Publ. Math. Debrecen
75/3-4 (2009), 387–399

On almost conservative matrix methods for double sequence spaces

By M. ZELTSER (Tallinn), M. MURSALEEN (Aligarh) and S. A. MOHIUDDINE (Aligarh)

Abstract. In this paper, we use the idea of almost convergence of double sequences to define and characterize the almost $\mathcal{C}_\nu$-conservative matrices, that is, those 4-dimensional matrices which transform ν-convergent double sequences into the almost convergent double sequences; where ν stands for p-, bp-, and r-convergence.

1. Introduction and preliminaries

Here we give notions and notation for double sequence spaces. For other notations we refer the reader to [1].

A double sequence $x = (x_{jk})$ of real or complex numbers is said to be *bounded* if

$$\|x\|_\infty = \sup_{j,k} |x_{jk}| < \infty.$$

The space of all bounded double sequences is denoted by $\mathcal{M}_u$.

Mathematics Subject Classification: 40B05, 40C05.
Key words and phrases: double sequences, matrix transformation, almost convergence, almost conservative 4-dimensional matrices.
Research of the first author was partially supported by the Estonian Sci. Foundation, grant 7033, and by the Estonian Min. of Educ. and Research, project SF0132723s06. Research of the second author is supported by the Department of Science and Technology, New Delhi, under grant No. SR/S4/MS:505/07. Research of the third author is supported by the Department of Atomic Energy, Government of India under the NBHM-Post Doctoral Fellowship programme number 28/2008/810.
The present paper was started when the first author visited Aligarh Muslim University during March 24-28, 2008.

A double sequence $x = (x_{jk})$ is said to *converge to the limit* L *in Pringsheim's sense* (shortly, *p-convergent to* L) [9] if for every $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case L is called the p-limit of x. If in addition $x \in \mathcal{M}_u$, then x is said to be *boundedly convergent to* L *in Pringsheim's sense* (shortly, *bp-convergent to* L).

A double sequence $x = (x_{jk})$ is said to *converge regularly to* L (shortly, *r-convergent to* L) if $x \in \mathcal{C}_p$ and the limits $x_j := \lim_k x_{jk}$ $(j \in \mathbb{N})$ and $x^k := \lim_j x_{jk}$ $(k \in \mathbb{N})$ exist. Note that in this case the limits $\lim_j \lim_k x_{jk}$ and $\lim_k \lim_j x_{jk}$ exist and are equal to the p-limit of x.

In general, for any notion of convergence ν, the space of all ν-convergent double sequences will be denoted by $\mathcal{C}_\nu$, the space of all ν-convergent to 0 double sequences by $\mathcal{C}_{\nu 0}$ and the limit of a ν-convergent double sequence x by ν-$\lim_{j,k} x_{jk}$, where $\nu \in \{p, bp, r\}$.

Let Ω denote the vector space of all double sequences with the vector space operations defined coordinatewise. Vector subspaces of Ω are called *double sequence spaces*. In addition to above-mentioned double sequence spaces we consider the double sequence space

$$\mathcal{L}_u := \Big\{ x \in \Omega \mid \|x\|_1 := \sum_{j,k} |x_{jk}| < \infty \Big\}$$

of all absolutely summable double sequences.

All considered double sequence spaces are supposed to contain

$$\Phi := \operatorname{span}\{\mathbf{e^{jk}} \mid j, k \in \mathbb{N}\},$$

where

$$\mathbf{e^{jk}_{il}} = \begin{cases} 1, & \text{if } (j,k) = (i,\ell), \\ 0, & \text{otherwise.} \end{cases}$$

We denote the pointwise sums $\sum_{j,k} \mathbf{e^{jk}}$, $\sum_j \mathbf{e^{jk}}$ $(k \in \mathbb{N})$, and $\sum_k \mathbf{e^{jk}}$ $(j \in \mathbb{N})$ by $\mathbf{e}$, $\mathbf{e^k}$ and $\mathbf{e_j}$ respectively.

Let E be the space of double sequences converging with respect to a convergence notion ν, F be a double sequence space, and $A = (a_{mnjk})$ be a 4-dimensional matrix of scalars. Define the set

$$F_A^{(\nu)} := \Big\{ x \in \Omega \mid [Ax]_{mn} := \nu - \sum_{j,k} a_{mnjk} x_{jk} \text{ exists and } Ax := ([Ax]_{mn})_{m,n} \in F \Big\}.$$

Then we say that A maps the space E into the space F if $E \subset F_A^{(\nu)}$ and denote by (E, F) the set of all 4-dimensional matrices A which map E into F.

