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# Jordan loop rings

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**Abstract.** A commutative loop or ring is said to be Jordan if it satisfies the identity $(x^2y)x = x^2(yx)$. We show that the loop ring of a Jordan loop $L$ is Jordan and not associative only if the characteristic of the coefficient ring is even and call such a loop *ring Jordan* (RJ, for short). While Jordan loops are in general not power associative, RJ loops are. We give various constructions of finite RJ loops and conjecture that these exist only when they have order divisible by four. We also conjecture that RJ loops are precisely those commutative loops in which squares are in the left nucleus.

## 1. Some history

The title of this paper is inspired by another, "Alternative Loop Rings," which appeared in this journal over twenty-five years ago [Goo83]. This was the first paper exhibiting a class of loop rings satisfying an "interesting" identity, other than associativity, and the present work is of a similar nature. In the interim, the subject of nonassociative[1] loop rings (and their underlying loops) has developed substantially. Whereas there was once reason to believe that nonassociative loop rings satisfying nonassociative identities could not exist, it turns out that virtually any identity of Bol–Moufang type is satisfied by some nonassociative loop ring [DG09]. There are RA loops, whose loop rings in all characteristics satisfy the Moufang identities (but not associativity), RA2 loops, whose loop rings have the

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[1]In this paper, "nonassociative" means "not associative."

same properties in characteristic 2, SRAR loops, whose loop rings satisfy the right Bol identity and, indeed, loops whose loop rings satisfy the C or extra identities as identified by Fenyves [Fen69]. Since loop rings cannot satisfy the identity $x^2 = 0$, Lie loop rings cannot exist. The remaining important nonassociative identity is the Jordan identity – $(x^2y)x = x^2(yx)$ – and, two years ago, it was shown that nonassociative Jordan loop rings do exist [GK08]. In this paper, we show that such rings must have even characteristic and that the underlying "RJ loops" are necessarily power associative, and we present several constructions which suggest that finite RJ loops might exist only with orders that are multiples of four. We also conjecture that RJ loops might simply be commutative loops in which squares are in the left nucleus.

## 2. Preliminaries

A *quasigroup* is a set $Q$ together with a binary operation $(a, b) \mapsto a \cdot b$ on $Q$ with the property that each of the equations $a \cdot x = b$ and $x \cdot a = b$ has a solution for any $a, b \in Q$. Thus there is a one-to-one correspondence between finite quasigroups of order $n$ and Latin squares of order $n$. A *loop* is a quasigroup with identity. The left, middle and right nuclei of a loop $L$ are the subsets

$$N_\lambda = \{a \in L \mid (ax)y = a(xy) \text{ for all } x, y \in L\} \quad \text{left nucleus}$$

$$N_\mu = \{a \in L \mid (xa)y = x(ay) \text{ for all } x, y \in L\} \quad \text{middle nucleus}$$

$$N_\rho = \{a \in L \mid (xy)a = x(ya) \text{ for all } x, y \in L\} \quad \text{right nucleus.}$$

In general, these are different associative subloops of $L$, but the situation is more restrictive when $L$ is commutative.

**Lemma 2.1.** *If $L$ is a commutative loop, then $N_\lambda = N_\rho \subseteq N_\mu$. Thus $N_\lambda$ is a normal subloop and, if $L$ is finite, $|N_\lambda| \mid |L|$.*

Proof. Let $a \in N_\lambda$ and let $x$, $y$ be any elements of $L$. Then $xy \cdot a = a \cdot xy = a \cdot yx = ay \cdot x = x \cdot ay = x \cdot ya$, so $a \in N_\rho$. Thus $N_\lambda \subseteq N_\rho$ and the other inclusion follows in a similar fashion. Furthermore, with $a$, $x$, $y$ as above, $xa \cdot y = ax \cdot y = a \cdot xy = a \cdot yx = ay \cdot x = x \cdot ay$, so $a \in N_\mu$, giving $N_\lambda \subseteq N_\mu$.

A subloop $H$ of a loop $L$ is normal if $Hx = xH$, $(Hx)y = H(xy)$, $x(yH) = (xy)H$ and $(xH)y = x(Hy)$ for all $x, y \in L$. With $L$ commutative and $H = N_\lambda$, these properties clearly hold. Also, since $N_\lambda(nx) = N_\lambda x$ for any $n \in N_\lambda$ and any $x \in L$, $L$ has a "coset expansion modulo" $N_\lambda$ [Bru58, §V.1], and this concludes the proof. □

An important idea that we use frequently throughout this paper is that if $Q$ is a finite Latin square of odd order with entries the elements of a set $S$, then each element of $S$ appears on the diagonal of $Q$ precisely once (because each occurrence of the element above the diagonal can be matched with an occurrence below the diagonal).

