

Lightlike curves in Lorentz manifolds

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To the memory of Professor András Rapcsák

The purpose of the present paper is to initiate a general study of differential geometry of lightlike curves in Lorentz manifolds. First, we construct a complementary vector subbundle to the tangent bundle of a lightlike curve. Then in section 2 we obtain the Frenet equations with respect to a general Frenet field of frames and prove theorems of reduction of the codimension of a lightlike curve (Theorems 3, 4 and 5). According to the Theorem 5, any lightlike curve of a Minkowski space whose the seventh curvature vanishes, lies in a 5-dimensional plane. This is a surprisingly result and it might have applications in multi-dimensional physical theories. Finally, we prove an existence and uniqueness theorem for lightlike curves in Lorentz manifolds.

§1. A complementary vector subbundle to the tangent bundle of a lightlike curve

Let M be a real $(m + 2)$ -dimensional Lorentz manifold, i.e., in M there exists a semi-Riemannian metric g of index $\nu = 1$, (cf. O'NEILL [6]). Suppose C is a differentiable curve in M locally given by

$$(1.1) \quad x^i = x^i(t), \quad t \in [a, b].$$

In case the tangent vector field

$$\frac{d}{dt} = \left(\frac{dx^1}{dt}, \dots, \frac{dx^{m+2}}{dt} \right),$$

has a non-null length with respect to g we have a complete study of the geometry of C (cf. SPIVAK [7]). Surprisingly, though theory of curves is one of the intensively studied theory of differential geometry, till now we do not have a method of studying curves whose tangent vector field is

lightlike, i.e., we have

$$(1.2) \quad g\left(\frac{d}{dt}, \frac{d}{dt}\right) = 0.$$

As far as we know, the results obtained on this class of curves refer to the case when the ambient space is one of the Minkowski spaces R_1^3 or R_1^4 (cf. CARTAN [3], CASTAGNINO [2], BONNOR [1], GRAVES [4], IKAWA [5]).

That is why, we consider as a need a general theory of such curves in a Lorentz manifold. The present paper is concerned with such a study and it might give more insights for a general study of lightlike submanifolds of semi-Riemannian manifolds.

We say that C is a *lightlike curve* in M if there exists a lightlike vector field ξ tangent to C , that is, we have

$$(1.3) \quad g(\xi, \xi) = 0.$$

Certainly, in this case there exists a differentiable function $k_0 \neq 0$ such that

$$(1.4) \quad \frac{d}{dt} = k_0 \xi,$$

and henceforth (1.2) and (1.3) are equivalent with each other. Denote by TC the tangent bundle of C and define as in case of nondegenerate curves

$$(1.5) \quad TC^\perp = \bigcup_{x \in C} TC_x^\perp; \quad TC_x^\perp = \{v \in T_x M, g(v, \xi_x) = 0\}.$$

Then $TC^\perp$ is a vector bundle over C whose fibres are $(m+1)$ -dimensional and ξ is a differentiable section of $TC^\perp$. Thus TC is a 1-dimensional vector subbundle of $TC^\perp$. Suppose sC is a complementary vector subbundle to TC in $TC^\perp$, i.e., we have

$$(1.6) \quad TC^\perp = TC \perp sC,$$

where $\perp$ means orthogonal direct sum. It follows that sC is a non-degenerate m -dimensional vector subbundle of TM . Then denote by $sC^\perp$ the 2-dimensional complementary orthogonal vector subbundle to sC in TM , i.e., we have

$$(1.7) \quad TM = sC \perp sC^\perp.$$

Throughout the paper we denote by $F(C)$ the algebra of differentiable functions on C and by $\Gamma(E)$ the $F(C)$ -module of differentiable sections of a vector bundle E over C . We use the same notation for any other vector bundle.

As in any theory of submanifolds appears as a necessity the construction of a complementary vector bundle of the tangent bundle of the submanifolds in the tangent bundle of the ambient space, we state first the following result.

