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# On a functional equation with a symmetric component

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*Dedicated to Professor Zoltán Daróczy on the occasion of his seventieth birthday*

**Abstract.** Let $I \subset \mathbb{R}$ be a nonvoid open interval and $r \neq 0, 1$, $q \in (0,1)$, such that $r \neq q$, $r \neq \frac{1}{2}$ and $q \neq \frac{1}{2}$. In this paper we give all the functions $f, g : I \to \mathbb{R}_+$ such that

$$f\left(\frac{x+y}{2}\right)[r(1-q)g(y) - (1-r)qg(x)] = \frac{r-q}{1-2q}[(1-q)f(x)g(y) - qf(y)g(x)]$$

for all $x, y \in I$.

## 1. Introduction

Let $J \subset \mathbb{R}$ be a nonvoid open interval and denote the class of continuous and strictly monotone real valued functions defined on the interval $J$ by $\mathcal{CM}(J)$. A function $M : J^2 \to J$ is called a *weighted quasi-arithmetic mean* on $J$ if there exist $0 < p < 1$ and $\varphi \in \mathcal{CM}(J)$ such that

$$M(x,y) = \varphi^{-1}(p\varphi(x) + (1-p)\varphi(y)) =: A_\varphi(x,y;p).$$

for all $x, y \in J$. The number $p$ is said to be the *weight* and the function $\varphi$ is called the *generating function* of the weighted quasi-arithmetic mean $M$.

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Now we can formulate the general problem as follows: determine all $M, N : J^2 \to J$ weighted quasi-arithmetic means and the constants $\mu \neq 0, 1$ and $r \neq 0, 1$, such that

$$\mu M(u,v) + (1-\mu)N(u,v) = ru + (1-r)v$$

holds for all $u, v \in J$. In detail this equation means the following: determine all the functions $\varphi, \psi \in \mathcal{CM}(J)$ and the constants $r \neq 0, 1$, $(p,q) \in (0,1)^2$, $\mu \neq 0, 1$ such that

$$\mu\varphi^{-1}(p\varphi(u) + (1-p)\varphi(v)) + (1-\mu)\psi^{-1}(q\psi(u) + (1-q)\psi(v)) = ru + (1-r)v$$

holds for all $u, v \in J$.

If we suppose that $\varphi, \psi \in \mathcal{CM}(J)$ are differentiable on $J$ and $\varphi'(u) > 0$, $\psi'(u) > 0$ for all $u \in J$, then with the notations $f := \varphi' \circ \varphi^{-1}$, $g := \psi' \circ \varphi^{-1}$, $I := \varphi(J)$ for the unknown functions $f, g : I \to \mathbb{R}_+$ and $\varphi(u) = x$ and $\varphi(v) = y$ $(x, y \in I)$, from the above equation we have

$$\begin{aligned} f(px + (1-p)y)&[r(1-q)g(y) - (1-r)qg(x)] \\ &= \mu[p(1-q)f(x)g(y) - (1-p)qf(y)g(x)] \qquad (1) \end{aligned}$$

for all $x, y \in I$. The functional equation (1) depends on the parameters $r \neq 0, 1$, $(p,q) \in (0,1)^2$ and $\mu \neq 0, 1$ for which, if $x = y$ in (1), by $f(x) > 0$, $g(x) > 0$ we have

$$\mu(p - q) = r - q. \qquad (2)$$

The functional equation (1) was studied in the following special cases:

(i) $p = q = r = \mu = 1/2$, by J. Matkowski [11], then by Z. Daróczy and Zs. Páles [5] under much weaker conditions.

(ii) $p = q$, $(p,q,r) \in (0,1)^3$ (then by (2) $r = q$) by Z. Daróczy and Zs. Páles in [6], [5].

(iii) $\mu = r$, $(p,q,r) \in (0,1)^3$ J. Jarczyk and J. Matkowski in [8], and J. Jarczyk [7], P. Burai [1].

