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On the maximal operator of the Marcinkiewicz–Fejér means of double Walsh–Kaczmarz-Fourier series

By USHANGI GOGINAVA (Tbilisi) and KÁROLY NAGY (Nyíregyháza)

Dedicated to Professor Zoltán Daróczy on the occasion of his seventieth birthday

Abstract. In the paper [3] we proved that the maximal operator of the Marcinkiewicz–Fejér means of the 2-dimensional Fourier series with respect to the Walsh–Kaczmarz system is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$.

Now, in this paper we prove a stronger result, that is there exists a martingale $f \in H_{2/3}$ such that the maximal Marcinkiewicz–Fejér operator with respect to Walsh–Kaczmarz system does not belong to the space $L_{2/3}$.

First, we give a brief introduction to the theory of dyadic analysis [8]. Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following

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way:

$$I_0(x) := G,$$

$$I_n(x) := I_n(x_0, \dots, x_{n-1}) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\},$$

$(x \in G,\ n \in \mathbb{N})$. These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of G, $I_n := I_n(0)$ $(n \in \mathbb{N})$. Set $e_n := (0, \dots, 0, 1, 0, \dots) \in G$, the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$.

For $k \in \mathbb{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the kth Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ $(i \in \mathbb{N})$, i.e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_k} \quad (x \in G,\ n \in \mathbb{P}).$$

The Walsh–Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_{|n|-k-1}}.$$

For $A \in \mathbb{N}$ define the transformation $\tau_A : G \to G$ by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_0, x_A, x_{A+1}, \dots).$$

By the definition of τ_A (see [11]), we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbb{N},\ x \in G).$$

The space $L_p(G^2)$, $0 < p \leq \infty$ with norms or quasi-norms $\|\cdot\|_p$ is defined in the usual way.

The Dirichlet kernels are defined by

$$D_n^{\alpha}(x) := \sum_{k=0}^{n-1} \alpha_k(x),$$

where $\alpha_k = w_k$ or κ_k. Recall that (see e.g. [8])

$$D_{2^n}(x) := D^w_{2^n}(x) = D^\kappa_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \notin I_n(0). \end{cases} \tag{1}$$

The two-dimensional dyadic cubes are of the form

$$I_{n,n}(x,y) := I_n(x) \times I_n(y).$$

The σ-algebra generated by the dyadic rectangles $\{I_{n,n}(x,y) : (x,y) \in G\times G\}$ is denoted by $F_{n,n}$.

Denote by $f = (f^{(n,n)},\ n \in N)$ a martingale with respect to $(F_{n,n},\ n \in \mathbb{N})$ (for details see, e.g. [15]). The maximal function of a martingale f is defined by

$$f^{\Box} = \sup_{n\in N} |f^{(n,n)}|.$$

In case $f \in L_1(G\times G)$, the maximal function can also be given by

$$f^{\Box}(x,y) = \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_{n,n}(x,y))} \left| \int_{I_{n,n}(x,y)} f(u,v)d\mu(u,v) \right|, \quad (x,y) \in G\times G.$$

For $0 < p < \infty$ the Hardy martingale space $H^{\Box}_p(G\times G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^{\Box}\|_p < \infty.$$

The Kroneker product $(\alpha_{m,n} : n, m \in \mathbb{N})$ of two Walsh(–Kaczmarz) system is said to be the two-dimensional Walsh(–Kaczmarz) system. Thus,

$$\alpha_{m,n}(x,y) = \alpha_n(x)\alpha_m(y).$$

If $f \in L_1(G^2)$, then the number $\hat{f}^\alpha(n,m) := \int_{G^2} f\alpha_{m,n}$ $(n, m \in \mathbb{N})$ is said to be the (n,m)th Walsh–(Kaczmarz)–Fourier coefficient of f. We can extend this definition to martingales in the usual way (see Weisz [14], [15]).