For more details on double sequences and 4-dimensional matrices, we refer to [5], [11], [12].

The idea of almost convergence for single sequences was introduced by LORENTZ [4] and for double sequences by MÓRICZ and RHOADES [6].

A double sequence $x = (x_{jk})$ of real numbers is said to be *almost convergent* to a limit L if

$$p\text{-}\lim_{p,q\to\infty} \sup_{m,n>0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} - L \right| = 0.$$

In this case L is called the f_2-limit of x and we shall denote by f_2 the space of all almost convergent double sequences.

Note that a convergent double sequence need not be almost convergent. However every bounded convergent double sequence is almost convergent and every almost convergent double sequence is bounded.

4-dimensional matrices mapping every almost convergent double sequence into a bp-convergent double sequence with the same limit were considered by MÓRICZ and RHOADES in [6]. 4-dimensional matrices mapping every almost convergent double sequence into an almost convergent double sequence with the same limit were characterized by MURSALEEN [8]).

In this paper, we characterize almost $\mathcal{C}_\nu$-conservative matrices, i.e. those 4-dimensional matrices $A = (a_{mnjk})$ which map the double sequence space $\mathcal{C}_\nu$ into the space f_2 where $\nu \in \{bp, r, p\}$.

To derive them we apply characterizations of 4-dimensional matrices from the class $(\mathcal{C}_\nu, \mathcal{C}_{bp})$ for $\nu \in \{bp, r, p\}$.

2. The class of matrices $(\mathcal{C}_\nu, \mathcal{C}_{bp})$

The conditions for a 4-dimensional matrix to map the spaces $\mathcal{C}_{bp}$, $\mathcal{C}_r$, $\mathcal{C}_p$ into the space $\mathcal{C}_{bp}$ are well known (see for example [2]).

Theorem 2.1. (a) *The matrix $A = (a_{mnjk})$ is in $(\mathcal{C}_r, \mathcal{C}_{bp})$ if and only if the following conditions hold:*

(i) $\sup_{m,n} \sum_{j,k} |a_{mnjk}| < \infty$,

(ii) *the limit* $bp\text{-}\lim_{m,n} a_{mnjk} = a_{jk}$ *exists* $(j, k \in \mathbb{N})$,

(iii) *the limit* $bp\text{-}\lim_{m,n} \sum_{j,k} a_{mnjk} = v$ *exists,*

(iv) *the limit* $bp\text{-}\lim_{m,n} \sum_j a_{mnjk_0} = u^{k_0}$ *exists* $(k_0 \in \mathbb{N})$,

(v) *the limit* $bp\text{-}\lim_{m,n}\sum_k a_{mnj_0k} = v_{j_0}$ *exists* $(j_0 \in \mathbb{N})$.

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, $(u^k), (v_j) \in \ell$ *and*

$$bp\text{-}\lim_{m,n}[Ax]_{m,n} = \sum_{j,k} a_{jk}x_{jk} + \sum_j \Big(v_j - \sum_k a_{jk}\Big)x_j + \sum_k \Big(u^k - \sum_j a_{jk}\Big)x^k$$
$$+ \Big(v + \sum_{j,k} a_{jk} - \sum_j v_j - \sum_k u^k\Big) r\text{-}\lim_{m,n} x_{mn} \quad (x \in \mathcal{C}_r).$$

(b) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_r, \mathcal{C}_{bp})$ *and* $bp\text{-}\lim Ax = r\text{-}\lim_{m,n} x_{mn}$ $(x \in \mathcal{C}_r)$ *if and only if the conditions* (i)–(v) *hold with* $a_{jk} = u^k = v_j = 0$ $(j, k \in \mathbb{N})$ *and* $v = 1$.

Theorem 2.2. (a) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_{bp}, \mathcal{C}_{bp})$ *if and only if it satisfies the conditions* (i), (ii), *and* (iii) *of Theorem 2.1 and*

(vi) $bp\text{-}\lim_{m,n}\sum_j |a_{mnjk_0} - a_{jk_0}| = 0$ $(k_0 \in \mathbb{N})$,

(vii) $bp\text{-}\lim_{m,n}\sum_k |a_{mnj_0k} - a_{j_0k}| = 0$ $(j_0 \in \mathbb{N})$.