Theorem 1. *Let C be a lightlike curve in a Lorentz manifold M . Then for a given vector subbundle sC as in (1.6) there exists a unique 1-dimensional vector subbundle nC of $sC^\perp$ such that on each neighbourhood of coordinates $\mathcal{U} \subset C$, for any $\xi \in \Gamma(TC|_{\mathcal{U}})$ there exists $N \in \Gamma(nC|_{\mathcal{U}})$ satisfying*

$$(1.8) \quad g(\xi, N) = 1,$$

and

$$(1.9) \quad g(N, N) = 0.$$

PROOF. Since TC is a vector subbundle of $sC^\perp$ we may consider a complementary vector subbundle δ of TC in $sC^\perp$. For any $\xi \in \Gamma(TC|_{\mathcal{U}})$ there exists $V \in \Gamma(\delta|_{\mathcal{U}})$ such that $g(\xi, V) \neq 0$, otherwise TM would be degenerate with respect to g . Then it follows that any $N \in \Gamma(sC|_{\mathcal{U}})$ satisfying (1.8) and (1.9) is given by

$$(1.10) \quad N = \frac{1}{g(\xi, V)} \left\{ V - \frac{g(V, V)}{2g(\xi, V)} \xi \right\}.$$

Moreover, it is easy to check that N depends neither on vector bundle δ nor on local section V . Hence for any $\xi \in \Gamma(TC|_{\mathcal{U}})$ there exists a unique vector field $N \in \Gamma(sC|_{\mathcal{U}})$ satisfying (1.8) and (1.9). Next, consider another neighborhood of coordinates $\mathcal{U}^* \subset C$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \emptyset$. Then $\xi^* = f\xi$, where f is nowhere zero differentiable function on $\mathcal{U} \cap \mathcal{U}^*$, and by using (1.10) obtain $N^* = \frac{1}{f}N$. Therefore we obtain a unique 1-dimensional vector subbundle nC of $sC^\perp$ whose local sections N satisfy (1.8) and (1.9). The proof is complete.

Next we consider the vector bundle

$$(1.11) \quad NC = nC \perp sC,$$

which according to the proof of Theorem 1 is complementary to TC in TM , i.e., we have

$$(1.12) \quad TM = (TC \oplus nC) \perp sC,$$

where $\oplus$ means direct sum but not orthogonal. It is important to note that the induced metrics on $TC \oplus nC$ and sC are of index $\nu = 1$ (Lorentz metric) and $\nu = 0$ (Riemannian metric) respectively.

§2. The Frenet equations for a lightlike curve in a Lorentz manifold

Let C be a lightlike curve of an $(m+2)$ -dimensional Lorentz manifold M and $\{\xi, N\}$ be the lightlike vector fields from Theorem 1. Suppose ∇

is the Levi-Civita connection on M . Then by using (1.3), (1.8) and (1.9) we obtain the following *general Frenet equations* for the lightlike curve C :

$$\begin{aligned}
 \nabla_\xi \xi &= A\xi + k_1 W_1 \\
 \nabla_\xi N &= -AN + k_2 W_1 + k_3 W_2 \\
 \nabla_\xi W_1 &= -k_2 \xi - k_1 N + k_4 W_2 + k_5 W_3 \\
 \nabla_\xi W_2 &= -k_3 \xi - k_4 W_1 + k_6 W_3 + k_7 W_4 \\
 \nabla_\xi W_3 &= -k_5 W_1 - k_6 W_2 + k_8 W_4 + k_9 W_5 \\
 &\vdots \\
 \nabla_\xi W_{m-2} &= -k_{2m-5} W_{m-4} - k_{2m-4} W_{m-3} + \\
 &\quad + k_{2m-2} W_{m-1} + k_{2m-1} W_m \\
 \nabla_\xi W_{m-1} &= -k_{2m-3} W_{m-3} - k_{2m-2} W_{m-2} + k_{2m} W_m \\
 \nabla_\xi W_m &= -k_{2m-1} W_{m-2} - k_{2m} W_{m-1},
 \end{aligned}
 \tag{2.1}$$

where A and $\{k_1, \dots, k_{2m}\}$ are differentiable functions and $\{W_1, \dots, W_m\}$ is an orthonormal basis of $\Gamma(sC)$. We call

$$\{\xi, N, W_1, \dots, W_m\}
 \tag{2.2}$$

the *lightlike Frenet field of frames* on M along C , and $\{k_1, \dots, k_{2m}\}$ the *curvature functions* of C . Then by using (1.3), (1.8) and (1.9) we obtain

$$\begin{cases}
 \bar{\xi} = f\xi; & \bar{N} = -\frac{f}{2} \sum_{\alpha=1}^m (c_\alpha)^2 + \frac{1}{f} N + \sum_{\alpha=1}^m c_\alpha W_\alpha \\
 \bar{W}_\alpha = \sum_{\beta=1}^m a_{\alpha\beta} (W_\beta - ac_\beta \xi),
 \end{cases}
 \tag{2.3}$$

where $f \neq 0$, c_α and $a_{\alpha\beta}$ are differentiable functions, and for any point x on C , $[a_{\alpha\beta}(x)]$ is an element of the orthogonal group $O(m)$, and $\{\bar{\xi}, \bar{N}, \bar{W}_1, \dots, \bar{W}_m\}$ is another lightlike Frenet field of frames on M along C .