Denote by $S^\alpha_{n,m}$ the (n,m)th rectangular partial sum of the Walsh–(Kaczmarz) –Fourier series of a martingale f. Namely,

$$S^\alpha_{n,m}(f;x,y) := \sum_{k=0}^{n-1}\sum_{i=0}^{m-1} \hat{f}^\alpha(k,i)\alpha_{k,i}(x,y).$$

The Marcinkiewicz–Fejér means of a martingale f are defined by

$$\mathcal{M}^\alpha_n(f;x,y) := \frac{1}{n}\sum_{k=0}^{n} S^\alpha_{k,k}(f,x,y).$$

The 2-dimensional Dirichlet kernels and Marcinkiewicz–Fejér kernels are defined by

$$D^{\alpha}_{k,l}(x,y) := D^{\alpha}_{k}(x)D^{\alpha}_{l}(y), \quad K^{\alpha}_{n}(x,y) := \frac{1}{n}\sum_{k=0}^{n} D^{\alpha}_{k,k}(x,y).$$

For the martingale f we consider the maximal operator

$$\mathcal{M}^{\kappa*}f(x,y) = \sup_{n} |\mathcal{M}^{\kappa}_{n}(f,x,y)|.$$

A bounded measurable function a is a p-atom, if there exists a dyadic 2-dimensional cube $I \times I$, such that

a) $\int_{I\times I} a d\mu = 0$;

b) $\|a\|_{\infty} \leq \mu(I \times I)^{-1/p}$;

c) $\operatorname{supp} a \subset I \times I$.

The basic result of atomic decomposition is the following one.

Theorem A (Weisz [15])**.** *A martingale $f = (f^{(n,n)} : n \in \mathbb{N})$ is in $H_p^{\square}$ $(0 < p \leq 1)$ if and only if there exists a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that for every $n \in \mathbb{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n,2^n} a_k = f^{(n,n)}, \qquad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{2}$$

Moreover,

$$\|f\|_{H_p^{\square}} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.$$

In 1939 for the two-dimensional trigonometric Fourier series Marcinkiewicz [7] has proved for $f \in L\log L([0,2\pi]^2)$ that the means

$$\mathcal{M}_n f = \frac{1}{n}\sum_{j=1}^{n-1} S_{j,j}(f)$$

converge a.e. to f as $n \to \infty$. Zhizhiashvili [16] improved this result for $f \in L([0,2\pi]^2)$.

For the two-dimensional Walsh–Fourier series Weisz [13] proved that the maximal operator

$$\mathcal{M}^{w*}f = \sup_{n\geq 1} \frac{1}{n}\left| \sum_{j=0}^{n-1} S^{w}_{j,j}(f) \right|$$

is bounded from the two-dimensional dyadic martingale Hardy space H_p to the space L_p for $p > 2/3$ and is of weak type $(1,1)$. The first author [4] proved that the assumption $p > 2/3$ is essential for the boundedness of the maximal operator $\mathcal{M}^{w*}$ from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.

In 1974 Schipp [9] and Young [12] proved that the Walsh–Kaczmarz system is a convergence system. Gát [1] proved, for any integrable functions, that the Fejér means with respect to the Walsh–Kaczmarz system converge almost everywhere to the function itself. Gát's Theorem was extended by Simon [10] to H_p spaces, namely that the maximal operator of Fejér means of one-dimensional Fourier series is bounded from Hardy space $H_p(G^2)$ into the space $L_p(G^2)$ for $p > 1/2$.

The second author [6] proved, that for any integrable functions, the Marcinkiewicz–Fejér means with respect to the two dimensional Walsh–Kaczmarz system converge almost everywhere to the function itself. This Theorem was extended in [2]. Namely, the following is true:

Theorem B. *Let $p > 2/3$, then the maximal operator $\mathcal{M}^{\kappa*}$ of the Marcinkiewicz–Fejér means of double Walsh–Kaczmarz-Fourier series is bounded from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.*

In the paper [3] it was proved that the assumption $p > 2/3$ is essential for the boundedness of the maximal operator $\mathcal{M}^{\kappa*}$ from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$. Namely,

Theorem C. *The maximal operator $\mathcal{M}^{\kappa*}$ of the Marcinkiewicz–Fejér means of double Walsh–Kaczmarz-Fourier series is not bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.*

We will prove a stronger theorem than Theorem C.