In this case, $a = (a_{jk}) \in \mathcal{L}_u$ *and*

$$bp\text{-}\lim_{m,n}[Ax]_{m,n} = \sum_{j,k} a_{jk}x_{jk} + \Big(v - \sum_{j,k} a_{jk}\Big) bp\text{-}\lim_{m,n} x_{mn} \quad (x \in \mathcal{C}_{bp}).$$

(b) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_{bp}, \mathcal{C}_{bp})$ *and* $bp\text{-}\lim Ax = bp\text{-}\lim_{m,n} x_{mn}$ $(x \in \mathcal{C}_{bp})$ *if and only if the conditions* (i), (ii), (iii) *of Theorem 2.1 and* (vi) *and* (vii) *hold with* $a_{jk} = 0$ $(j, k \in \mathbb{N})$ *and* $v = 1$.

Theorem 2.3. (a) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_p, \mathcal{C}_{bp})$ *if and only if the conditions* (i)–(iii) *of Theorem 2.1 hold and*

(viii) *for every* $j \in \mathbb{N}$, *there exists* $K \in \mathbb{N}$ *such that* $a_{mnjk} = 0$ *for* $k > K$ $(m, n \in \mathbb{N})$,

(ix) *for every* $k \in \mathbb{N}$, *there exists* $J \in \mathbb{N}$ *such that* $a_{mnjk} = 0$ *for* $j > J$ $(m, n \in \mathbb{N})$,

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, $(a_{jk_0})_j, (a_{j_0k})_k \in \varphi$ $(j_0, k_0 \in \mathbb{N})$ *and*

$$bp\text{-}\lim_{m,n}[Ax]_{m,n} = \sum_{j,k} a_{jk}x_{jk} + \sum_j \Big(v - \sum_{j,k} a_{jk}\Big) p\text{-}\lim_{m,n} x_{mn} \quad (x \in \mathcal{C}_p).$$

(b) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_p, \mathcal{C}_{bp})$ *and* $bp\text{-}\lim Ax = p\text{-}\lim_{m,n} x_{mn}$ $(x \in \mathcal{C}_p)$ *if and only if the conditions* (ii), (iii) *of Theorem 2.1 and* (viii)–(xiii) *hold with* $a_{jk} = 0$ $(j, k \in \mathbb{N})$ *and* $v = 1$.

3. Almost $\mathcal{C}_\nu$-conservative matrices

In this section, we define and characterize almost $\mathcal{C}_\nu$-conservative and almost $\mathcal{C}_\nu$-regular matrices.

Definition 3.1. A four dimensional matrix $A = (a_{mnjk})$ is said to be *almost $\mathcal{C}_\nu$-conservative* if it transforms every ν-convergent double sequence $x = (x_{jk})$ into the almost convergent double sequence, where $\nu \in \{p, bp, r\}$; that is, $A \in (\mathcal{C}_\nu, f_2)$.

Definition 3.2. A four dimensional matrix $A = (a_{mnjk})$ is said to be *almost $\mathcal{C}_\nu$-regular* if it is almost $\mathcal{C}_\nu$-conservative and $f_2\text{-}\lim Ax = \nu\text{-}\lim x$ for every $x \in \mathcal{C}_\nu$.

Comparing this definition with the definition of almost regular 4-dimensional matrix by Mursaleen and Savaş ([7]) we see that the authors in fact considered almost $\mathcal{C}_{bp}$-regular matrices.

Almost conservative and almost regular matrices for single sequences were characterized by King [3].

Theorem 3.1. (a) *A matrix $A = (a_{mnjk})$ is almost $\mathcal{C}_{bp}$-conservative if and only if the following conditions hold:*

(i) $\sup_{m,n} \sum_{j,k} |a_{mnjk}| =: M < \infty$,

(ii) *the limit $bp\text{-}\lim_{p,q} \alpha(j,k,p,q,s,t) = a_{jk}$ exists $(j,k \in \mathbb{N})$ uniformly in $s,t \in \mathbb{N}$,*

(iii) *the limit $bp\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t) = u$ exists uniformly in $s,t \in \mathbb{N}$,*

(iv) *the limit $bp\text{-}\lim_{p,q} \sum_{k} |\alpha(j,k,p,q,s,t) - a_{jk}| = 0$ exists $(j \in \mathbb{N})$ uniformly in $s,t \in \mathbb{N}$,*

(v) *the limit $bp\text{-}\lim_{p,q} \sum_{j} |\alpha(j,k,p,q,s,t) - a_{jk}| = 0$ exists $(k \in \mathbb{N})$ uniformly in $s,t \in \mathbb{N}$,*

where

$$\alpha(j,k,p,q,s,t) = \frac{1}{pq} \sum_{m=s}^{s+p-1} \sum_{n=t}^{t+q-1} a_{mnjk}.$$

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, and

$$f_2\text{-}\lim Ax = \sum_{j,k} a_{jk} x_{jk} + \left(u - \sum_{j,k} a_{jk}\right) bp\text{-}\lim_{i,l} x_{il}, \tag{1}$$

that is,

$$bp\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t) x_{jk} = \sum_{j,k} a_{jk} x_{jk} + \left(u - \sum_{j,k} a_{jk}\right) bp\text{-}\lim_{i,l} x_{il}$$

uniformly in $s, t \in \mathbb{N}$.

