

On the weighted ℓ^p –space of a discrete group

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Abstract. Let G be a locally compact group and $1 < p < \infty$. The L^p -conjecture asserts that $L^p(G)$ is closed under the convolution if and only if G is compact. For $2 < p < \infty$, we have recently shown that $f * g$ exists and belongs to $L^\infty(G)$ for all $f, g \in L^p(G)$ if and only if G is compact. Here, we consider the weighted case of this result for a discrete group G and a weight function ω on G ; we prove that $f * g$ exists and belongs to $\ell^\infty(G, 1/\tilde{\omega})$ for all $f, g \in \ell^p(G, \omega)$ if and only if $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\tilde{\omega})$, the dual of $\ell^p(G, \tilde{\omega})$.

1. Introduction

Let G be a locally compact group with a fixed left Haar measure λ . For $1 \leq p < \infty$ and a measurable function $\alpha : G \rightarrow (0, \infty)$, we denote by $L^p(G, \alpha)$ the space of all measurable complex-valued functions f on G such that $f\alpha \in L^p(G)$, the usual Lebesgue space on G with respect to λ ; see [7] for more details. Then $L^p(G, \alpha)$ with the norm $\|\cdot\|_{p, \alpha}$ defined by

$$\|f\|_{p, \alpha} := \|f\alpha\|_p$$

for all $f \in L^p(G, \alpha)$ is a Banach space. Let us remark that when G is discrete, $L^p(G, \alpha)$ is the space $\ell^p(G, \alpha)$ of all complex-valued functions f on G such that

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$f \alpha \in \ell^p(G)$, the space of all complex-valued functions g on G with

$$\|g\|_p := \left(\sum_{x \in G} |g(x)|^p \right)^{1/p} < \infty.$$

We also denote by $L^\infty(G, 1/\alpha)$ the space of all measurable complex valued functions f on G such that $f/\alpha \in L^\infty(G)$, the space as defined in [7]. Then $L^\infty(G, 1/\alpha)$ with the norm $\|\cdot\|_{\infty, 1/\alpha}$ defined by

$$\|f\|_{\infty, 1/\alpha} := \|f/\alpha\|_\infty$$

for all $f \in L^\infty(G, 1/\alpha)$. Furthermore, for $1 \leq p < \infty$, the continuous dual of $L^p(G, \alpha)$ coincides with $L^q(G, 1/\alpha)$, where q is the conjugate number $p/(p-1)$ of p ; that is, $1 < q < \infty$ and $p+q = pq$. In fact, the mapping τ from $L^q(G, 1/\alpha)$ to $L^p(G, \alpha)^*$ defined by

$$\langle \tau(f), \varphi \rangle = \int_G f(x) \varphi(x) d\lambda(x) \quad (1)$$

is an isometric isomorphism. For measurable functions f and g on G , the convolution multiplication

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) d\lambda(y).$$

is defined at each point $x \in G$ for which this makes sense; i.e. the function $y \mapsto f(y) g(y^{-1}x)$ is λ -integrable. Then $f * g$ is said to exist if $(f * g)(x)$ exists for almost all $x \in G$.

For $1 < p < \infty$, the L^p -conjecture states that $L^p(G)$ is closed under the convolution if and only if G is compact. The first result related to this conjecture is due to ZELAZKO [21] and URBANIK [20] in 1961; they proved that the conjecture is true for all locally compact abelian groups. However, this conjecture was first formulated for a locally compact group G by RAJAGOPALAN in his Ph.D. thesis in 1963.

The truth of the conjecture has been established for $p > 2$ by ZELAZKO [22] and RAJAGOPALAN [14] independently; see also RAJAGOPALAN's works [13] for the case where $p \geq 2$ and G is discrete, [14] for the case where $p = 2$ and G is totally disconnected, and [15] for the case where $p > 1$ and G is either nilpotent or a semidirect product of two locally compact groups. In the joint work [16], they showed that the conjecture is true for $p > 1$ and amenable groups; this

result can be also found in GREENLEAF's book [6]. RICKERT [18] confirmed the conjecture for $p = 2$. For related results on the subject see also CROMBEZ [2] and [3], GAUDET and GAMLEN [5], JOHNSON [8], KUNZE and STEIN [9], LOHOUÉ [11], MILNES [12], RICKERT [17], and ZELAZKO [23]. Finally, in 1990, SAEKI [19] gave an affirmative answer to the conjecture by a completely self-contained proof; see also KUZNETSOVA [10] and EL KINANI and BENAZZOUZ [4] for some results on the weighted L^p -spaces of locally compact groups.