Next, from the first Frenet equation of (2.1) written for both Frenet fields of frames we obtain

$$\begin{cases}
 \bar{A} = \bar{k}_1 \sum_{\alpha=1}^m a_{1\alpha} c_\alpha + \xi(f) + fA; & \bar{k}_1 a_{11} = f^2 k_1 \\
 \bar{k}_1 a_{12} = \dots = \bar{k}_1 a_{1m} = 0.
 \end{cases}
 \tag{2.4}$$

Suppose $k_1 = 0$ on C . Then $\bar{k}_1 = 0$ on C , otherwise from (2.4) obtain $a_{11} = a_{12} = \dots = a_{1m} = 0$, which is a contradiction because $[a_{\alpha\beta}]$ is an

orthogonal matrix. A lightlike curve C on which $k_1 = 0$ is called a *lightlike geodesic* of the Lorentz manifold M . In this case the first Frenet equation becomes

$$(2.5) \quad \nabla_{\bar{\xi}} \bar{\xi} = \bar{A} \bar{\xi},$$

with respect to the lightlike Frenet field of frames $\{\bar{\xi}, \bar{N}, \bar{W}_1, \dots, \bar{W}_m\}$. Now, choose f from the first relation (2.3) as a solution of partial differential equation

$$(2.6) \quad \xi(f) + fA = 0.$$

Then $\bar{A} = 0$ and (2.5) becomes

$$(2.7) \quad \nabla_{\bar{\xi}} \bar{\xi} = 0.$$

Suppose $\frac{d}{dt} = \bar{\varepsilon} \bar{\xi}$ and consider u given by $\frac{du}{dt} = \bar{\varepsilon}$ as a new parameter on C , provided $\bar{\varepsilon} > 0$ on C . Then $\frac{d}{du} = \bar{\xi}$ and (2.7) becomes

$$(2.8) \quad \frac{d^2 x^i}{du^2} + \Gamma_j^i{}^k \frac{dx^j}{du} \frac{dx^k}{du} = 0,$$

where $\Gamma_j^i{}^k$ are the Christoffel symbols induced by ∇ . Then by using (2.8) we obtain

Theorem 2. *A lightlike curve C of R_1^{m+2} is a straight line if and only if $k_1 = 0$ on C .*

For the particular case $m = 2$, Theorem 2 is due to BONNOR [1]. We call u the *pseudo-arc* on C (cf. VESSIOT [8]).

The above study enables us to suppose, from now on, $k_1 \neq 0$ at every point of the lightlike curve C . Then $k_1 \neq 0$ and (2.4) becomes

$$(2.9) \quad \begin{cases} a_{11} = 1, \quad a_{1\alpha} = a_{\alpha 1} = 0, \quad \alpha \in \{2, \dots, m\} \\ \bar{A} = f^2 k_1 c_1 + \xi(f) + fA; \quad \bar{k}_1 = f^2 k_1. \end{cases}$$

Remark 1. For the particular case $m = 2$, that is, for a lightlike curve C of a 4-dimensional Lorentz manifold the transformation of lightlike Frenet fields of frames (2.3) becomes

$$(2.10) \quad \begin{cases} \bar{\xi} = f\xi; \quad \bar{N} = -\frac{f}{2}((c_1)^2 + (c_2)^2)\xi + \frac{1}{f}N + c_1 W_1 + c_2 W_2, \\ \bar{W}_1 = W_1 - f c_1 \xi; \quad \bar{W}_2 = W_2 - f c_2 \xi. \end{cases}$$

According to the above remark we are further concerned with the study of relationships between $\{A, k_1, \dots, k_{2m}\}$ and $\{\bar{A}, \bar{k}_1, \dots, \bar{k}_{2m}\}$ with respect to the transformation of lightlike Frenet field of frames:

$$(2.11) \quad \begin{cases} \bar{\xi} = f\xi; & \bar{N} = -\frac{f}{2} \sum_{\alpha=1}^m (c_\alpha)^2 \xi + \frac{1}{f} N + \sum_{\alpha=1}^m c_\alpha W_\alpha \\ \bar{W}_\alpha = W_\alpha - f c_\alpha \xi, & \alpha \in \{1, \dots, m\}. \end{cases}$$

