1), \\ f^{n+1}(x, \omega_1, \omega_2, \dots) &= f(f^n(x, \omega_1, \omega_2, \dots), \omega_{n+1}) \end{aligned}$$

for x from X and $(\omega_1, \omega_2, \dots)$ from Ω^∞ defined as $\Omega^\mathbb{N}$. Note that $f^n : X \times \Omega^\infty \rightarrow X$ is an rv-function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More exactly, the n -th iterate f^n is $\mathcal{B} \otimes \mathcal{A}_n$ -measurable, where $\mathcal{A}_n$ denotes the σ -algebra of all the sets of the form

$$\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : (\omega_1, \dots, \omega_n) \in A\}$$

with A from the product σ -algebra $\mathcal{A}^n$. (See [6; Section 1.4], [3], [4].)

Fix an rv-function $f : X \times \Omega \rightarrow X$.

According to R. KAPICA [4; Theorem 2] the probability distribution of the limit in measure of $(f^n(x, \cdot))_{n \in \mathbb{N}}$ always produces a bounded solution of the equation

$$\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega) \quad (1)$$

which in addition is nonconstant provided the limit really depends on x . We generalize this theorem to the weak convergence of distributions and show that a simple additional condition guarantees that both the limit distribution and solutions of (1) generated by this limit are Lipschitzian.

By a distribution (on X) we mean any probability measure defined on $\mathcal{B}$. Recall that a sequence $(\pi_n)_{n \in \mathbb{N}}$ of distributions converges weakly to a distribution π if

$$\lim_{n \rightarrow \infty} \int_X u(x) \pi_n(dx) = \int_X u(x) \pi(dx)$$

for any continuous and bounded function $u : X \rightarrow \mathbb{R}$. It is well known (see, [1; Theorem 11.3.3]) that this convergence is metrizable by the (Fortet–Mourier, Lévy–Prohorov, Wasserstein) metric:

$$\|\pi_1 - \pi_2\|_W = \sup \left\{ \left| \int_X u d\pi_1 - \int_X u d\pi_2 \right| : u \in \text{Lip}_1(X), \|u\|_\infty \leq 1 \right\},$$

where

$$\text{Lip}_1(X) = \{u : X \rightarrow \mathbb{R} \mid |u(x) - u(z)| \leq \varrho(x, z) \text{ for } x, z \in X\}$$

and $\|u\|_\infty = \sup\{|u(x)| : x \in X\}$ for a bounded $u : X \rightarrow \mathbb{R}$.

2. Let $\pi_n(x, \cdot)$ denote the distribution of $f^n(x, \cdot)$, i.e.,

$$\pi_n(x, B) = P^\infty(f^n(x, \cdot) \in B) \quad (2)$$

for $n \in \mathbb{N}$, $x \in X$ and $B \in \mathcal{B}$. Clearly, $\pi_1(x, \cdot)$ is the distribution of $f(x, \cdot)$:

$$\pi_1(x, B) = P(f(x, \cdot) \in B) \quad \text{for } x \in X \text{ and } B \in \mathcal{B}. \quad (3)$$

We start with the following lemma.

Lemma 1. *For any $n \in \mathbb{N}$ and $B \in \mathcal{B}$ the function $\pi_n(\cdot, B)$ given by (2) is Borel and*

$$\pi_{n+1}(x, B) = \int_{\Omega} \pi_n(f(x, \omega), B) P(d\omega) \quad \text{for } x \in X; \quad (4)$$

moreover, if $u : X \rightarrow \mathbb{R}$ is Borel and bounded, then the function

$$x \mapsto \int_X u(y) \pi_n(x, dy), \quad x \in X, \quad (5)$$

is Borel, for every $x \in X$ the function

$$\omega \mapsto \int_X u(y) \pi_n(f(x, \omega), dy), \quad \omega \in \Omega, \quad (6)$$

is $\mathcal{A}$ -measurable and

$$\int_X u(y) \pi_{n+1}(x, dy) = \int_{\Omega} \left(\int_X u(y) \pi_n(f(x, \omega), dy) \right) P(d\omega). \quad (7)$$

PROOF. Since

$$C := \{(x, \omega) \in X \times \Omega^\infty : f^n(x, \omega) \in B\} \in \mathcal{B} \otimes \mathcal{A}^\infty,$$

the function

$$x \mapsto P^\infty(C_x), \quad x \in X,$$

i.e. $\pi_n(\cdot, B)$, is (see, e.g., [7; Theorem 6.3.1]) Borel. To get (4) note that (by induction)

$$f^{n+1}(x, \omega_1, \omega_2, \dots) = f^n(f(x, \omega_1), \omega_2, \omega_3, \dots)$$

for $x \in X$ and $(\omega_1, \omega_2, \dots) \in \Omega^\infty$, and observe that

$$\begin{aligned} \pi_{n+1}(x, B) &= P^\infty(\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : f^n(f(x, \omega_1), \omega_2, \omega_3, \dots) \in B\}) \\ &= P^\infty(\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : (\omega_2, \omega_3, \dots) \in C_{f(x, \omega_1)}\}) \\ &= \int_{\Omega} P^\infty(C_{f(x, \omega_1)})(P(d\omega_1)) = \int_{\Omega} \pi_n(f(x, \omega_1), B) P(d\omega_1) \end{aligned}$$

for $x \in X$.