Theorem 1. *There exists a martingale $f \in H_{2/3}(G \times G)$ such that*

$$|\mathcal{M}^{\kappa,*} f\|_{L_{2/3}} = +\infty.$$

Proof. Let $\{m_k : k \in \mathbb{N}\}$ be an increasing sequence of positive integers such that

$$\sum_{k=0}^{\infty} \frac{1}{m_k^{2/3}} < \infty, \tag{3}$$

$$\sum_{l=0}^{k-1} \frac{2^{8m_l}}{m_l} < \frac{2^{8m_k}}{m_k}, \tag{4}$$

$$\frac{2^{8m_{k-1}}}{m_{k-1}} < \frac{2^{m_k}}{k m_k}. \tag{5}$$

We note that such an increasing sequence $\{m_k : k \in \mathbb{N}\}$ which satisfies condition (3)–(5) can be constructed. Let

$$f^{(A,A)}(x,y) := \sum_{k, 2m_k < A} \lambda_k a_k(x,y), \quad \text{where } \lambda_k := \frac{1}{m_k}$$

and

$$a_k(x,y) := 2^{2m_k}(D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x))(D_{2^{2m_k+1}}(y) - D_{2^{2m_k}}(y)).$$

The martingale $f := (f^{(0,0)}, f^{(1,1)}, \dots, f^{(A,A)}, \dots) \in H^{\square}_{2/3}(G \times G)$. Indeed,

$$S_{2^A,2^A} a_k(x,y) = \begin{cases} 0, & \text{if } A \leq 2m_k, \\ a_k(x,y), & \text{if } A > 2m_k, \end{cases}$$

$$f^{(A,A)}(x) = \sum_{k, 2m_k < A} \lambda_k a_k(x,y) = \sum_{k=0}^{\infty} \lambda_k S_{2^A,2^A} a_k(x,y),$$

from (3) and Theorem A we conclude that $f \in H^{\square}_{2/3}(G \times G)$.

Now, we investigate the Fourier coefficients. Since,

$$\int_{G \times G} f^{(A)}(x,y) \kappa_i(x) \kappa_j(y) d\mu(x,y)$$

$$= \begin{cases} 0, & (i,j) \notin \bigcup\limits_{k=0}^{\infty} \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\}, \\ 0, & (i,j) \in \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\}, \\ & \quad A = 0, 1, \dots, 2m_k, \\ \dfrac{2^{2m_k}}{m_k}, & (i,j) \in \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\}, \\ & \quad A > 2m_k, \end{cases}$$

we can write

$$\widehat{f}^{\kappa}(i,j)$$

$$= \begin{cases} \dfrac{2^{2m_k}}{m_k}, & (i,j) \in \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\}, \\ 0, & (i,j) \notin \bigcup\limits_{k=1}^{\infty} \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \dots, 2^{2m_k+1} - 1\}. \end{cases} \tag{6}$$

Set $q_{A,s} := 2^{2A} + 2^{2s}$ for any $A > s$.

We decompose the $(q_{m_k,s})$th Marcinkiewicz–Fejér means as follows

$$\mathcal{M}^{\kappa}_{q_{m_k,s}} f(x,y) = \frac{1}{q_{m_k,s}} \sum_{j=1}^{q_{m_k,s}} S^{\kappa}_{j,j} f(x,y) = \frac{1}{q_{m_k,s}} \sum_{j=1}^{2^{2m_k}-1} S^{\kappa}_{j,j} f(x,y) + \frac{1}{q_{m_k,s}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} S^{\kappa}_{j,j} f(x,y) =: I + II. \tag{7}$$

Let $j \in \{0,1,\dots,2^{2m_k}-1\}$ for some k. Then from (6) and (4), it is easy to show that

$$|S^{\kappa}_{j,j} f(x,y)| \le \sum_{l=0}^{k-1} \sum_{\nu=2^{2m_l}}^{2^{2m_l+1}-1} \sum_{\mu=2^{2m_l}}^{2^{2m_l+1}-1} |\widehat{f}^{\kappa}(\nu,\mu)| \le \sum_{l=0}^{k-1} \frac{2^{8m_l}}{m_l} \le \frac{C2^{m_k}}{km_k}.$$