Thus by direct calculations using (2.11) and the last $m+1$ equations in (2.1) for both lightlike Frenet field of frames we obtain

$$(2.12) \quad \begin{cases} \bar{k}_2 = k_2 + f c_1 A + \xi(f c_1) + \frac{f^2 k_1}{2} ((c_1)^2 - (c_2)^2 - (c_3)^2) - \\ \quad - f(c_2 k_4 + c_3 k_5) \\ \bar{k}_3 = k_3 + f c_2 A + \xi(f c_2) + f^2 c_1 c_2 k_1 + f c_1 k_4 - f c_3 k_6 \\ \bar{k}_4 = f(k_4 + f c_2 k_1); \quad \bar{k}_5 = f(k_5 + f c_3 k_1), \\ \bar{k}_\alpha = f k_\alpha, \quad \alpha \in \{6, \dots, m\}, \end{cases}$$

and

$$(2.13) \quad \begin{aligned} f A c_3 + \xi(f c_3) + f^2 c_1 c_3 k_3 + f c_1 k_5 + f c_2 k_6 &= 0 \\ c_2 k_7 + c_3 k_8 &= 0; \quad c_3 k_9 = 0; \quad c_\alpha = 0, \quad \alpha \in \{4, \dots, m\}. \end{aligned}$$

Now we choose c_1 given by

$$(2.14) \quad c_1 = -\frac{1}{f^2 k_1} (\xi(f) + f A).$$

Then from (2.9) it follows $\bar{A} = 0$. Therefore, we always may consider a lightlike Frenet field of frames $\left\{ \frac{d}{du} = \xi, N, W_1, \dots, W_m \right\}$ with respect to with the Frenet equations are given by

$$(2.15) \quad \begin{aligned} \frac{D\xi}{Du} &= k_1 W_1 \\ \frac{DN}{Du} &= k_2 W_1 + k_3 W_2 \\ \frac{DW_1}{Du} &= -k_2 \xi - k_1 N + k_4 W_2 + k_5 W_3 \\ \frac{DW_2}{Du} &= -k_3 \xi - k_4 W_1 + k_6 W_3 + k_7 W_4 \end{aligned}$$

$$\begin{aligned}
\frac{DW_3}{Du} &= -k_5W_1 - k_6W_2 + k_8W_4 + k_9W_5 \\
&\vdots \\
\frac{DW_{m-2}}{Du} &= -k_{2m-5}W_{m-4} - k_{2m-4}W_{m-3} + \\
&\quad + k_{2m-2}W_{m-1} + k_{2m-1}W_m \\
\frac{DW_{m-1}}{Du} &= -k_{2m-3}W_{m-3} - k_{2m-2}W_{m-2} + k_{2m}W_m \\
\frac{DW_m}{Du} &= -k_{2m-1}W_{m-2} - k_{2m}W_{m-1},
\end{aligned}$$

where $\frac{D}{Du} = \nabla_{\frac{d}{du}}$.

Theorem 3. *Let C be a lightlike curve of a $(m+2)$ -dimensional ($m > 4$) Lorentz manifold M with $k_1 \neq 0$ on C . Then with respect to the lightlike Frenet field of frames $\{\xi, N, W_1, \dots, W_m\}$ we have $k_8 = k_9 = 0$ on C .*

PROOF. We choose $c_3 = 1$, $c_2 = 0$ and f as a non-null solution of (2.6). Then from (2.14) we obtain $c_1 = 0$ and the assertion of the theorem follows from (2.13).

Theorem 4. *Let C be a lightlike curve of a Minkowski space R_1^{m+2} ($m > 3$) with k_α , $\alpha \in \{1, \dots, 2p-4\}$, $p \in \{3, \dots, m\}$ nowhere zero and k_{2p-3} , k_{2p-2} and k_{2p-1} everywhere zero on C . Then C lies in some p -dimensional Minkowski space of R_1^{m+2} . In case k_{2m-1} and k_{2m} are everywhere zero, C lies in a Minkowski hyperplane of R_1^{m+2} .*