(iv) $\mu = r$ and $p = 1/2$, $q \neq 1/2$, $(q,r) \in (0,1)^2$ by Z. Daróczy in [3] without any conditions.

(v) $p = 1/2$, $q \neq 1/2$ and $q, r \in (0,1)^2$, $r \neq q$, $r \neq 1/2$ and $\mu = \frac{2(r-q)}{1-2q}$ by Z. Daróczy and J. Dascăl in [4].

In this paper we study the functional equation (1) in the case $p = 1/2$ and $p \neq q$. Hence, by (2) we have $r \neq q$ and $r \neq \frac{1}{2}$ and

$$\mu = \frac{r-q}{\frac{1}{2} - q} = 2 \cdot \frac{r-q}{1-2q}.$$

This means we have to determine all the functions $f, g : I \to \mathbb{R}_+$ ($I \subset \mathbb{R}$ nonvoid open interval) and the constants $r \neq 0, 1$, $q \in (0, 1)$, such that

$$f\left(\frac{x+y}{2}\right)[r(1-q)g(y) - (1-r)qg(x)] = \frac{r-q}{1-2q}[(1-q)f(x)g(y) - qf(y)g(x)] \tag{3}$$

holds for all $x, y \in I$.

## 2. Main result

**Theorem 1.** *Let $I \subset \mathbb{R}$ be a nonvoid open interval and $r \neq 0, 1$, $q \in (0, 1)$, such that $r \neq q$, $r \neq \frac{1}{2}$ and $q \neq \frac{1}{2}$. If the functions $f, g : I \to \mathbb{R}_+$ are solutions of the functional equation* (3) *then the following cases are possible:*

(1) *If $r \neq \frac{q^2}{q^2+(1-q)^2}$ and $r \neq \frac{q}{2q-1}$ then there exist constants $a, b \in \mathbb{R}_+$ such that*

$$f(x) = a \quad \text{and} \quad g(x) = b \quad \text{for all } x \in I;$$

(2) *If $r = \frac{q^2}{q^2+(1-q)^2}$ then there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ and positive real numbers $c_1, c_2$ such that*

$$g(x) = c_1 e^{A(x)} \quad \text{and} \quad f(x) = c_2 e^{2A(x)} \quad \text{for all } x \in I;$$

(3) *If $r = \frac{q}{2q-1}$ then there exist real numbers $d_1$, $d_2$, $d_3$ such that*

$$g(x) = \frac{1}{d_1 x + d_2} > 0 \quad \text{and} \quad f(x) = d_3 \frac{1}{d_1 x + d_2} > 0 \quad \text{for all } x, y \in I.$$

*Conversely, the functions given in the above cases are solutions of equation* (3).

To prove Theorem 1 we need the following lemmas.

**Lemma 1.** *Let $I \subset \mathbb{R}$ be a nonvoid open interval and $r \neq 0, 1$, $0 < q < 1$, $r \neq q$, $r, q \neq 1/2$. If the functions $f, g : I \to \mathbb{R}_+$ satisfy the functional equation* (3) *then*

$$f\left(\frac{x+y}{2}\right)[g(x) + g(y)] = [f(x)g(y) + f(y)g(x)] \tag{4}$$

*holds for all $x, y \in I$.*

**Lemma 2.** *Let $I \subset \mathbb{R}$ be a nonvoid open interval and $r \neq 0, 1$, $0 < q < 1$, $r \neq q$, $r, q \neq 1/2$. If the functions $f, g : I \to \mathbb{R}_+$ satisfy the functional equation (3) then*

$$f(x)g(y)\{q(1-q)(1-2r)g(y) - [r(1-2q) - q^2(1-2r)]g(x)\}$$
$$= f(y)g(x)\{q(1-q)(1-2r)g(x) - [r(1-2q) - q^2(1-2r)]g(y)\} \quad (5)$$

*holds for all $x, y \in I$.*

These lemmas are proved in [4].