Consequently, we have

$$|I| \le \frac{1}{q_{m_k,s}} \sum_{j=1}^{2^{2m_k}-1} |S^{\kappa}_{j,j} f(x,y)| \le \frac{c}{q_{m_k,s}} \sum_{j=1}^{2^{2m_k}-1} \frac{2^{m_k}}{km_k} \le \frac{c2^{2m_k}}{q_{m_k,s}} \frac{2^{m_k}}{km_k} \le \frac{c2^{m_k}}{km_k}. \tag{8}$$

Now, we discuss II.
Let $i \in \{2^{2m_k},\dots,q_{m_k}-1\}$. Then from (6) we have

$$\begin{aligned} S^{\kappa}_{i,i} f\ (x,y) &= \sum_{\nu=0}^{i-1} \sum_{\mu=0}^{i-1} \widehat{f}^{\kappa}(\nu,\mu)\kappa_\nu(x)\kappa_\mu(y) \\ &= \sum_{l=0}^{k-1} \sum_{\nu=2^{2m_l}}^{2^{2m_l+1}-1} \sum_{\mu=2^{2m_l}}^{2^{2m_l+1}-1} \widehat{f}^{\kappa}(\nu,\mu)\kappa_\nu(x)\kappa_\mu(y) \\ &\quad + \sum_{\nu=2^{2m_k}}^{i-1} \sum_{\mu=2^{2m_k}}^{i-1} \widehat{f}^{\kappa}\,(\nu,\mu)\,\kappa_\nu(x)\kappa_\mu(y) \\ &= \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l} (D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x))(D_{2^{2m_l+1}}(y) - D_{2^{2m_l}}(y)) \\ &\quad + \frac{2^{2m_k}}{m_k} (D^{\kappa}_i(x) - D_{2^{2m_k}}(x))(D^{\kappa}_i(y) - D_{2^{2m_k}}(y)) \end{aligned} \tag{9}$$

and

$$II = \frac{1}{q_{m_k,s}}(q_{m_k,s} - 2^{2m_k} + 1)\sum_{l=0}^{k-1}\frac{2^{2m_l}}{m_l}$$
$$\times (D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x))(D_{2^{2m_l+1}}(y) - D_{2^{2m_l}}(y))$$
$$+ \frac{1}{q_{m_k,s}}\frac{2^{2m_k}}{m_k}\left(\sum_{i=2^{2m_k}}^{q_{m_k,s}}(D_i^\kappa(x) - D_{2^{2m_k}}(x))(D_i^\kappa(y) - D_{2^{2m_k}}(y))\right) =: II_1 + II_2.$$

By (4), (5) and $|D_{2^n}(x)| \leq 2^n$, we get that

$$|II_1| \leq C\sum_{l=0}^{k-1}\frac{2^{8m_l}}{m_l} \leq C\frac{2^{m_k}}{km_k}$$

and

$$|\mathcal{M}^\kappa_{q_{m_k,s}}f(x,y)| \geq |II_2| - \frac{C2^{m_k}}{km_k}.$$

We can write the nth Dirichlet kernel with respect to the Walsh–Kaczmarz system in the following form:

$$\begin{aligned} D_n^\kappa(x) &= D_{2^{|n|}}(x) + \sum_{k=2^{|n|}}^{n-1} r_{|k|}(x) w_{k-2^{|n|}}(\tau_{|k|}(x)) \\ &= D_{2^{|n|}}(x) + r_{|n|}(x) D^w_{n-2^{|n|}}(\tau_{|n|}(x)). \end{aligned} \tag{10}$$

By the help of this equation we immediately have for II_2 that

$$\begin{aligned} II_2 &= \frac{2^{2m_k}}{q_{m_k,s}m_k} r_{2m_k}(x) r_{2m_k}(y) \sum_{i=0}^{2^{2s}} D_i^w(\tau_{2m_k}(x)) D_i^w(\tau_{2m_k}(y)) \\ &= \frac{2^{2m_k}}{q_{m_k,s}m_k} r_{2m_k}(x) r_{2m_k}(y) 2^{2s} K^w_{2^{2s}}(\tau_{2m_k}(x)), \tau_{2m_k}(y)). \end{aligned}$$

This implies

$$|\mathcal{M}^\kappa_{q_{m_k,s}}f(x,y)| \geq \frac{2^{2s}}{m_k}|K^w_{2^{2s}}(\tau_{2m_k}(x)), \tau_{2m_k}(y))| - \frac{C2^{m_k}}{km_k}.$$