PROOF. Let $\{\xi = \frac{d}{du}, N, W_1, \dots, W_{p-2}\}$ be a part of a lightlike Frenet field of frames along C , and $\Delta(u) \subset T_{x(u)}R_1^{m+2}$ be the p -dimensional subspace spanned by $\{\xi(u), N(u), W_1(u), \dots, W_{p-2}(u)\}$. All these subspace are parallel as p -dimensional planes of R_1^{m+2} . In order to prove this we first note that $\frac{DX}{Du}$ is just $X'(u)$ in R_1^{m+2} and by using (2.15) obtain

$$(2.16) \quad W'_i(u) = \sum_{j=1}^p A_{ij}(u)W_j(u), \quad i \in \{1, \dots, p\},$$

where $W_{p-1} = N$ and $W_p = \xi$. Suppose now C is given by the equations

$$x^i = x^i(u), \quad u \in [a, b],$$

and V is a constant vector field on C such that

$$(2.17) \quad g(W_i(a), V) = 0, \quad i \in \{1, \dots, p\}.$$

Then by using (2.16) we obtain the system

$$(2.18) \quad \frac{d}{du}(g(W_i(u), V) = g(W'_i(u), V) = \sum_{j=1}^p A_{ij}(u)g(W_j(u), V),$$

with initial conditions (2.17). By the uniqueness of solutions of (2.18) we infer $g(W_i(u), V) = 0$ for all u . Hence all p -planes $\Delta(u)$ are parallel with $\Delta(a)$. The proof is complete by the following general result.

Proposition 1. (Spivak [7], p.39). *Let $C : x^i = x^i(u)$, $u \in [a, b]$ be an immersed curve of R^{m+2} such that $\frac{dx^i}{du} \in \Delta(u)$ for all u , where $\Delta(u)$ are parallel p -dimensional planes of R^{m+2} . Then C is a curve in some p -dimensional plane of R^{m+2} .*

Remark 2. The p -dimensional plane H wherein C lies is a Minkowski space since both linear independent lightlike vector fields ξ and N belong to H . The second assertion of the theorem follows in a similar way as the first one.

From Theorems 3 and 4 we obtain the following surprising result.

Theorem 5. *Let C be a lightlike curve of a Minkowski space R_1^{m+2} ($m > 4$) with $\{k_1, \dots, k_6\}$ nowhere zero and k_7 everywhere zero on C . Then C lies in a 5-dimensional plane of R_1^{m+2} .*

§3. The fundamental existence and uniqueness theorem for lightlike curves

Let M be a $(m+2)$ -dimensional Lorentz manifold. In the previous sections we have seen that the lightlike Frenet field of frames $\{\xi, N, W_1, \dots, W_m\}$ constructed along a lightlike curve is *quasi-orthonormal* (cf. VRĂNCEANU-ROȘCA [9]), that is, $\{W_1, \dots, W_m\}$ is an orthonormal basis and ξ and N are lightlike vector fields satisfying (1.8).

Consider R_1^{m+2} with the Lorentz metric

$$(3.1) \quad g(x, y) = \sum_{\alpha=1}^{m+1} x^\alpha y^\alpha - x^{m+2} y^{m+2}.$$

Then we define the quasi-orthonormal basis

$$(3.2) \quad \begin{cases} \mathring{W}_1 = (1, 0, \dots, 0), \dots, \mathring{W}_m = (0, \dots, 1, 0, 0), \\ \mathring{W}_{m+1} = (0, \dots, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \mathring{W}_{m+2} = (0, \dots, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \end{cases}$$

that is we have

$$(3.3) \quad \begin{cases} g(\mathring{W}_\alpha, \mathring{W}_\beta) = \delta_{\alpha,\beta}; \quad g(\mathring{W}_{m+1}, \mathring{W}_{m+1}) = g(\mathring{W}_{m+2}, \mathring{W}_{m+2}) = 0 \\ g(\mathring{W}_{m+1}, \mathring{W}_{m+2}) = 1, \quad \alpha, \beta \in \{1, \dots, m\}. \end{cases}$$

It is easy to see that

$$(3.4) \quad \sum_{\alpha=1}^m \mathring{W}_\alpha^i \mathring{W}_\alpha^j + \mathring{W}_{m+1}^i \mathring{W}_{m+2}^j + \mathring{W}_{m+1}^j \mathring{W}_{m+2}^i = g^{ij},$$

where we put

$$g^{ij} = \begin{cases} 1, & i = j \neq m+2 \\ -1, & i = j = m+2 \\ 0, & i \neq j. \end{cases}$$