Motivated by the L^p -conjecture, in [1], we have considered only the property that $f * g$ exists for all $f, g \in L^p(G)$ and proved the following result which was indeed our purpose of that work.

Theorem 1.1. *Let G be a locally compact group and $2 < p < \infty$. If $f * g$ exists for all $f, g \in L^p(G)$, then G is compact.*

On the other hand, it is well-known from [13] that for $2 < p < \infty$, $L^p(G) \subseteq L^q(G)$ if and only if G is compact. As a consequence of these observations, we have the following result.

Theorem 1.2. *Let G be a locally compact group and $2 < p < \infty$. Then the following assertions are equivalent.*

- (a) $L^p(G) \subseteq L^q(G)$.
- (b) $f * g$ exists and belongs to $L^\infty(G)$ for all $f, g \in L^p(G)$.

In the case where G is discrete and $1 \leq p \leq 2$ we have $\ell^p(G) \subseteq \ell^2(G)$, and hence $f * g$ exists and belongs to $\ell^\infty(G)$ for all $f, g \in \ell^p(G)$ by the Holder inequality; moreover, $p \leq q$, and so $\ell^p(G) \subseteq \ell^q(G)$. These facts together with 1.2 led us to the following result.

Corollary 1.1. *Let G be a discrete group and $1 \leq p < \infty$. Then the following assertions are equivalent.*

- (a) $\ell^p(G) \subseteq \ell^q(G)$.
- (b) $f * g$ exists and belongs to $\ell^\infty(G)$ for all $f, g \in \ell^p(G)$.

In fact, this result gives a necessary and sufficient condition for that

$$\ell^p(G) * \ell^p(G) \subseteq \ell^\infty(G).$$

It states that the assertions (a) and (b) are always true if $1 < p \leq 2$, and that they are equivalent to finiteness of G if $2 < p < \infty$.

In this paper, we investigate this equivalence in the weighted case and prove an analogue of that result in terms of a weight function ω on G . We also observe that ω plays a significant role in this respect; in fact, we show that there is an infinite group for which the weighted forms of these equivalent assertions hold.

2. The results

Throughout this section, let G be a discrete group and ω be a weight function on G ; that is, a measurable function with $\omega(x) > 0$ and $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. Also, let $\tilde{\omega}$ be the weight function defined by $\tilde{\omega}(x) = \omega(x^{-1})$ for all $x \in G$. We commence with the following lemma.

Lemma 2.1. *Let G be a discrete group, ω be a weight function on G and $1 \leq p < \infty$. If $f * g \in \ell^\infty(G, 1/\tilde{\omega})$ for all $f, g \in \ell^p(G, \omega)$, then the map $(f, g) \mapsto f * g$ from $\ell^p(G, \omega) \times \ell^p(G, \omega)$ into $\ell^\infty(G, 1/\tilde{\omega})$ is separately continuous.*

PROOF. Fix a positive function $g \in \ell^p(G, \omega)$, and suppose on the contrary that the map $f \mapsto f * g$ from $\ell^p(G, \omega)$ into $\ell^\infty(G, 1/\tilde{\omega})$ is not continuous. Then there exists a sequence $(f_n) \subseteq \ell^p(G, \omega)$ of positive functions with $\|f_n\|_{p, \omega} \leq 1$ such that

$$\|f_n * g\|_{\infty, 1/\tilde{\omega}} \geq n^3$$

for all $n \in \mathbb{N}$. So, if we put

$$f := \sum_{n=1}^{\infty} \frac{f_n}{n^2},$$

then $f \in \ell^p(G, \omega)$; indeed,

$$\|f\|_{p, \omega} \leq \sum_{n=1}^{\infty} \frac{\|f_n\|_{p, \omega}}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