(b) $A = (a_{mnjk})$ *is almost* $\mathcal{C}_{bp}$*-regular if and only if the conditions* (i)–(v) *hold with* $a_{jk} = 0$ $(j, k \in \mathbb{N})$ *and* $u = 1$.

Proof. (a) *Necessity.* Let $A \in (\mathcal{C}_{bp}, f_2)$. The condition (i) follows, since $(\mathcal{C}_{bp}, f_2) \subset (\mathcal{C}_{bp}, \mathcal{M}_u)$ (see [2], §5, 5). Since $\mathbf{e^{jk}}$ and $\mathbf{e}$ are in $\mathcal{C}_{bp}$, the conditions (ii) and (iii) follow respectively.

It is obvious that if $A \in (\mathcal{C}_{bp}, f_2)$, then the matrix $B^{st} := (b^{st}_{pqjk})_{p,q,j,k} := (\alpha(j,k,p,q,s,t))_{p,q,j,k}$ is in $(\mathcal{C}_{bp}, \mathcal{C}_{bp})$ for every $s, t \in \mathbb{N}$. In particular, the double sequence $b^{st} = (b^{st}_{jk})$ with $b^{st}_{jk} := bp\text{-}\lim_{p,q} b^{st}_{pqjk} = a_{jk}$ is in $\mathcal{L}_u$ and

$$bp\text{-}\lim_{p,q} \sum_k |b^{st}_{pqjk} - b^{st}_{jk}| = bp\text{-}\lim_{p,q} \sum_k |\alpha(j,k,p,q,s,t) - a_{jk}| = 0$$

for every $s, t \in \mathbb{N}$.

To verify the conditions (iv) and (v), we need to prove that these limits are uniform in $s, t \in \mathbb{N}$. Suppose on contrary that for given $j_0 \in \mathbb{N}$

$$bp\text{-}\lim_{p,q} \sup_{s,t} \sum_k |\alpha(j_0,k,p,q,s,t) - a_{j_0k}| \neq 0.$$

Then there exists $\varepsilon > 0$ and index sequences $(p_i), (q_i)$ such that

$$\sup_{s,t} \sum_k |\alpha(j_0,k,p_i,q_i,s,t) - a_{j_0k}| \geq \varepsilon \quad (i \in \mathbb{N}).$$

So for every $i \in \mathbb{N}$, we can choose $s_i, t_i \in \mathbb{N}$ such that

$$\sum_k |\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}| \geq \varepsilon \quad (i \in \mathbb{N}).$$

Since

$$\sum_k |\alpha(j_0,k,p_i,q_i,s_i,t_i)| \leq \sup_{m,n} \sum_{j,k} |a_{mnjk}| < \infty,$$

$(a_{jk}) \in \mathcal{L}_u$ and by (ii) going to a subsequence of $(p_i,\ q_i,\ s_i,\ t_i)$ on need we may find an index sequence (k_i) such that

$$\sum_{k=1}^{k_i} |\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}| \leq \frac{\varepsilon}{8} \quad \text{and}$$

$$\sum_{k=k_{i+1}+1}^{\infty} |\alpha(j_0,k,p_i,q_i,s_i,t_i)| + \sum_{k=k_{i+1}+1}^{\infty} |a_{j_0k}| \leq \frac{\varepsilon}{8} \quad (i \in \mathbb{N}).$$

So
$$\sum_{k=k_i+1}^{k_{i+1}} |\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}| \ge \frac{3\varepsilon}{4} \quad (i \in \mathbb{N}).$$

We define the double sequence $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} (-1)^i \mathrm{sgn}(\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}) & \text{for } k_i < k \le k_{i+1}\ (i \in \mathbb{N}),\ j = j_0 \\ 0 & \text{for } j \ne j_0. \end{cases}$$