Theorem 6. *Let M be a Lorentz manifold, let $k_1, \dots, k_{2m}: [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}$ be everywhere continuous functions and let $\{\mathring{W}_1, \dots, \mathring{W}_{m+2}\}$ from (3.2) as a basis of $T_{x_0}M$. Then there exists a unique pseudo-arc parametrized lightlike curve $C : x^i = x^i(u)$, $u \in [-\varepsilon, \varepsilon]$, such that $x^i(0) = x_0^i$, whose curvature functions are $k_1, \dots, k_{2m}$ and whose lightlike Frenet field of frames $\{\xi, N, W_1, \dots, W_m\}$ satisfies*

$$\xi(0) = \mathring{W}_{m+1}, \quad N(0) = \mathring{W}_{m+2}, \quad W_\alpha(0) = \mathring{W}_\alpha, \quad \alpha \in \{1, \dots, m\}.$$

PROOF. First we note that without loss of generality we may suppose M is the Minkowski space R_1^{m+2} . Then consider the system of differential equations

$$(3.5) \quad \begin{aligned} W'_{m+1}(u) &= k_1 W_1 \\ W'_{m+2}(u) &= k_2 W_1 + k_3 W_2 \\ W'_1(u) &= -k_2 W_{m+1} - k_1 W_{m+2} + k_4 W_2 + k_5 W_3 \\ &\vdots \\ W'_{m-2}(u) &= -k_{2m-5} W_{m-4} - k_{2m-4} W_{m-3} + k_{2m-2} W_{m-1} + k_{2m-1} W_m \end{aligned}$$

$$\begin{aligned} W'_{m-1}(u) &= -k_{2m-3}W_{m-3} - k_{2m-2}W_{m-2} + k_{2m}W_m \\ W'_m(u) &= -k_{2m-1}W_{m-2} - k_{2m}W_{m-1}, \end{aligned}$$

and based on a well known result on the existence and uniqueness of its solutions, there exists a unique solution $(W_1, \dots, W_{m+2})$ satisfying initial conditions $W_\alpha(0) = \overset{\circ}{W}_\alpha$, $\alpha \in \{1, \dots, m+2\}$. Now we claim that $\{W_1(u), \dots, W_{m+2}(u)\}$ is a quasi-orthonormal basis for any $u \in [-\varepsilon, \varepsilon]$. To this end, by direct calculations, using (3.4) we obtain

$$(3.6) \quad \frac{d}{du} \left(\sum_{\alpha=1}^m W_\alpha^i W_\alpha^j + W_{m+1}^i W_{m+2}^j + W_{m+1}^j W_{m+2}^i \right) = 0.$$

As for $u = 0$ we have (3.4), from (3.6) it follows

$$(3.7) \quad \sum_{\alpha=1}^m W_\alpha^i(u) W_\alpha^j(u) + W_{m+1}^i(u) W_{m+2}^j(u) + W_{m+1}^j(u) W_{m+2}^i(u) = g^{ij}$$

Further we construct the field of frames

$$(3.8) \quad \begin{cases} V_{m+1} = \frac{1}{\sqrt{2}} (W_{m+1} + W_{m+2}); & V_{m+2} = \frac{1}{\sqrt{2}} (W_{m+1} - W_{m+2}) \\ V_\alpha = W_\alpha, & \alpha \in \{1, \dots, m\}. \end{cases}$$

Then (3.7) becomes

$$(3.9) \quad \sum_{\alpha=1}^{m+1} V_\alpha^i(u) V_\alpha^j(u) - V_{m+2}^i(u) V_{m+2}^j(u) = g^{ij}.$$

Following BONNOR [1], we define the matrix $[b^{ij}]$ as follows

$$(3.10) \quad \begin{cases} b^{\alpha\beta} = V_\beta^\alpha, & \alpha, \beta \in \{1, \dots, m+1\}; & b^{\alpha m+2} = -\sqrt{-1} V_{m+2}^\alpha \\ b^{(m+2)\alpha} = \sqrt{-1} V_\alpha^{m+2}; & b^{(m+2)(m+2)} = V_{m+2}^{m+2}. \end{cases}$$

It is easy to check that $[b^{ij}]$ is an orthogonal matrix. This implies $\{V_1, \dots, V_{m+2}\}$ is an orthonormal basis with respect to the metric (3.1) of R_1^{m+2} . Hence $\{W_1, \dots, W_{m+2}\}$ is a quasi-orthonormal basis for any $u \in [-\varepsilon, \varepsilon]$. The lightlike curve C is obtained by integrating the system

$$(3.11) \quad \frac{dx^i}{du} = W_{m+1}^i(u).$$

It follows that C is pseudo-arc parametrized with curvature functions $\{k_1, \dots, k_{2m}\}$ with respect to the quasi-orthonormal field of frames $\{\xi = W_{m+1}, N = W_{m+2}, W_1, \dots, W_m\}$. The proof is complete.

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