For each $n \in \mathbb{N}$ we have $f \geq n^{-2}f_n$ and thus $f * g \geq n^{-2}f_n * g$. Therefore

$$\|f * g\|_{\infty, 1/\tilde{\omega}} \geq n^{-2} \|f_n * g\|_{\infty, 1/\tilde{\omega}} \geq n;$$

in particular, $f * g \notin \ell^\infty(G, 1/\tilde{\omega})$, a contradiction. It follows that the map $f \mapsto f * g$ from $\ell^p(G, \omega)$ into $\ell^\infty(G, 1/\tilde{\omega})$ is continuous. Similarly, the map $f \mapsto g * f$ from $\ell^p(G, \omega)$ into $\ell^\infty(G, 1/\tilde{\omega})$ is continuous. $\square$

Now, we are ready to state and prove the main result of the paper which gives a necessary and sufficient condition for $\ell^p(G, \omega) * \ell^p(G, \omega) \subseteq \ell^\infty(G, 1/\tilde{\omega})$.

Theorem 2.1. *Let G be a discrete group, ω a weight function on G and $1 \leq p < \infty$. Then the following assertions are equivalent.*

- (a) $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\tilde{\omega})$.
- (b) $f * g$ exists and belongs to $\ell^\infty(G, 1/\tilde{\omega})$ for all $f, g \in \ell^p(G, \omega)$.

PROOF. Suppose that (a) holds and let $f, g \in \ell^p(G, \omega)$ and $x \in G$. Define the function $R_x g$ on G by $R_x g(y) = g(yx)$ for all $y \in G$ and note that

$$\|R_x g\|_{p, \omega} = \left(\sum_{y \in G} |g(yx)|^p \omega(y)^p \right)^{1/p} = \left(\sum_{y \in G} |g(y)|^p \omega(yx^{-1})^p \right)^{1/p} \leq \tilde{\omega}(x) \|g\|_{p, \omega}.$$

We therefore have $R_x g \in \ell^p(G, \omega)$ and thus $\widetilde{R_x g} / \omega \in \ell^q(G)$ by (a). Since $\omega f \in \ell^p(G)$, the Holder inequality implies that $f \widetilde{R_x g} \in \ell^1(G)$ and thus

$$\|f \widetilde{R_x g}\|_1 \leq \|f\|_{p, \omega} \|\widetilde{R_x g}\|_{q, 1/\omega} = \|f\|_{p, \omega} \|R_x g\|_{q, 1/\tilde{\omega}}.$$

Since

$$\frac{|(f * R_x g)(e)|}{\tilde{\omega}(e)} \leq |(f * R_x g)(e)| = \left| \sum_{y \in G} f(y) \widetilde{R_x g}(y) \right|,$$

it follows that

$$\frac{|(f * R_x g)(e)|}{\tilde{\omega}(e)} \leq \|f\|_{p, \omega} \|R_x g\|_{q, 1/\tilde{\omega}}.$$

This together with

$$\|R_x g\|_{q, 1/\tilde{\omega}} = \left(\sum_{y \in G} \frac{|g(yx)|^q}{\tilde{\omega}(y)^q} \right)^{1/q} = \left(\sum_{y \in G} \frac{|g(y)|^q}{\tilde{\omega}(yx^{-1})^q} \right)^{1/q} \leq \tilde{\omega}(x) \|g\|_{q, 1/\tilde{\omega}}$$

yield that

$$\begin{aligned} |(f * g)(x)| &= \left| \sum_{y \in G} f(y) g(y^{-1}x) \right| = \left| \sum_{y \in G} f(y) \widetilde{R_x g}(y) \right| = |(f * R_x g)(e)| \\ &\leq \omega(e) \|f\|_{p, \omega} \|R_x g\|_{q, 1/\tilde{\omega}} \leq \omega(e) \tilde{\omega}(x) \|f\|_{p, \omega} \|g\|_{q, 1/\tilde{\omega}}. \end{aligned}$$

Hence

$$\|f * g\|_{\infty, 1/\tilde{\omega}} \leq \omega(e) \|g\|_{p, \omega} \|f\|_{q, 1/\tilde{\omega}}.$$

Therefore $f * g$ exists and belongs to $\ell^\infty(G, 1/\tilde{\omega})$.