Then $x \in \mathcal{C}_{bp0}$ with $\|x\|_\infty \le 1$, but for i even we have

$$\begin{aligned} &\frac{1}{p_iq_i}\sum_{m=s_i}^{s_i+p_i-1}\sum_{n=t_i}^{t_i+q_i-1}(Ax)_{mn} - \sum_{j,k} a_{jk}x_{jk} = \sum_k \alpha(j_0,k,p_i,q_i,s_i,t_i)x_{j_0k} \\ &\quad - \sum_k a_{j_0k}x_{j_0k} \ge \sum_{k=k_i+1}^{k_{i+1}} (\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k})x_{j_0k} \\ &\quad - \sum_{k=1}^{k_i} |\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}| - \sum_{k=k_{i+1}+1}^{\infty} |\alpha(j_0,k,p_i,q_i,s_i,t_i)| \\ &\quad - \sum_{k=k_{i+1}+1}^{\infty} |a_{j_0k}| \ge \sum_{k=k_i+1}^{k_{i+1}} |\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}| - \frac{\varepsilon}{8} - \frac{\varepsilon}{8} \ge \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

Analogously for i odd we get

$$\frac{1}{p_iq_i}\sum_{m=s_i}^{s_i+p_i-1}\sum_{n=t_i}^{t_i+q_i-1}(Ax)_{mn} - \sum_{j,k} a_{jk}x_{jk} \le -\frac{\varepsilon}{2}.$$

Hence $\frac{1}{pq}\sum_{m=s}^{s+p-1}\sum_{n=t}^{t+q-1}(Ax)_{mn}$ does not converge as $p,q \to \infty$ uniformly in $s,t \in \mathbb{N}$, that is, $Ax \notin f_2$, giving the contradiction. Hence (iv) holds. In the same way we get that (v) holds.

Sufficiency. Let the conditions (i)–(v) hold. Then for any s, t the matrix $B^{st} := (\alpha(j,k,p,q,s,t))_{p,q,j,k}$ is in $(\mathcal{C}_{bp},\mathcal{C}_{bp})$. In particular

$$bp\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)x_{jk} = \sum_{j,k}a_{jk}x_{jk} + \Big(u - \sum_{j,k}a_{jk}\Big) bp\text{-}\lim_{i,l} x_{il} \quad (s,t \in \mathbb{N}).$$

To prove that the limit is uniform in $s,t \in \mathbb{N}$, we consider

$$\sum_{j,k}(\alpha(j,k,p,q,s,t) - a_{jk})(x_{jk} - bp\text{-}\lim_{i,l} x_{il}).$$

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$|x_{jk} - bp\text{-}\lim_{i,l} x_{il}| \le \frac{\varepsilon}{8M} \quad \text{for } j,k \ge N.$$

By (ii), (iv) and (v) we can choose $P \in \mathbb{N}$ such that for $p,q \ge P$ and every $s,t \in \mathbb{N}$ we have

$$\sum_{j=1}^{N-1} \sum_{k=1}^{N-1} |\alpha(j,k,p,q,s,t) - a_{jk}| \le \frac{\varepsilon}{8\|x\|_\infty}$$

$$\sum_{j=1}^{N-1} \sum_{k} |\alpha(j,k,p,q,s,t) - a_{jk}| \le \frac{\varepsilon}{8\|x\|_\infty}$$

$$\sum_{k=1}^{N-1} \sum_{j=N}^{\infty} |\alpha(j,k,p,q,s,t) - a_{jk}| \le \frac{\varepsilon}{8\|x\|_\infty}.$$

Then

$$\begin{aligned}
&\Big|\sum_{j,k} (\alpha(j,k,p,q,s,t) - a_{jk})(x_{jk} - bp\text{-}\lim_{i,l} x_{il})\Big| \\
&\qquad \le 2 \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} |\alpha(j,k,p,q,s,t) - a_{jk}| \, \|x\|_\infty \\
&\qquad\quad + 2 \sum_{j=1}^{N-1} \sum_{k} |\alpha(j,k,p,q,s,t) - a_{jk}| \, \|x\|_\infty \\
&\qquad\quad + 2 \sum_{k=1}^{N-1} \sum_{j=N}^{\infty} |\alpha(j,k,p,q,s,t) - a_{jk}| \, \|x\|_\infty \\
&\qquad\quad + \sum_{j=N}^{\infty} \sum_{k=N}^{\infty} (|\alpha(j,k,p,q,s,t)| + |a_{jk}|) \big|x_{jk} - bp\text{-}\lim_{i,l} x_{il}\big| \\
&\qquad \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2M \frac{\varepsilon}{8M} = \varepsilon \qquad (s,t \in \mathbb{N}).
\end{aligned}$$