Given a commutative and associative ring $R$ and a loop $L$, one forms the *loop ring* $RL$ precisely as in the case that $L$ is a group. Specifically, $RL$ is the free $R$-module with basis $L$ and multiplication defined by extending the product in $L$ using the distributive laws. We say that a loop or ring is *Jordan* if it is commutative and satisfies the Jordan identity, $(x^2y)x = x^2(yx)$. This paper is motivated by the existence of and a desire to investigate nonassociative Jordan loop rings (and the loops by which they are defined).

## 3. Even characteristic is a must

Recall that the *characteristic* of a ring $R$ is the unique nonnegative integer $n$, denoted $\operatorname{char} R$, that generates the ideal $\{k \in \mathsf{Z} \mid kx = 0 \text{ for all } x \in R\}$.

Lowell Paige (with a minor correction by Marshall Osborn) showed that in characteristic relatively prime to 6, a commutative power associative loop ring is necessarily a group ring [Pai55], [Osb84]. We have a result of a similar nature about Jordan loop rings.

**Theorem 3.1.** *Let $R$ be a commutative associative ring with 1 and of characteristic relatively prime to 2 and let $L$ be a loop. If $RL$ is a Jordan ring, then $L$ is an abelian group.*

Proof. We begin by noting that in characteristic different from 2, a Jordan ring is power associative [Sch66, §IV.1] so our assumption about characteristic tells us immediately that $RL$ is power associative. Were $\operatorname{char} R$ to be relatively prime to 3 as well as 2, then the Paige–Osborn theorem would give the desired result, so it remains only to show that a Jordan loop ring is associative in odd characteristic not relatively prime to 3. Assume then that $\operatorname{char} R = 3n$ with $n$ odd.

We "linearize" the Jordan identity: specifically, we replace $x$ by $x + z$ in the Jordan identity $(x^2y)x = x^2(yx)$ and, after cancelling two pairs of terms that are equal by virtue of the Jordan identity, obtain

$$x^2y \cdot z + 2(xz \cdot y)x + 2(xz \cdot y)z + z^2y \cdot x = x^2 \cdot yz + 2xz \cdot yx + 2xz \cdot yz + z^2 \cdot yx.$$

Defining the *ring associator* of elements $a$, $b$, $c$ to be $(ab)c - a(bc)$ and denoting this element $[a, b, c]$, our new identity reads

$$[x^2, y, z] + 2[xz, y, x] + 2[xz, y, z] + [z^2, y, x] = 0. \tag{3.1}$$

Replacing $x$ by $2x$ gives $4[x^2, y, z] + 8[xz, y, x] + 4[xz, y, z] + 2[z^2, y, x] = 0$ and subtracting twice (3.1) gives $2[x^2, y, z] + 4[xz, y, x] = 0$. In odd characteristic, we may divide by 2, so $[x^2, y, z] + 2[xz, y, x] = 0$. Replacing $x$ by $x + w$ gives $2[xw, y, z] + 2[xz, y, w] + 2[wz, y, x] = 0$, so

$$[xw, y, z] + [xz, y, w] + [wz, y, x] = 0.$$

Here we set $z$ and $w$ equal to $y$ obtaining $2[xy, y, y] + [y^2, y, x] = 0$ which, expanded, is $2(xy \cdot y)y - 2(xy)y^2 + y^3x - y^2(yx)$. Remembering that $RL$ is commutative, we get $2(xy \cdot y)y + xy^3 = 3(xy)y^2$ and then, multiplying by $n$, $2n(xy \cdot y)y + nxy^3 = 0$ (because $3nRL = 0$). Since $-2na = na$ for any $a \in RL$, we have $nxy^3 = n(xy \cdot y)y$ for all $x, y \in RL$.

Again we linearize, replacing $y$ by $y + z$ and use $3nRL = 0$ to obtain

$$\begin{aligned} nx(y^3 + z^3) = {} & n(xy \cdot y)y + n(xy \cdot y)z + n(xy \cdot z)y + n(xy \cdot z)z \\ & + n(xz \cdot y)y + n(xz \cdot y)z + n(xz \cdot z)y + n(xz \cdot z)z \end{aligned}$$

and, after two cancellations,

$$n(xy \cdot y)z + n(xy \cdot z)y + n(xy \cdot z)z + n(xz \cdot y)y + n(xz \cdot y)z + n(xz \cdot z)y = 0.$$

Here we replace $y$ by $2y$ and find

$$4n(xy \cdot y)z + 4n(xy \cdot z)y + 2n(xy \cdot z)z + 4n(xz \cdot y)y + 2n(xz \cdot y)z + 2n(xz \cdot z)y = 0.$$