Conversely, suppose that (b) holds, and let $f \in \ell^p(G, \omega)$. We show that $f \in \ell^q(G, 1/\tilde{\omega})$. Lemma 2.1 implies that the map

$$\ell^p(G, \omega) \times \ell^p(G, \omega) \rightarrow \ell^\infty(G, 1/\tilde{\omega})$$

with the formula $(f, g) \mapsto f * g$ is separately continuous. By the uniform boundedness theorem, there exists a constant $M > 0$ such that

$$\|f * g\|_{\infty, 1/\tilde{\omega}} \leq M \|f\|_{p, \omega} \|g\|_{p, \omega}$$

for all $f, g \in \ell^p(G, \omega)$. For every $g \in \ell^p(G, \tilde{\omega})$ we have

$$\begin{aligned} \sum_{y \in G} f(y) g(y) &= \sum_{y \in G} f(y) \tilde{g}(y^{-1}) = \sum_{y \in G} f(y) \tilde{g}(y^{-1}e) = (f * \tilde{g})(e) \\ &\leq \omega(e) \|f * \tilde{g}\|_{\infty, 1/\tilde{\omega}} \leq M \omega(e) \|f\|_{p, \omega} \|\tilde{g}\|_{p, \omega} \\ &= M \omega(e) \|f\|_{p, \omega} \|g\|_{p, \tilde{\omega}}. \end{aligned}$$

Thus the functional $\tau(f) : g \mapsto \sum_{y \in G} f(y) g(y)$ is bounded on $\ell^p(G, \tilde{\omega})$. Since $\ell^p(G, \tilde{\omega})^* = \ell^q(G, 1/\tilde{\omega})$, it follows that $f \in \ell^q(G, 1/\tilde{\omega})$. $\square$

Corollary 2.1. *Let G be a discrete group and ω be a weight function such that $\omega \geq 1$, $\omega = \tilde{\omega}$ and $1 \in \ell^q(G, 1/\omega)$, and $1 \leq p < \infty$. Then $f * g$ exists and belongs to $\ell^\infty(G, 1/\tilde{\omega})$ for all $f, g \in \ell^p(G, \omega)$.*

PROOF. If $f \in \ell^p(G, \omega)$, then $f\omega \in \ell^p(G)$. Since $1/\omega \in \ell^q(G)$, $f = (f\omega)(1/\omega) \in \ell^1(G)$. Since G is discrete and $\omega \geq 1$,

$$\ell^1(G) \subseteq \ell^q(G) \subseteq \ell^q(G, 1/\omega).$$

Thus $f \in \ell^q(G, 1/\omega)$ and therefore $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\omega)$. Now, we only need to invoke Theorem 2.1. $\square$

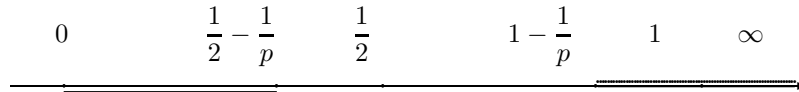
Corollary 2.2. *Let G be a discrete group, ω be a weight function on G and $1 \leq p \leq 2$. Then $f * g$ exists and belongs to $\ell^\infty(G, 1/\tilde{\omega})$ for all $f, g \in \ell^p(G, \omega)$.*

PROOF. Since $\omega\tilde{\omega} \geq 1$, it follows that $\ell^q(G, \omega) \subseteq \ell^q(G, 1/\tilde{\omega})$; also $p \leq q$, and so $\ell^p(G, \omega) \subseteq \ell^q(G, \omega)$. Consequently, $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\tilde{\omega})$. Now the result follows from Lemma 2.1 and Theorem 2.1. $\square$

Our next result introduces a large class of weight functions ω on an infinite group G for which $f * g$ exists and belongs to $\ell^\infty(G, 1/\tilde{\omega})$ for all $f, g \in \ell^p(G, \omega)$. In particular, this shows that Corollary 2.2 is not true for $2 < p < \infty$.