Hence

$$p\text{-}\lim_{p,q} \sum_{j,k} (\alpha(j,k,p,q,s,t) - a_{jk})(x_{jk} - bp\text{-}\lim_{i,l} x_{il}) = 0$$

uniformly in s,t, that is

$$f_2\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t) x_{jk} = \sum_{j,k} a_{jk} x_{jk} + \Big(u - \sum_{j,k} a_{jk}\Big) bp\text{-}\lim_{i,l} x_{il}.$$

(b) The sufficiency follows from (1) and the necessity follows from the inclusion $\{\mathbf{e^{jk}},\ \mathbf{e} \mid j,k \in \mathbb{N}\} \subset \mathcal{C}_{bp}$. □

Theorem 3.2. (a) *A matrix $A = (a_{mnjk})$ is almost $\mathcal{C}_r$-conservative if and only if the conditions* (i)–(iii) *of Theorem 3.1 hold and*

(iv) *the limit bp-$\lim_{p,q} \sum_j \alpha(j, k_0, p, q, s, t) = u^{k_0}$ exists uniformly in s, t $(k_0 \in \mathbb{N})$,*

(v) *the limit bp-$\lim_{p,q} \sum_k \alpha(j_0, k, p, q, s, t) = v_{j_0}$ exists uniformly in s, t $(j_0 \in \mathbb{N})$.*

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, $(u^k), (v_j) \in \ell$ and

$$f_2\text{-}\lim Ax = \sum_{j,k} a_{jk}x_{jk} + \sum_j \Big(v_j - \sum_k a_{jk}\Big)x_j + \sum_k \Big(u^k - \sum_j a_{jk}\Big)x^k + \Big(u + \sum_{j,k} a_{jk} - \sum_j v_j - \sum_k u^k\Big)r\text{-}\lim x. \tag{2}$$

(b) *$A = (a_{mnjk})$ is almost $\mathcal{C}_\nu$-regular if and only if the conditions* (i)–(iii) *of Theorem 3.1 and* (iv), (v) *hold with $u_{jk} = u^k = v_j = 0$ $(j, k \in \mathbb{N})$ and $u = 1$.*

Proof. (a) *Necessity.* The condition (i) holds, since $(\mathcal{C}_r, f_2) \subset (\mathcal{C}_r, \mathcal{M}_u)$. The conditions (ii), (iii), (iv) and (v) follow, since $\mathbf{e^{jk}}$, $\mathbf{e}$, $\mathbf{e^k}$, $\mathbf{e_j} \in \mathcal{C}_r$ $(j, k \in \mathbb{N})$.

Sufficiency. Let the conditions (i)–(v) hold and suppose first that $x = (x_{jk}) \in \mathcal{C}_r$ satisfies $x_j = x^k = 0$ $(j, k \in \mathbb{N})$. Then also r-$\lim x = 0$.

By Theorem 2.1 the matrix $B^{st} := (\alpha(j, k, p, q, s, t))_{p,q,j,k}$ is in $(\mathcal{C}_r, \mathcal{C}_{bp})$ for any $s, t \in \mathbb{N}$. In particular

$$bp\text{-}\lim_{p,q} \sum_{j,k} \alpha(j, k, p, q, s, t)x_{jk} = \sum_{j,k} a_{jk}x_{jk} \quad (s, t \in \mathbb{N}).$$

To prove that the limit is uniform in $s, t \in \mathbb{N}$, we consider

$$\sum_{j,k} (\alpha(j, k, p, q, s, t) - a_{jk})x_{jk}.$$

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$|x_{jk}| \le \frac{\varepsilon}{4M} \quad \text{for } j \ge N \text{ or } k \ge N \quad (j, k \in \mathbb{N}).$$

By (ii) we can choose $P \in \mathbb{N}$ such that for $p, q \ge P$ and any $s, t \in \mathbb{N}$ we have

$$\sum_{j=1}^{N-1} \sum_{k=1}^{N-1} |\alpha(j, k, p, q, s, t) - a_{jk}| \le \frac{\varepsilon}{2\|x\|_\infty}.$$

Then

$$\Big|\sum_{j,k}(\alpha(j,k,p,q,s,t)-a_{jk})x_{jk}\Big| \leq \sum_{j=1}^{N-1}\sum_{k=1}^{N-1}|\alpha(j,k,p,q,s,t)-a_{jk}|\,\|x\|_\infty$$

$$+\sum_{(j,k)\in\mathbb{N}^2\setminus[1,N-1]^2}(|\alpha(j,k,p,q,s,t)|+|a_{jk}|)|x_{jk}| \leq \frac{\varepsilon}{2}+2M\frac{\varepsilon}{4M}=\varepsilon \quad (s,t\in\mathbb{N}).$$