Subtracting these equations gives

$$3n(xy \cdot y)z + 3n(xy \cdot z)y + n(xy \cdot z)z + 3n(xz \cdot y)y + n(xz \cdot y)z + n(xz \cdot z)y = 0,$$

so

$$n(xy \cdot z)z + n(xz \cdot y)z + n(xz \cdot z)y = 0. \tag{3.2}$$

Now let $x$, $y$ and $z$ be elements of $L$. Since the coefficients on the left are nonzero, equation (3.2) asserts the linear dependence of three loop elements. This cannot happen if the loop elements are distinct. In fact, the loop elements here must all be the same because no equation of the form $2n\ell_1 + n\ell_2 = 0$, $\ell_1, \ell_2 \in L$, $\ell_1 \neq \ell_2$ can hold in $RL$. In particular, we must have $(xy \cdot z)z = (xz \cdot y)z$ which, in a loop, gives $xy \cdot z = xz \cdot y$ which is $z \cdot xy = zx \cdot y$ because of commutativity. This equation holds for any $x$, $y$, $z$ in $L$, so the loop is associative, as claimed. □

*Remark 3.2.* Both the Paige–Osborn result and our Theorem 3.1 are decidedly false without the assumption on characteristic. Indeed, there exists a com-

mutative loop $L$ of order 6 with $x^2 = 1$ for all $x \in L$ (so $L$ is not associative). If $F$ is the field of two elements and $\alpha \in FL$, it is easy to see that $\alpha^2 = 0$ or $\alpha^2 = 1$, so $FL$ is both Jordan and power associative.

## 4. RJ loops

By analogy with the term "RA loop," RA for ring alternative, to describe a loop whose loop ring is alternative, but not associative, we call a loop *RJ (ring Jordan)* if it has a loop ring that is nonassociative Jordan.

The following theorem is an easy consequence of Theorem 1.1 of [GK08].

**Theorem 4.1.** *A loop $L$ is RJ if and only if it is Jordan, not associative and, for any $x, y, z \in L$, either*

J1: $x^2y \cdot z = x^2 \cdot yz$ *and* $x \cdot yz^2 = xy \cdot z^2$, *or*

J2: $x^2y \cdot z = xy \cdot z^2$ *and* $x \cdot yz^2 = x^2 \cdot yz$, *or*

J3: $x^2y \cdot z = x \cdot yz^2$ *and* $x^2 \cdot yz = xy \cdot z^2$.

*Remark 4.2.* It is easily seen that any two of the conditions J1, J2, J3 imply that all four elements $x^2y \cdot z$, $x^2 \cdot yz$, $x \cdot yz^2$, $xy \cdot z^2$ are the same. Thus the third condition is satisfied too. So for any $x, y, z$ in an RJ loop precisely one of J1, J2, J3 hold or all three of these conditions hold.

We use the notation $(a, b, c)$ for the *associator* of three elements $a$, $b$, $c$ in a loop. Recall that this is the unique element $f$ satisfying $ab \cdot c = (a \cdot bc)f$ so that $a$, $b$, $c$ associate if and only if $(a, b, c) = 1$.

**Corollary 4.3.** *If $L$ is an RJ loop and $x, y, z \in L$, then $(x^2, y, z) = 1$ if and only if $(x, y, z^2) = 1$. In particular, if $L$ has odd order, then $x \in N_\lambda(L)$ if and only if $x^2 \in N_\lambda(L)$.*

Proof. Let $x, y, z \in L$. Then we have J1 or J2 or J3. If J1 is the case, then $(x^2, y, z) = (x, y, z^2) = 1$. If J2 is the case and $(x^2, y, z) = 1$, then $xy \cdot z^2 = x^2y \cdot z = x^2 \cdot yz = x \cdot yz^2$; that is, $(x, y, z^2) = 1$. Similarly $(x, y, z^2) = 1$ gives $(x^2, y, z) = 1$. The argument assuming J3 is similar. The second statement follows from the first and the fact that $L^2 = L$ when $L$ has odd order. □

While Jordan rings are almost always power associative,[2] in general they are not. The table below defines products of basis elements in a 6-dimensional Jordan

[2]in characteristic different from 2 [Sch66, §IV.1], and in characteristic 2 for algebras over fields containing at least four elements [Kok55]

algebra over the field of two elements and we note that $((((a\cdot a)a)a)a)a = f$ while $a^3 \cdot a^3 = c^2 = 0$.

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 0 |
| $b$ | $c$ | $d$ | $e$ | $f$ | 0 | 0 |
| $c$ | $d$ | $e$ | 0 | 0 | 0 | 0 |
| $d$ | $e$ | $f$ | 0 | 0 | 0 | 0 |
| $e$ | $f$ | 0 | 0 | 0 | 0 | 0 |
| $f$ | 0 | 0 | 0 | 0 | 0 | 0 |

We do not know if a Jordan *loop ring* must be power associative (but see Section 7). This uncertainty, together with the fact that Jordan loops are in general not power associative (see [GK08, §2]), makes the next result both interesting and suggestive. We recognize that the theorem would be superfluous were it known that a Jordan loop ring is power associative.