If $B \in \mathcal{B}$ and $u = \mathbf{1}_B$, then (5) is the function $\pi_n(\cdot, B)$ – and we already have shown that it is Borel – whereas (6) is the function $\omega \mapsto \pi_n(f(x, \omega), B)$, $\omega \in \Omega$, which is clearly $\mathcal{A}$ -measurable, and (7) reduces to (4) for every $x \in X$. A pass to the general case is standard. $\square$

Now we assume the following condition.

(H) For every $x \in X$ the sequence $(\pi_n(x, \cdot))_{n \in \mathbb{N}}$ defined by (2) converges weakly to a distribution $\pi(x, \cdot)$.

The following theorem generalizes [4; Theorem 2].

Theorem 1. *If (H) holds, then for every Borel and bounded $u : X \rightarrow \mathbb{R}$ the function $\varphi : X \rightarrow \mathbb{R}$ given by*

$$\varphi(x) = \int_X u(y) \pi(x, dy) \quad (8)$$

is a Borel and bounded solution of (1); in particular, for any $B \in \mathcal{B}$ the function $\pi(\cdot, B)$ is a Borel solution of (1).

PROOF. Assume first that $u : X \rightarrow \mathbb{R}$ is continuous and bounded. Since for every $n \in \mathbb{N}$ the function (5) is Borel, so is (see [1; Theorem 4.2.2]) the pointwise limit

$$x \mapsto \int_X u(y) \pi(x, dy), \quad x \in X, \quad (9)$$

of the sequence built of these functions. Moreover, making use of (7) and applying the Lebesgue dominated theorem we have also

$$\int_X u(y) \pi(x, dy) = \int_\Omega \left(\int_X u(y) \pi(f(x, \omega), dy) \right) P(d\omega) \quad \text{for } x \in X, \quad (10)$$

which means that $\varphi : X \rightarrow \mathbb{R}$ given by (8) solves (1).

Fix now a Borel and bounded function $u_0 : X \rightarrow \mathbb{R}$, put

$$M = \|u_0\|_\infty$$

and consider the family $\mathbf{U}$ of all Borel functions $u : X \rightarrow [-M, M]$ such that the function (9) is Borel and (10) holds. The previous part of the proof shows that any continuous $u : X \rightarrow [-M, M]$ is in $\mathbf{U}$. Moreover, from the Lebesgue dominated convergence theorem it follows that $\mathbf{U}$ contains the limit of any pointwise convergent sequence of functions in $\mathbf{U}$. Consequently (see [7; Theorem 4.5.2]) every Borel function $u : X \rightarrow [-M, M]$ belongs to $\mathbf{U}$. In particular, $u_0 \in \mathbf{U}$. This proves the main part of Theorem 1.

To finish the proof observe that if $B \in \mathcal{B}$, then putting $u = \mathbf{1}_B$ in (8) we get $\varphi = \pi(\cdot, B)$. $\square$

Remark 1. Assume (H), let $A \in \mathcal{B}$ and

$$P(f(x, \cdot) \in A) = 1 \quad \text{for } x \in A. \quad (11)$$

If $p \in [0, 1]$, $F \subset X$ is closed and

$$P(f(x, \cdot) \in F) \geq p \quad \text{for } x \in A, \quad (12)$$

then

$$\pi(x, F) \geq p \quad \text{for } x \in A. \quad (13)$$

PROOF. By induction, making use of (3) and the recurrence (4), we obtain

$$\pi_n(x, F) \geq p \quad \text{for } x \in A.$$

Since F is closed, this jointly with (H) gives (see [1; Theorem 11.1.1])

$$p \leq \limsup_{n \rightarrow \infty} \pi_n(x, F) \leq \pi(x, F)$$

for $x \in A$. □

Remark 2. Assume (H) and let a finite $A \subset X$ satisfies (11). If $x_0 \in X$,

$$f(x_0, \cdot) = x_0 \quad \text{a.s.} \quad (14)$$

and

$$P(f(x, \cdot) = x_0) < 1 \quad \text{for } x \in A, \quad (15)$$

then

$$\pi(x, \cdot) \neq \pi(x_0, \cdot) \quad \text{for } x \in A.$$

PROOF. Let $B(x_0, r)$ denote the open ball with center at x_0 and radius r . From (15) it follows that

$$0 < P(f(x, \cdot) \in X \setminus \{x_0\}) = \lim_{n \rightarrow \infty} P\left(f(x, \cdot) \in X \setminus B\left(x_0, \frac{1}{n}\right)\right)$$

for $x \in A$. Since A is finite it shows that there is a positive integer n such that (12) holds with

$$F = X \setminus B\left(x_0, \frac{1}{n}\right) \quad \text{and} \quad p = \min\{P(f(x, \cdot) \in F) : x \in A\} > 0.$$