Proposition 2.1. *Let $2 < p < \infty$, $\mathbb{Z}$ be the additive group of integer numbers and $\omega_\alpha(n) = (1 + |n|)^\alpha$ for all $n \in \mathbb{Z}$, where $0 < \alpha < \infty$. Then*

- (1) $\ell^p(\mathbb{Z}, \omega_\alpha) \not\subseteq \ell^q(\mathbb{Z}, 1/\tilde{\omega}_\alpha)$ for all $0 < \alpha < \frac{1}{2} - \frac{1}{p}$.
- (2) $\ell^p(\mathbb{Z}, \omega_\alpha) \subseteq \ell^q(\mathbb{Z}, 1/\tilde{\omega}_\alpha)$ for all $1 - \frac{1}{p} < \alpha < \infty$.



PROOF. (1). Suppose that $0 < \alpha < \frac{1}{2} - \frac{1}{p}$. Then $\alpha + \frac{1}{p} < \frac{1}{q} - \alpha$ and consequently there exists a real number β such that

$$\alpha + \frac{1}{p} < \beta < \frac{1}{q} - \alpha$$

We therefore have $p\beta - p\alpha > 1$ and $q\beta + q\alpha < 1$. Now, let f be the function on $\mathbb{Z}$ defined by

$$f(n) = (1 + |n|)^{-\beta} \quad (n \in \mathbb{Z}).$$

It follows that

$$\begin{aligned} \|f\|_{p,\omega}^p &= \sum_{n=-\infty}^{\infty} f(n)^p \omega_{\alpha}(n)^p = 1 + 2 \sum_{n=1}^{\infty} (1 + |n|)^{-p\beta} (1 + |n|)^{p\alpha} \\ &= 1 + 2 \sum_{n=1}^{\infty} \frac{1}{(1 + |n|)^{p\beta - p\alpha}} \end{aligned}$$

whence $\|f\|_{p,\omega_{\alpha}}^p < \infty$, and so $f \in \ell^p(\mathbb{Z}, \omega_{\alpha})$. But

$$\begin{aligned} \|f\|_{q,1/\omega}^q &= 1 + 2 \sum_{n=1}^{\infty} \frac{f(n)^q}{\omega_{\alpha}(n)^q} = 1 + 2 \sum_{n=1}^{\infty} (1 + |n|)^{-q\beta} (1 + |n|)^{-q\alpha} \\ &= 1 + 2 \sum_{n=1}^{\infty} (1 + |n|)^{-q\alpha - q\beta} \end{aligned}$$

whence $\|f\|_{q,1/\omega_{\alpha}}^q = \infty$ and therefore $\ell^p(\mathbb{Z}, \omega_{\alpha}) \not\subseteq \ell^q(\mathbb{Z}, 1/\tilde{\omega}_{\alpha})$.

(2). Suppose that $1 - \frac{1}{p} < \alpha < \infty$. Then $\omega_{\alpha} \geq 1$, $\omega_{\alpha} = \tilde{\omega}_{\alpha}$ and $1 \in \ell^q(G, 1/\omega_{\alpha})$. It follows from the proof of Corollary 2.1 that $\ell^p(G, \omega_{\alpha}) \subseteq \ell^q(G, 1/\omega_{\alpha})$. $\square$

A combination of Theorems 1.1 and 2.1 imply that G is finite if $\ell^p(G) \subseteq \ell^q(G)$. As an immediate consequence of Proposition 2.1, we have the following result which shows that this result is not valid in the weighted case.

Corollary 2.3. *For each $1 \leq p < \infty$, there exist an infinite countable discrete group G and a weight function ω on G with $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\tilde{\omega})$.*

Let us point out that if there is a bounded or multiplicative weight function ω on G with $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\tilde{\omega})$, then G is finite. Our next result gives an analogue of this fact without the hypothesis that ω is bounded or multiplicative.

Proposition 2.2. *Let G be a discrete group and $2 < p < \infty$. If there is a weight function ω on G with $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\tilde{\omega})$, then G is countable.*