**Theorem 4.4.** *An RJ loop is power associative.*

Proof. Let $L$ be an RJ loop. As is customary, for any $x \in L$, we define nonnegative powers of $x$ inductively by $x0 = 1$ and, for $n > 0$, $x^n = x^{n-1}x$ and prove that $L$ is power associative by showing that $x^r x^s = x^{r+s}$ for any $r, s \geq 0$. This is clear if $r = 0, 1$ or $s = 0, 1$ so we assume in what follows that $r, s \geq 2$. We assume initially as well that $r \neq s$, so $r > s$, without loss of generality.

Assume $r+s > 3$ and that $x^n$ is well defined for any $n < r+s$. (In particular, $x^r$ and $x^s$ are well defined.) According to Theorem 4.1, a loop is RJ if and only if any three elements satisfy J1 or J2 or J3. We examine these conditions as they apply to the triple $(x, x^{r-2}, x^s)$ and, in each case, use the induction hypothesis liberally and tacitly.

Suppose J1 is the case. The first equation of this condition gives $x^r x^s = (x^2x^{r-2})x^s = x2(x^{r-2}x^s) = x^2x^{r+s-2} = x2(x^{r+s-3}x) = (x2x^{r+s-3})x$ (by the Jordan identity) $= x^{r+s-1}x = x^{r+s}$.

Suppose J2 is the case. The first equation of this condition says $x^r x^s = x^{r-1}x^{2s}$, and $x^{2s}$ is well defined because of our assumption $s < r$. Thus J2 gives $x^r x^s = x^{r-1}x^{2s} = (x^s x^{r-s-1})x^{2s} = x^s(x^{r-s-1}x^{2s})$, using the Jordan identity. Cancelling $x^s$, we have $x^r = x^{r-s-1}x^{2s} = x^{r+s-1} = x^r x^{s-1}$, so $x^{s-1} = 1$ and $x^s = x$. Thus $x^r x^s = x^r x = x^{r+1}$, whereas $x^{r+s} = x^{r+s-1}x = (x^r x^{s-1})x = x^r x = x^{r+1}$ too.

If J3 is the case, the second equation of this condition says $x^2x^{r+s-2} = x^{r-1}x^{2s}$, so $x^{r-1}x^{2s} = x2(x^{r+s-3}x) = (x^2x^{r+s-3})x = x^{r+s-1}x$. Also, $x^{r-1}x^{2s} = (x^s x^{r-s-1})x^{2s} = x^s(x^{r-s-1}x^{2s}) = x^s x^{r+s-1}$, and so $x^{r+s-1}x = x^s x^{r+s-1}$. It

follows that $x^s = x$. Thus $x^r x^s = x^r x = x^{r+1}$, whereas $x^{r+s} = x^{r+s-1}x = (x^{r-1}x^s)x = x^r x = x^{r+1}$ too. In every situation, J1 or J2 or J3, so we obtain the desired result.

It remains to consider the case $r = s$. If $r = s = 2k$ is even, $x^r x^s = x^{2k}x^{2k} = x^{2k}(x^k x^k) = (x^{2k}x^k)x^k$ (by the Jordan identity) $= x^{3k}x^k = x^{4k} = x^{r+s}$ by the case $r \neq s$ already established. If $r = s = 2k+1$ is odd, we examine conditions J1, J2, J3 as they apply to the triple $(x^k, x, x^{2k+1})$ and use extensively the fact that the result we want holds when the exponents are different. Assuming J1, $x^{2k+1}x^{2k+1} = x^{2k}x^{2k+2} = x^{2k}(x^{k+2}x^k) = (x^{2k}x^{k+2})x^k = x^{3k+2}x^k = x^{4k+2}$ as desired (by the $r > s$ case already settled); thus $x^r x^s = x^{r+s}$. Assuming J2, the second equation says $x^k[x(x^{2k+1})2] = x^{2k}x^{2k+2} = x^{4k+2} = x^{3k+2}x^k$. Cancelling $x^k$, we get $(x^{2k+1})^2 x = x^{3k+2} = x^{3k+1}x$, so $(x^{2k+1})2 = x^{3k+1} = x^{2k+1}x^k$. Thus $x^k = x^{2k+1} = x^k x^{k+1}$, so $x^{k+1} = 1$ and $x^{2k+1} = x^{k+1}x^k = x^k$. Now $x^r x^s = x^{2k+1}x^{2k+1} = x^k x^k = x^{2k}$, while $x^{r+s} = x^{4k+2} = x^{2k}x^{2k+2} = x^{2k}(x^{k+1})2 = x^{2k}$ as well. Finally, assume J3 holds for $(x^k, x, x^{2k+1})$. Then $x^{2k}x^{2k+2} = x^{k+1}(x^{2k+1})2$, from the second equation, so $x^{k+1}(x^{2k+1})2 = x^{2k}(x^{k+2}x^k) = (x^{2k}x^{k+2})x^k$ (Jordan identity) $= x^{3k+2}x^k = x^{4k+2} = x^{k+1}x^{3k+1}$, giving $(x^{2k+1})2 = x^{3k+1} = x^{2k+1}x^k$ and $x^{2k+1} = x^k$. So $x^k = x^k x^{k+1}$ and $x^{k+1} = 1$, hence $x^{4k+2} = x^{3k+1}x^{k+1} = x^{3k+1}$. Using the first equation of J3, however, we find $x^{2k+1}x^{2k+1} = x^k[x(x^{2k+1})2] = x^k(xx^{2k}) = x^k x^{2k+1} = x^{3k+1}$ as well. This completes the proof. □