By Remark 1 we have (13). On the other hand, from (3) and (14),

$$\pi_1(x_0, B) = P(x_0 \in B) = \mathbf{1}_B(x_0) = \delta_{x_0}(B)$$

for $B \in \mathcal{B}$ which jointly with (4) and (14) shows that $\pi_n(x_0, \cdot) = \delta_{x_0}$ for $n \in \mathbb{N}$. Consequently also

$$\pi(x_0, \cdot) = \delta_{x_0} \quad (16)$$

and since $x_0 \notin F$ we see that

$$\pi(x_0, F) = 0 < p \leq \pi(x, F)$$

for $x \in A$. □

3. To obtain more information about the limit distribution and solutions of (1) generated by this limit we assume that

$$\int_{\Omega} \varrho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \lambda \varrho(x, z) \quad \text{for } x, z \in X. \quad (17)$$

Theorem 2. Assume (H). If (17) holds with a $\lambda \in (0, \infty)$, then:

- (i) for every Lipschitzian and bounded $u : X \rightarrow \mathbb{R}$ the function $\varphi : X \rightarrow \mathbb{R}$ given by (8) is of the first Baire class and a bounded solution of (1);
- (ii) if $x, z \in X$ and $\pi(x, \cdot) \neq \pi(z, \cdot)$, then (1) has a bounded solution $\varphi : X \rightarrow \mathbb{R}$ of the first Baire class such that $\varphi(x) \neq \varphi(z)$.

PROOF. From (17) it follows by induction that

$$\int_{\Omega^\infty} \varrho(f^n(x, \omega), f^n(z, \omega)) P^\infty(d\omega) \leq \lambda^n \varrho(x, z) \quad \text{for } x, z \in X \text{ and } n \in \mathbb{N}.$$

Hence, if $u : X \rightarrow \mathbb{R}$ is bounded and Lipschitzian with a Lipschitz constant L , then

$$\left| \int_{\Omega^\infty} u(f^n(x, \omega)) P^\infty(d\omega) - \int_{\Omega^\infty} u(f^n(z, \omega)) P^\infty(d\omega) \right| \leq L \lambda^n \varrho(x, z),$$

i.e., by (2),

$$\left| \int_X u(y) \pi_n(x, dy) - \int_X u(y) \pi_n(z, dy) \right| \leq L \lambda^n \varrho(x, z) \quad \text{for } x, z \in X \quad (18)$$

and for $n \in \mathbb{N}$. This shows that the function $\varphi : X \rightarrow \mathbb{R}$ given by (8) is the pointwise limit of Lipschitzian functions

$$x \mapsto \int_X u(y) \pi_n(x, dy), \quad x \in X,$$

hence of the first Baire class. This and Theorem 1 give (i).

To get (ii) it is enough to observe that if $\pi(x, \cdot) \neq \pi(z, \cdot)$, then (see [1, Proposition 11.3.2]) there exists a bounded $u \in \text{Lip}_1(X)$ such that

$$\int_X u(y)\pi(x, dy) \neq \int_X u(y)\pi(z, dy)$$

and to apply part (i). $\square$

Corollary 1. *Assume (H) and let X be compact. If (17) holds with a $\lambda \in (0, \infty)$, then for every continuous $u : X \rightarrow \mathbb{R}$ the function $\varphi : X \rightarrow \mathbb{R}$ given by (8) is of the first Baire class and a bounded solution of (1).*

PROOF. Fix a continuous function $u : X \rightarrow \mathbb{R}$ and (see [1; Theorem 11.2.4]) let $(u_n)_{n \in \mathbb{N}}$ be a sequence of Lipschitzian mappings of X into $\mathbb{R}$ uniformly convergent to u . Defining $\varphi_n : X \rightarrow \mathbb{R}$ by

$$\varphi_n(x) = \int_X u_n(y)\pi(x, dy) \quad (19)$$

for $n \in \mathbb{N}$ we see that $(\varphi_n)_{n \in \mathbb{N}}$ uniformly converges to the function $\varphi : X \rightarrow \mathbb{R}$ given by (8). It follows from Theorem 2(i) that φ_n is of the first Baire class for every $n \in \mathbb{N}$, and so is (see [7; Theorem 3.5.2]) the uniform limit φ . This and Theorem 1 end the proof. $\square$