PROOF. It suffices to show that the set $\{x \in G : \omega(x) \omega(x^{-1}) \leq k\}$ is finite for all $k \geq 2$. To that end, suppose on the contrary that there exist $k_0 \geq 1$ and a sequence (x_n) of distinct elements of G with $\omega(x_n) \omega(x_n^{-1}) \leq k_0$ for all $n \geq 1$. Define the function $f : G \rightarrow \mathbb{R}$ by

$$f(x_n) = \frac{1}{\omega(x_n)\sqrt{n}} \quad \text{and} \quad f(x_n^{-1}) = \frac{1}{\omega(x_n^{-1})\sqrt{n}}$$

for all $n \geq 1$ and $f(x) = 0$ otherwise. Clearly $f \in \ell^p(G, \omega)$ and we have

$$(f * f)(e) = \sum_{n=1}^{\infty} f(x_n) f(x_n^{-1}) = \sum_{n=1}^{\infty} \frac{1}{\omega(x_n) \omega(x_n^{-1}) n} \geq \frac{1}{k_0} \sum_{n=1}^{\infty} \frac{1}{n}.$$

It follows that $f * f$ does not exist. This contradicts Theorem 2.1 together with the assumption that $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\tilde{\omega})$. $\square$

Let us present some examples illustrating our observations.

Example 1. (a) Let $\mathbb{R}$ be the additive group of real numbers and $2 < p < \infty$. Then there is no weight functions ω on $\mathbb{R}$ with $\ell^p(\mathbb{R}, \omega) \subseteq \ell^q(\mathbb{R}, 1/\tilde{\omega})$ by Proposition 2.2. In particular, $\ell^p(\mathbb{R}, \omega_\alpha) \not\subseteq \ell^q(\mathbb{R}, 1/\tilde{\omega}_\alpha)$, where $\omega_\alpha(x) = (1 + |x|)^\alpha$ for all $x \in \mathbb{R}$, and $\alpha > 0$. So, there exist $f, g \in \ell^p(\mathbb{R}, \omega_\alpha)$ such that $f * g$ does not belong to $\ell^\infty(\mathbb{R}, 1/\tilde{\omega}_\alpha)$.

(b) Let $\mathbb{Z}$ be the additive group of integer numbers, $\alpha > 0$ and $\omega_\alpha(n) = (1 + |n|)^\alpha$ for all $n \in \mathbb{Z}$. Then $\ell^3(\mathbb{Z}, \omega_\alpha) \subseteq \ell^{3/2}(\mathbb{Z}, 1/\tilde{\omega}_\alpha)$ for all $\alpha > 2/3$ and $\ell^3(\mathbb{Z}, \omega_\alpha) \not\subseteq \ell^{3/2}(\mathbb{Z}, 1/\tilde{\omega}_\alpha)$ for all $0 < \alpha < 1/6$.

(c) Let $\mathbb{Z}$ be the additive group of integer numbers, $2 < p < \infty$ and $\omega(n) = \exp(|n|)$ for all $n \in \mathbb{Z}$. Then ω is a weight function on $\mathbb{Z}$ such that $\omega \geq 1$, $\omega = \tilde{\omega}$ and $1/\omega \in \ell^q(G)$. Then $\ell^p(\mathbb{Z}, \omega) \subseteq \ell^q(\mathbb{Z}, 1/\tilde{\omega})$.

(d) Let $\mathbb{Z}$ be the additive group of integer numbers, $2 < p < \infty$ and $\omega(n) = \exp(n)$ for all $n \in \mathbb{Z}$. Then ω is a multiplicative weight function on $\mathbb{Z}$, and thus $\ell^p(\mathbb{Z}, \omega) \not\subseteq \ell^q(\mathbb{Z}, 1/\tilde{\omega})$.

We end this work by some questions which arises from our results.

Question 1. Does Theorem 2.1 remains valid for all locally compact groups G ? In fact, for a measurable weight function ω on G , is the following assertions equivalent?

- (a) $L^p(G, \omega) \subseteq L^q(G, 1/\tilde{\omega})$.
- (b) $f * g$ exists and belongs to $L^\infty(G, 1/\tilde{\omega})$ for all $f, g \in L^p(G, \omega)$.

Question 2. In Proposition 2.1, what happens when $1/2 - 1/p \leq \alpha \leq 1 - 1/p$? In fact, does the inclusion $\ell^p(\mathbb{Z}, \omega_\alpha) \subseteq \ell^q(\mathbb{Z}, 1/\tilde{\omega}_\alpha)$ hold for all such α ?

Question 3. Is the converse of Proposition 2.2 true? In fact, if G is an infinite countable discrete group and $2 < p < \infty$, is there a weight function ω on G with $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\tilde{\omega})$?

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