We decompose the set G as the following disjoint union:

$$G = I_A \cup \bigcup_{t=0}^{A-1} J_t^A,$$

where $A > t \geq 1$ and $J_t^A := \{x \in G : x_{A-1} = \cdots = x_{A-t} = 0, x_{A-t-1} = 1\}$, $J_0^A := \{x \in G : x_{A-1} = 1\}$. Notice that, by the definition of τ_A we have $\tau_A(J_t^A) = I_t \backslash I_{t+1}$. By Corollary 2.4 in [5], for $(x, y) \in I_A \times I_A$

$$\mathcal{K}_{2^A}^w(x, y) = \frac{(2^A + 1)(2^{A+1} + 1)}{6}. \tag{11}$$

Therefore, for $k > C$ we write

$$\begin{aligned}
\int_{G\times G} |\mathcal{M}^{\kappa *}|^{2/3} d\mu &\geq \sum_{t=1}^{2m_k-1} \int_{J_t^{2m_k} \times J_t^{2m_k}} |\mathcal{M}^{\kappa *}|^{2/3} d\mu \\
&\geq \sum_{s=[\frac{m_k}{6}]+1}^{m_k-1} \int_{J_{2s}^{2m_k} \times J_{2s}^{2m_k}} |\mathcal{M}^{\kappa *}|^{2/3} d\mu \\
&\geq \sum_{s=[\frac{m_k}{6}]+1}^{m_k-1} \int_{J_{2s}^{2m_k} \times J_{2s}^{2m_k}} |\mathcal{M}^{\kappa}_{q_{m_k,s}}|^{2/3} d\mu \\
&\geq \sum_{s=[\frac{m_k}{6}]+1}^{m_k-1} \int_{J_{2s}^{2m_k} \times J_{2s}^{2m_k}} \left(\frac{2^{2s}}{m_k} |K_{2^{2s}}^w \circ (\tau_{2m_k} \times \tau_{2m_k})| - \frac{C2^{m_k}}{km_k} \right)^{2/3} d\mu \\
&\geq \sum_{s=[\frac{m_k}{6}]+1}^{m_k-1} \int_{(I_{2s} \backslash I_{2s+1}) \times (I_{2s} \backslash I_{2s+1})} \left(\frac{2^{2s}}{m_k} |K_{2^{2s}}^w| - \frac{C2^{m_k}}{km_k} \right)^{2/3} d\mu,
\end{aligned}$$

and (11) gives

$$\begin{aligned}
\int_{G\times G} |\mathcal{M}^{\kappa *}|^{2/3} d\mu &\geq \sum_{s=[\frac{m_k}{6}]+1}^{m_k-1} \int_{(I_{2s} \backslash I_{2s+1}) \times (I_{2s} \backslash I_{2s+1})} \left| \frac{2^{6s}}{m_k} - \frac{C2^{m_k}}{km_k} \right|^{2/3} d\mu \\
&\geq c \sum_{s=[\frac{m_k}{6}]+1}^{m_k-1} \int_{(I_{2s} \backslash I_{2s+1}) \times (I_{2s} \backslash I_{2s+1})} \left(\frac{2^{6s}}{m_k} \right)^{2/3} d\mu \\
&\geq c \sum_{s=[\frac{m_k}{6}]+1}^{m_k-1} \frac{2^{4s}}{m_k^{2/3}} 2^{-4s} \geq c m_k^{1/3} \to \infty \text{ as } k \to \infty.
\end{aligned}$$

This completes the proof of the main theorem. $\square$

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USHANGI GOGINAVA
INSTITUTE OF MATHEMATICS
FACULTY OF EXACT AND NATURAL SCIENCES
TBILISI STATE UNIVERSITY
CHAVCHAVADZE STR. 1, TBILISI 0128
GEORGIA

E-mail: z_goginava@hotmail.com

KÁROLY NAGY
INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCES
COLLEGE OF NYÍREGYHÁZA
H-4400, NYÍREGYHÁZA, P.O. BOX 166
HUNGARY

E-mail: nkaroly@nyf.hu

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