Proof of Theorem 1:

The proof of cases (1) and (2) is the same as the proof of Theorem 1 from [4]. In case (3), when $r = \frac{q}{2q-1}$, by Lemma 2 the equation (5) becomes

$$f(x)g(y)\frac{q(1-q)}{1-2q}[g(x)+g(y)] = f(y)g(x)\frac{q(1-q)}{1-2q}[g(x)+g(y)].$$

for all $x, y \in I$. Hence $f(x)g(y) = f(y)g(x)$, thus

$$f(x) = d_3 g(x) \quad \text{for some } d_3 > 0 \quad \text{and for all } x \in I. \quad (6)$$

Replacing this form of $f$ in (4) we have

$$g\left(\frac{x+y}{2}\right) = \frac{2}{\frac{1}{g(x)} + \frac{1}{g(y)}},$$

consequently, by [9], [10] there exist an additive function $B : \mathbb{R} \to \mathbb{R}$ and a real number $d_2$ such that $\frac{1}{g(x)} = B(x) + d_2 > 0$, thus $g(x) = \frac{1}{B(x)+d_2} > 0$ for all $x \in I$, that is, there exists $d_1 \in \mathbb{R}$ such that $B(x) = d_1 x$ for all $x \in I$, thus $g(x) = \frac{1}{d_1 x + d_2}$ for all $x \in I$. Finally, (6) completes the proof of case (3).

## 3. Application

Returning to the generalized problem we need the following definitions.

*Definition 1.* Let $\varphi, \psi \in \mathcal{CM}(J)$. If there exist $a \neq 0$ and $b$ such that

$$\psi(x) = a\varphi(x) + b \quad \text{if } x \in J$$

then we say that $\varphi$ is equivalent to $\psi$ on $J$ and denote it by $\varphi(x) \sim \psi(x)$ if $x \in J$ or in short $\varphi \sim \psi$ on $J$.

It is well-known that if $0 < p < 1$ and $\varphi, \psi \in \mathcal{CM}(J)$, then $A_\varphi(x, y; p) = A_\psi(x, y; p)$ for all $x, y \in J$ if and only if $\varphi \sim \psi$ on $J$.

We define the following sets:

$$T_+(J) := \{t \in \mathbb{R} \mid J + t \subset \mathbb{R}_+\}$$
$$T_-(J) := \{t \in \mathbb{R} \mid -J + t \subset \mathbb{R}_+\}.$$

With the help of these notations , set

$$\gamma_t^+(x) := \sqrt{x+t} \quad \text{if} \ \ t \in T_+(J) \quad (x \in J)$$
$$\gamma_t^-(x) := \sqrt{-x+t} \quad \text{if} \ \ t \in T_-(J) \quad (x \in J).$$

The general problem is as follows: determine all the functions $\varphi, \psi \in \mathcal{CM}(J)$ and the constants $r \neq 0, 1$, $(p, q) \in (0, 1)^2$, $\mu \neq 0, 1$ such that

$$\mu\varphi^{-1}(p\varphi(u) + (1-p)\varphi(v)) + (1-\mu)\psi^{-1}(q\psi(u) + (1-q)\psi(v)) = ru + (1-r)v$$

holds for all $u, v \in J$. If either $p$ or $q$ equals $1/2$, the following theorem gives the solutions of this equation. If $(\varphi, \psi)$ is the solution of the above functional equation with $p = 1/2$, $q \neq 1/2$, then $(\psi, \varphi)$ is the solution of the equation with $p \neq 1/2$, $q = 1/2$. So it is enough to state our theorem for the case $p = 1/2$, $q \neq 1/2$. In [4] the above equation (with $p = 1/2$) is solved for $0 < r < 1$, but here we take $r \neq 0, 1$ and we get further solutions, which solutions are also found by Z. Daróczy in [2] without the assumption of differentiability of the functions $\varphi$ and $\psi$.