## 5. A construction: $J(G, \alpha, \beta)$ loops

We present a construction that generalizes that of [GK08, Theorem 2.1] and allows us to exhibit various RJ loops.

**Theorem 5.1.** *Let $(G, \cdot)$ be an abelian group and $u$ an indeterminate. Let $\alpha, \beta : G \times G \to G$ be symmetric maps with the property that for each $g \in G$, the functions $\alpha_g, \beta_g : G \to G$ defined by $\alpha_g(x) = \alpha(g, x)$ and $\beta_g(x) = \beta(g, x)$ are bijections and $\beta_1(g) = g$ for all $g \in G$. Let $L = G \cup Gu$ and extend the product in $G$ to $L$ using the rules $(gu)(hu) = \alpha(g, h)$ and $g(hu) = (gu)h = \beta(g, h)u$. Then*

(1) *$(L, \cdot)$ is a commutative loop which is Jordan if and only if*
   - $\beta(g, \beta(g^2, h)) = \beta(g^2, \beta(g, h))$,
   - $\beta(\alpha(g, g)h, g) = \beta(\alpha(g, g), \beta(g, h))$ *and*
   - $\alpha(\beta(\alpha(g, g), h), g) = \alpha(g, g)\alpha(g, h)$

   *for all $g$, $h$ and $k$ in $G$;*

(2) $(L,\cdot)$ *is a group if and only if, for all* $g,h\in G$, $\beta(g,h)=gh$ *and there exists* $a\in G$ *such that* $\alpha(g,h)=agh$.

PROOF. (1) The operation in $L$ is commutative and the condition $\beta_1(g)=g$ for all $g$ implies that the identity of $G$ is an identity for $L$ so, to see that $(L,\cdot)$ is a loop, we have only to note that it is a quasigroup: The unique solutions to $ax=b$ and $xa=b$ appear in Table 1, while Table 2 shows why the conditions equivalent to the Jordan identity are as stated.

| $a$ | $x$ | $b$ |
|---|---|---|
| $g$ | $g^{-1}h$ | $h$ |
| $g$ | $\beta_g^{-1}(h)u$ | $hu$ |
| $gu$ | $\alpha_g^{-1}(h)u$ | $h$ |
| $gu$ | $\beta_g^{-1}(h)$ | $hu$ |

| $x$ | $a$ | $b$ |
|---|---|---|
| $hg^{-1}$ | $g$ | $h$ |
| $\beta_g^{-1}(h)u$ | $g$ | $hu$ |
| $\alpha_g^{-1}(g)u$ | $gu$ | $h$ |
| $\beta_g^{-1}(h)$ | $gu$ | $hu$ |

*Table 1.* The unique solutions to $ax=b$ and $xa=b$ for $a,b\in\{g,h,gu,hu\mid g,h\in G\}$.

| $x$ | $y$ | $(x^2y)x$ | $x^2(yx)$ |
|---|---|---|---|
| $g$ | $h$ | $g^3h$ | $g^3h$ |
| $g$ | $hu$ | $\beta(g,\beta(g^2,h))u$ | $\beta(g^2,\beta(h,g))u$ |
| $gu$ | $h$ | $\beta(\alpha(g,g)h,g)u$ | $\beta(\alpha(g,g),\beta(h,g))u$ |
| $gu$ | $hu$ | $\alpha(\beta(\alpha(g,g),h),g)$ | $\alpha(g,g)\alpha(h,g)$ |

*Table 2.* $g$ and $h$ are elements of the group $G$.