Hence

$$p\text{-}\lim_{p,q}\sum_{j,k}(\alpha(j,k,p,q,s,t)-a_{jk})x_{jk}=0$$

uniformly in s, t, that is

$$f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)x_{jk}=\sum_{j,k}a_{jk}x_{jk}.$$

Now let $x=(x_{jk})$ be any element of $\mathcal{C}_r$ with $\xi := r\text{-}\lim x$, then for the double sequence $z:=(z_{jk})$ with $z_{jk}:=x_{jk}-x_j-x^k+\xi$ we have $\lim_k z_{jk}=0$ $(j\in\mathbb{N})$ and $\lim_j z_{jk}=0$ $(k\in\mathbb{N})$. Hence

$$f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)(x_{jk}-x_j-x^k+\xi)=f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)z_{jk}$$

$$=\sum_{j,k}a_{jk}z_{jk}=\sum_{j,k}a_{jk}(x_{jk}-x_j-x^k+\xi).$$

The existence of the limit

$$f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)x_{jk}=\sum_{j,k}a_{jk}z_{jk}+f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)(x_j-\xi)$$

$$+f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)(x^k-\xi)+f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)\xi$$

then would follow if the limits on the right side exist.

The third limit

$$f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)\xi=\xi v$$

exists by (iii).

We will show that the first limit equals to $\sum_j v_j(x_j-\xi)$. For that end let $\varepsilon>0$ and $N\in\mathbb{N}$ such that

$$|x_j-\xi|\leq\frac{\varepsilon}{4M}\quad \text{for } j\geq N.$$

By (v) we can choose $P \in \mathbb{N}$ such that for $p, q \geq P$ and any $s, t \in \mathbb{N}$ we have

$$\sum_{j=1}^{N-1} \left| \sum_k \alpha(j,k,p,q,s,t) - v_j \right| \leq \frac{\varepsilon}{4\|x\|_\infty}.$$

Then

$$\left| \sum_j \Big(\sum_k \alpha(j,k,p,q,s,t) - v_j \Big)(x_j - \xi) \right| \leq 2 \sum_{j=1}^{N-1} \left| \sum_k \alpha(j,k,p,q,s,t) - v_j \right| \|x\|_\infty$$
$$+ \sum_{j=N}^{\infty} \left(\sum_k |\alpha(j,k,p,q,s,t)| + |v_j| \right) |x_{jk}| \leq \frac{\varepsilon}{2} + 2M \frac{\varepsilon}{4M} = \varepsilon \quad (s, t \in \mathbb{N}).$$

Hence

$$f_2\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t)(x_j - \xi) = \sum_j v_j (x_j - \xi).$$

Analogously

$$f_2\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t)(x^k - \xi) = \sum_k u^k (x^k - \xi).$$

Hence the limit

$$f_2\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t) x_{jk}$$

exists and the formula (2) holds.

(b) The sufficiency follows from (2) and the necessity follows from the inclusion $\{\mathbf{e^{jk}}, \mathbf{e}, \mathbf{e^k}, \mathbf{e_j} \mid j, k \in \mathbb{N}\} \subset \mathcal{C}_r$. $\square$

Theorem 3.3. (a) *A matrix $A = (a_{mnjk})$ is almost $\mathcal{C}_p$-conservative if and only if the conditions* (i)–(iii) *of Theorem 3.1 and* (viii), (ix) *of Theorm 2.3 hold.*

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, $(a_{jk_0})_j, (a_{j_0k})_k \in \varphi$ $(j_0, k_0 \in \mathbb{N})$ and

$$f_2\text{-}\lim Ax = \sum_{j,k} a_{jk} x_{jk} + \Big(u - \sum_{j,k} a_{jk} \Big) p\text{-}\lim_{i,l} x_{il}, \tag{3}$$

(b) *$A = (a_{mnjk})$ is almost $\mathcal{C}_p$-regular if and only if the conditions* (i)–(iii) *of Theorem 3.1 and* (viii), (ix) *of Theorem 2.3 hold with $a_{jk} = 0$ $(j, k \in \mathbb{N})$ and $u = 1$.*

Proof. (a) *Necessity* of conditions (i)–(iii) follows in the same way as in Theorem 3.1. The conditions (viii), (ix) of Theorem 2.3 follow since $(\mathcal{C}_p, f_2) \subset (\mathcal{C}_p, \mathcal{M}_u)$ (see [2], §5, 6).