Corollary 2. *Assume (H). If (17) holds with a $\lambda \in (0, \infty)$, then for every closed subset F of X the function $\pi(\cdot, F)$ is of the second Baire class and a solution of (1).*

PROOF. Fix a closed $F \subset X$ and for every $n \in \mathbb{N}$ define $u_n, \varphi_n : X \rightarrow [0, 1]$ by

$$u_n(x) = 1 - \min\{1, n\varrho(x, F)\}$$

and (19). Since u_n is Lipschitzian, by Theorem 2(i) the function φ_n is of the first Baire class for $n \in \mathbb{N}$, and since $(u_n)_{n \in \mathbb{N}}$ pointwise converges to $\mathbf{1}_F$, by the Lebesgue dominated theorem $(\varphi_n)_{n \in \mathbb{N}}$ pointwise converges to $\pi(\cdot, F)$. $\square$

Assuming (17) with $\lambda = 1$ we can obtain much more.

Theorem 3. *Assume (H). If*

$$\int_{\Omega} \varrho(f(x, \omega), f(z, \omega))P(d\omega) \leq \varrho(x, z) \quad \text{for } x, z \in X, \quad (20)$$

then

$$\|\pi(x, \cdot) - \pi(z, \cdot)\|_W \leq \varrho(x, z) \quad \text{for } x, z \in X \quad (21)$$

and

- (i) for every Lipschitzian and bounded $u : X \rightarrow \mathbb{R}$ the function $\varphi : X \rightarrow \mathbb{R}$ given by (8) is a Lipschitzian and bounded solution of (1);
- (i) if $x, z \in X$ and $\pi(x, \cdot) \neq \pi(z, \cdot)$, then (1) has a Lipschitzian and bounded solution $\varphi : X \rightarrow \mathbb{R}$ such that $\varphi(x) \neq \varphi(z)$.

PROOF. If $u \in \text{Lip}_1(X)$ is bounded, then we have (18) with $L = 1$ and $\lambda = 1$ for every $n \in \mathbb{N}$ and passing to the limit we see that (21) holds.

If $u : X \rightarrow \mathbb{R}$ is bounded and Lipschitzian with a Lipschitz constant L , then we have (18) with $\lambda = 1$ and passing to the limit we see that $\varphi : X \rightarrow \mathbb{R}$ given by (8) is Lipschitzian with a Lipschitz constant L .

To get (ii) we argue as in the proof of Theorem 2. $\square$

Remark 3 (cf. [2; Theorem 5.1]). Assume (H). If $\varphi : X \rightarrow \mathbb{R}$ is a continuous and bounded solution of (1), then

$$\varphi(x) = \int_X \varphi(y) \pi(x, dy) \quad \text{for } x \in X; \quad (22)$$

in particular, if $x, z \in X$ and $\pi(x, \cdot) = \pi(z, \cdot)$, then $\varphi(x) = \varphi(z)$.

PROOF. It follows from (1) and (2) that

$$\varphi(x) = \int_{\Omega^\infty} \varphi(f^n(x, \omega)) P^\infty(d\omega) = \int_X \varphi(y) \pi_n(x, dy)$$

for $x \in X$ and $n \in \mathbb{N}$. Passing to the limit we obtain (22). $\square$

Corollary 3. Assume (H) and (20). If (1) has a nonconstant continuous and bounded solution $\varphi : X \rightarrow \mathbb{R}$, then it has also a nonconstant Lipschitzian and bounded solution $\varphi : X \rightarrow \mathbb{R}$.

PROOF. It is enough to apply Remark 3 and Theorem 3(ii). $\square$

Corollary 4. Assume (H) and (20). If (14) holds for an $x_0 \in X$ and any Lipschitzian and bounded solution $\varphi : X \rightarrow \mathbb{R}$ of (1) is a constant function, then for every $x \in X$ the sequence $(f^n(x, \cdot))_{n \in \mathbb{N}}$ converges to x_0 in probability.

PROOF. From (16) and Theorem 3(ii) it follows that

$$\pi(x, \cdot) = \delta_{x_0} \quad \text{for } x \in X.$$

Applying [1; Proposition 11.1.3] we obtain the assertion. $\square$

Remark 4. In case of the convergence in probability [5; Theorem 3.4] brings completely different conditions ensuring that for some continuous and bounded $u : X \rightarrow \mathbb{R}$ the function $\varphi : X \rightarrow \mathbb{R}$ given by (8) is continuous and nonconstant. In view of Theorem 1, as the proof of [5; Theorem 3.4(iii)] shows it remains valid also in case (H) and for Borel and bounded $u : X \rightarrow \mathbb{R}$ satisfying (3.14) of [5].

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