(2) Looking at Table 3, we see that associativity is equivalent to

- $\beta(gh,k)=\beta(g,\beta(h,k))=\beta(\beta(g,h),k)=\beta(g,hk)$,
- $\alpha(\beta(g,h),k)=g\alpha(h,k)=\alpha(g,\beta(h,k))=\alpha(g,h)k$ and
- $\beta(\alpha(g,h),k))=\beta(g,\alpha(h,k))$

for all $g,h,k\in G$.

| $x$ | $y$ | $z$ | $(xy)z$ | $x(yz)$ |
|---|---|---|---|---|
| $g$ | $h$ | $k$ | $(gh)k$ | $g(hk)$ |
| $g$ | $h$ | $ku$ | $\beta(gh,k)u$ | $\beta(g,\beta(h,k))u$ |
| $g$ | $hu$ | $k$ | $\beta(\beta(g,h),k)u$ | $\beta(g,\beta(h,k))u$ |
| $g$ | $hu$ | $ku$ | $\alpha(\beta(g,h),k)$ | $g\alpha(h,k)$ |
| $gu$ | $h$ | $k$ | $\beta(\beta(g,h),k)u$ | $\beta(g,hk)u$ |
| $gu$ | $h$ | $ku$ | $\alpha(\beta(g,h),k)$ | $\alpha(g,\beta(h,k))$ |
| $gu$ | $hu$ | $k$ | $\alpha(g,h)k$ | $\alpha(g,\beta(h,k))$ |
| $gu$ | $hu$ | $ku$ | $\beta(\alpha(g,h),k))u$ | $\beta(g,\alpha(h,k))u$ |

*Table 3.* $g$, $h$ and $k$ are elements of $G$.

If $\beta(g,h) = gh$ and $\alpha(g,h) = agh$ for some $a \in G$ and all $g,h \in G$, then these conditions are satisfied, so the loop $L$ is associative and hence a group. Conversely, assume that $L$ is a group. Since $\beta_k$ is a bijection for any $k \in G$, the condition $\beta_k(gh) = \beta_k(\beta(g,h))$ says $\beta(g,h) = gh$ for all $g,h \in G$. Now, the conditions for associativity become

$$\alpha(gh,k) = g\alpha(h,k) = \alpha(g,hk) = \alpha(g,h)k = g\alpha(h,k)$$

for all $g,h,k \in G$. Setting $h = 1$ in the first equation gives $\alpha(g,k) = g\alpha(1,k)$ which, with $k = 1$ says $\alpha(g,1) = g\alpha(1,1)$. So $\alpha(g,k) = g\alpha(1,k) = gk\alpha(1,1)$ for all $g$ and $k$. This gives the desired result with $a = \alpha(1,1)$. □

*Notation.* We use the notation $J(G,\alpha,\beta)$ to describe the loop constructed in Theorem 5.1.

We identify two special cases.

**Corollary 5.2** ([GK08])**.** *Let $(G,\cdot)$ be an abelian group and $u$ an indeterminate. Let $\alpha : G \times G \to G$ be a symmetric map such that, for each $g \in G$, the function $\alpha_g : G \to G$ defined by $\alpha_g(x) = \alpha(g,x)$ is a bijection. Let $L = G \cup Gu$ with multiplication defined by the rules $(gu)(hu) = \alpha(g,h)$ and $g(hu) = (gu)h = (gh)u$. Then $L$ is a commutative loop that is Jordan if and only if $\alpha(\alpha(g,g)h,g) = \alpha(g,g)\alpha(g,h)$ for all $g,h \in G$ and a group if and only if there exists $a \in G$ such that $\alpha(g,h) = agh$ for all $g,h \in G$.*

Proof. This is just Theorem 5.1 in the special case $\beta(g,h) = gh$ for $g,h \in G$. □

The loop just described was introduced in [GK08, Theorem 2.1] and is denoted $J(G,\alpha)$. By analogy, we label $J(G,\beta)$ loops described by the next corollary.

**Corollary 5.3.** *Let $(G,\cdot)$ be an abelian group and $u$ an indeterminate. Let $\beta : G \times G \to G$ be a symmetric map such that for each $g \in G$ the function $\beta_g : G \to G$ defined by $\beta_g(x) = \beta(g,x)$ is a bijection and $\beta_1(x) = x$ for all $x \in G$. Let $L = G \cup Gu$ with multiplication given by the rules $(gu)(hu) = gh$ and $g(hu) = (gu)h = \beta(g,h)u$. Then $(L,\cdot)$ is a commutative loop that is Jordan if and only if*

$$\beta(g^2,h) = g^2h \quad \text{and} \quad \beta(g^2h,g) = g^2\beta(g,h)$$

*for all $g,h \in G$, and a group if and only if $\beta(g,h) = gh$ for all $g,h \in G$, in particular, if $G$ has finite odd order.*

Proof. This is Theorem 5.1 in the special case $\alpha(g,h) = gh$ for $g,h \in G$. The last remark follows because in a group of odd order, every element is a square, so $\beta(g^2,h) = g^2h$ for all $g$, $h$ says $\beta(g,h) = gh$ for all $g$, $h$. □

Now we turn our attention to the possibility that a loop $J(G,\alpha,\beta)$ is ring Jordan.