**Theorem 2.** *Let $J \subset \mathbb{R}$ be a nonvoid open interval and $r \neq 0, 1$, $0 < q < 1$, $r, q \neq \frac{1}{2}$, $r \neq q$. If $\varphi, \psi \in \mathcal{CM}(J)$ are solutions of the functional equation*

$$\frac{2(r-q)}{1-2q}\varphi^{-1}\left(\frac{\varphi(u)+\varphi(v)}{2}\right) + \left(1 - \frac{2(r-q)}{1-2q}\right)\psi^{-1}(q\psi(u) + (1-q)\psi(v)) = ru + (1-r)v \tag{7}$$

*for all $u, v \in J$ and $\varphi, \psi$ are differentiable on $J$ and $\varphi'(u) > 0$, $\psi'(u) > 0$ for all $u \in J$ then $\varphi \sim \mathrm{id}$ and $\psi \sim \mathrm{id}$ on $J$, furthermore in the case $r = \frac{q^2}{q^2+(1-q)^2}$ the following cases are also possible:*

$$\varphi \sim \log\gamma_t^+,\ \psi \sim \gamma_t^+ \quad \textit{if } t \in T_+(J) \qquad \textit{or} \qquad \varphi \sim \log\gamma_t^-,\ \psi \sim \gamma_t^- \quad \textit{if } t \in T_-(J)$$

*and in the case $r = \frac{q}{2q-1}$ the following cases are also possible:*

$$\varphi \sim \gamma_t^+,\ \psi \sim \gamma_t^+ \quad \textit{if } t \in T_+(J) \qquad \textit{or} \qquad \varphi \sim \gamma_t^-,\ \psi \sim \gamma_t^- \quad \textit{if } t \in T_-(J).$$

Proof. It is enough to solve the functional equation (7) up to the equivalence of the functions $\varphi$ and $\psi$. With the notations $f := \varphi' \circ \varphi^{-1}$, $g := \psi' \circ \varphi^{-1}$, $I := \varphi(J)$ we get that equation (3) holds. Due to the definition of $f$, we obtain the differential equation for the function $\varphi$:

$$\varphi'(x) = f(\varphi(x)) \quad x \in J. \tag{8}$$

By Theorem 1, the case $r \neq \frac{q^2}{q^2+(1-q)^2}$, $r \neq \frac{q}{2q-1}$ gives the constant solutions, which implies that $\varphi \sim \mathrm{id}$, $\psi \sim \mathrm{id}$.

If $r = \frac{q^2}{q^2+(1-q)^2}$ the proof is found in [4].

If $r = \frac{q}{2q-1}$ then

$$f(x) = d_3 \frac{1}{d_1 x + d_2} \text{ and } g(x) = \frac{1}{d_1 x + d_2} \quad \text{for all } x \in I, \tag{9}$$

where $d_1, d_2, d_3 \in \mathbb{R}$, $d_3 > 0$.

In the case $d_1 = 0$, $\varphi \sim \mathrm{id}$ and $\psi \sim \mathrm{id}$.

In the case $d_1 \neq 0$ from (8) we have

$$\varphi'(u) = d_3 \frac{1}{d_1 \varphi(u) + d_2} > 0 \quad \text{for all } u \in J,$$

which implies that $\varphi(u) \sim \sqrt{C_2 u + C_3}$, from which we deduce that either there exists $t \in T_+(J)$ such that $\varphi \sim \gamma_t^+$ on $J$ or there exists $t \in T_-(J)$ such that $\varphi \sim \gamma_t^-$ on $J$.

Due to the definition of $g$, by (9) we obtain that

$$\psi'(u) = \frac{1}{d_1 \varphi(u) + d_2} > 0 \quad \text{for all } u \in J,$$

which implies that either there exists $t \in T_+(J)$ such that $\psi \sim \gamma_t^+$ on $J$ or there exists $t \in T_-(J)$ such that $\psi \sim \gamma_t^-$ on $J$. □

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