Sufficiency. First note that the condition (viii) of Theorem 2.3 implies that $\alpha(j_0, k, p, q, s, t) = 0$ for given $j_0 \in \mathbb{N}$, $k > K$ and any $p, q, s, t \in \mathbb{N}$. Hence also $a_{j_0 k} = 0$ for $k > K$. Now in view of (ii) the condition (iv) of Theorem 3.1 follows. Analogously the condition (v) of Theorem 3.1 as well as $(a_{jk_0})_j \in \varphi$ $(k_0 \in \mathbb{N})$ follows from the condition (ix) of Theorem 2.3. So in view of Theorem 3.1 A is almost $\mathcal{C}_{bp}$-conservative.

Now let $x \in \mathcal{C}_p$. Then there exists $N \in \mathbb{N}$ such that

$$\sup_{k,l>N} |x_{kl}| < \infty.$$

We consider x as a decomposition $x = y + z$ where y is an element of $\mathcal{C}_{bp}$ defined by $y_{kl} := x_{kl}$ for $k, l > N$ and $y_{kl} := 0$ for $k \le N$ or $l \le N$ and $z := x - y$. So $Ay \in f_2$ and

$$f_2\text{-}\lim Ay = \sum_{j,k>N} a_{jk} x_{jk} + \Big(u - \sum_{j,k} a_{jk}\Big) p\text{-}\lim_{i,l} x_{il}.$$

To prove that $Ax \in f_2$ we need to verify that $Az \in f_2$. For that end let $K \in \mathbb{N}$ be such that $a_{mnjk} = 0$ for $k > K$, $j = 1, \dots, N$ and any $m, n \in \mathbb{N}$. Let also $J \in \mathbb{N}$ be such that $a_{mnjk} = 0$ for $j > J$, $k = 1, \dots, N$ and any $m, n \in \mathbb{N}$. Then

$$Az = \sum_{j=1}^{N} \sum_{k=1}^{K} z_{jk} A\mathbf{e^{jk}} + \sum_{k=1}^{N} \sum_{j=N+1}^{J} z_{jk} A\mathbf{e^{jk}} \in f_2$$

and

$$\begin{aligned} f_2\text{-}\lim Az &= \sum_{j=1}^{N} \sum_{k=1}^{K} a_{jk} z_{jk} + \sum_{k=1}^{N} \sum_{j=N+1}^{J} a_{jk} z_{jk} \\ &= \sum_{j=1}^{N} \sum_{k} a_{jk} z_{jk} + \sum_{k=1}^{N} \sum_{j=N+1}^{\infty} a_{jk} z_{jk}. \end{aligned}$$

Hence $Ax = Ay + Az \in f_2$ and the formula (3) holds.

(b) can be proved in the same way as in Theorem 3.1. □

References

[1] J. Boos, Classical and Modern Methods in Summability, *Oxford University Press*, *New York, Oxford*, 2000.

[2] H. J. Hamilton, Transformations of multiple sequences, *Duke Math. J.* **2** (1996), 29–60.

[3] J. P. King, Almost summable sequences, *Proc. Amer. Math. Soc.* **17** (1966), 1219–1225.

[4] G. G. Lorentz, A contribution to theory of divergent sequences, *Acta Math.* **80** (1948), 167–190.

[5] F. Móricz, Extensions of the spaces c and c_o from single to double sequences, *Acta Math. Hungar.* **57** (1991), 129–136.

[6] F. Móricz and B. E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.* **104** (1988), 283–294.

[7] M. Mursaleen and E. Savaş, Almost regular matrices for double sequences, *Studia Sci. Math. Hung.* **40** (2003), 205–212.

[8] M. Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.* **293** (2004), 523–531.

[9] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, *Math. Z.* **53** (1900), 289–321.

[10] G. M. Robinson, Divergent double sequences and series, *Trans. Amer. Math. Soc.* **28** (1926), 50–73.

[11] M. Zeltser, Investigation of double sequence spaces by soft and hard analytical methods, Dissertationes Mathematicae Universitatis Tartuensis 25, *Tartu University Press*, 2001.

[12] M. Zeltser, On conservative matrix methods for double sequence spaces, *Acta Math. Hungar.* **95** (2002), 225–242.

M. ZELTSER
DEPARTMENT OF MATHEMATICS
TALLINN UNIVERSITY
NARVA MNT. 25
10120, TALLINN
ESTONIA

E-mail: mariaz@tlu.ee

M. MURSALEEN
DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002
INDIA

E-mail: mursaleenm@gmail.com

S. A. MOHIUDDINE
DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002
INDIA

E-mail: mohiuddine@gmail.com

(Received September 15, 2008)