**Theorem 5.4.** *A loop $L = J(G,\alpha,\beta)$ is RJ if and only if it is not associative and, for any $g,h,k \in G$, the following conditions are satisfied:*

(1) $\beta(g^2h,k) = g^2\beta(h,k)$

(2) $\alpha(g^2h,k) = g^2\alpha(h,k)$

(3) $\beta(\beta(\alpha(g,g),h),k) = \beta(\alpha(g,g),\beta(h,k))$

(4) $\beta(\alpha(g,g)h,k)=\beta(\alpha(g,g),\beta(h,k))$ *and* $\beta(g,h\alpha(k,k))=\beta(\beta(g,h),\alpha(k,k))$ *or* $\beta(\alpha(g,g)h,k)=\beta(g,h\alpha(k,k))$ *and* $\beta(\alpha(g,g),\beta(h,k))=\beta(\beta(g,h),\alpha(k,k))$ *or* $\beta(\alpha(g,g)h,k)=\beta(\beta(g,h),\alpha(k,k))$ *and* $\beta(\alpha(g,g),\beta(h,k)) = \beta(g,h\alpha(k,k))$

(5) $\alpha(\beta(\alpha(g,g),h),k)=\alpha(g,g)\alpha(h,k)$ *and* $\alpha(g,\beta(h,\alpha(k,k)))=\alpha(g,h)\alpha(k,k)$ *or* $\alpha(\beta(\alpha(g,g),h),k)=\alpha(g,\beta(h,\alpha(k,k)))$ *and* $\alpha(g,g)\alpha(h,k)=\alpha(g,h)\alpha(k,k)$ *or* $\alpha(\beta(\alpha(g,g),h),k)=\alpha(g,h)\alpha(k,k)$ *and* $\alpha(g,g)\alpha(h,k)=\alpha(g,\beta(h,\alpha(k,k)))$.

Proof. We consider Theorem 4.1, along with Tables 4 and 5 and note immediately that the Jordan identity is equivalent to the following nine conditions. The first six conditions are consequences of the fact that an element of $G$ cannot equal an element of $Gu$ so that, for example, the $g$, $h$, $ku$ lines of Table 4 and Table 5 imply that $\beta(g^2h,k) = \beta(g^2,\beta(h,k))$.

(1) $\beta(g^2h,k) = \beta(g^2,\beta(h,k))$

(2) $\alpha(\beta(g^2,h),k) = g^2\alpha(h,k)$

(3) $\beta(g,\beta(h,\alpha(k,k))) = \beta(\beta(g,h),\alpha(k,k))$

(4) $\beta(g,hk^2) = \beta(\beta(g,h),k^2)$

(5) $\beta(\beta(\alpha(g,g),h),k) = \beta(\alpha(g,g),\beta(h,k))$

(6) $\alpha(g,\beta(h,k^2)) = \alpha(g,h)k^2$

(7) $\beta(\beta(g^2,h),k) = \beta(g^2,\beta(h,k))$ and $\beta(g,\beta(h,k^2)) = \beta(\beta(g,h),k^2)$, or $\beta(\beta(g^2,h),k) = \beta(g,\beta(h,k^2))$ and $\beta(g^2,\beta(h,k)) = \beta(\beta(g,h),k^2)$, or $\beta(\beta(g^2,h),k) = \beta(\beta(g,h),k^2)$ and $\beta(g^2,\beta(h,k)) = \beta(g,\beta(h,k^2))$,

(8) $\beta(\alpha(g,g)h,k)=\beta(\alpha(g,g),\beta(h,k))$ and $\beta(g,h\alpha(k,k))=\beta(\beta(g,h),\alpha(k,k))$ or $\beta(\alpha(g,g)h,k)=\beta(g,h\alpha(k,k))$ and $\beta(\alpha(g,g),\beta(h,k))=\beta(\beta(g,h),\alpha(k,k))$ or $\beta(\alpha(g,g)h,k)=\beta(\beta(g,h),\alpha(k,k))$ and $\beta(\alpha(g,g),\beta(h,k))=\beta(g,h\alpha(k,k))$,

(9) $\alpha(\beta(\alpha(g,g),h),k)=\alpha(g,g)\alpha(h,k)$ and $\alpha(g,\beta(h,\alpha(k,k)))=\alpha(g,h)\alpha(k,k)$ or $\alpha(\beta(\alpha(g,g),h),k)=\alpha(g,\beta(h,\alpha(k,k)))$ and $\alpha(g,g)\alpha(h,k)=\alpha(g,h)\alpha(k,k)$ or $\alpha(\beta(\alpha(g,g),h),k)=\alpha(g,h)\alpha(k,k)$ and $\alpha(g,g)\alpha(h,k)=\alpha(g,\beta(h,\alpha(k,k)))$.

| $x$ | $y$ | $z$ | $(x^2y)z$ | $x^2(yz)$ |
|---|---|---|---|---|
| $g$ | $h$ | $k$ | $(g^2h)k$ | $g^2(hk)$ |
| $g$ | $h$ | $ku$ | $\beta(g^2h,k)u$ | $\beta(g^2,\beta(h,k))u$ |
| $g$ | $hu$ | $k$ | $\beta(\beta(g^2,h),k)u$ | $\beta(g^2,\beta(h,k))u$ |
| $g$ | $hu$ | $ku$ | $\alpha(\beta(g^2,h),k)$ | $g^2\alpha(h,k)$ |
| $gu$ | $h$ | $k$ | $(\alpha(g,g)h)k$ | $\alpha(g,g)(hk)$ |
| $gu$ | $h$ | $ku$ | $\beta(\alpha(g,g)h,k)u$ | $\beta(\alpha(g,g),\beta(h,k))u$ |
| $gu$ | $hu$ | $k$ | $\beta(\beta(\alpha(g,g),h),k)u$ | $\beta(\alpha(g,g),\beta(h,k))u$ |
| $gu$ | $hu$ | $ku$ | $\alpha(\beta(\alpha(g,g),h),k)$ | $\alpha(g,g)\alpha(h,k)$ |

*Table 4.* $g$, $h$ and $k$ are elements of $G$.

| $x$ | $y$ | $z$ | $x(yz^2)$ | $(xy)z^2$ |
|---|---|---|---|---|
| $g$ | $h$ | $k$ | $g(hk^2)$ | $(gh)k^2$ |
| $g$ | $h$ | $ku$ | $g(h\alpha(k,k))$ | $(gh)\alpha(k,k)$ |
| $g$ | $hu$ | $k$ | $\beta(g,\beta(h,k^2))u$ | $\beta(\beta(g,h),k^2)u$ |
| $g$ | $hu$ | $ku$ | $\beta(g,\beta(h,\alpha(k,k)))u$ | $\beta(\beta(g,h),\alpha(k,k))u$ |
| $gu$ | $h$ | $k$ | $\beta(g,hk^2)u$ | $\beta(\beta(g,h),k^2)u$ |
| $gu$ | $h$ | $ku$ | $\beta(g,h\alpha(k,k))u$ | $\beta(\beta(g,h),\alpha(k,k))u$ |
| $gu$ | $hu$ | $k$ | $\alpha(g,\beta(h,k^2))$ | $\alpha(g,h)k^2$ |
| $gu$ | $hu$ | $ku$ | $\alpha(g,\beta(h,\alpha(k,k)))$ | $\alpha(g,h)\alpha(k,k)$ |

*Table 5.* $g$, $h$ and $k$ are elements of $G$.

Condition (1) implies $\beta(k^2h, g) = \beta(k^2, \beta(h, g)) = \beta(\beta(g, h), k^2)$ using twice the symmetry of $\beta$. Thus condition (1) implies condition (4), and conversely. Condition (3) implies $\beta(k, \beta(h, \alpha(g, g))) = \beta(\beta(k, h), \alpha(g, g))$ and hence $\beta(h, \beta(k, \alpha(g, g))) = \beta(\beta(h, k), \alpha(g, g))$ which, by symmetry, is (5). Conversely condition (5) implies condition (3), so conditions (3) and (5) are equivalent, as are conditions (2) and (6). Thus, removing redundancies, we have conditions (1), (2), (5), (7), (8), (9).

Setting $k = 1$ in (1) and remembering that $\beta(x, 1) = x$ identically gives $g^2h = \beta(g^2, h)$, so $\beta(g^2h, k) = g^2\beta(h, k)$ for all $g$ $h$, $k$ and (2) becomes $\alpha(g^2h, k) = g^2\alpha(h, k)$. Conversely, $\beta(g^2h, k) = g^2\beta(h, k)$ and $\alpha(g^2h, k) = g^2\alpha(h, k)$ imply conditions (1) and (2) and the first of the three alternatives in condition (7), so the proof is complete. □

Once again, we specialize $\alpha$ and $\beta$.

**Corollary 5.5.** (1) *The loop $L = J(G, \alpha)$ is RJ if and only if it is not associative and, for any $g, h, k \in G$, $\alpha(g^2h, k) = g^2\alpha(h, k)$ and one of the following must hold:*

- $\alpha(\alpha(g, g)h, k) = \alpha(g, g)\alpha(h, k)$ *and* $\alpha(g, h\alpha(k, k)) = \alpha(g, h)\alpha(k, k)$, *or*
- $\alpha(\alpha(g, g)h, k) = \alpha(g, h\alpha(k, k))$ *and* $\alpha(g, g)\alpha(h, k) = \alpha(g, h)\alpha(k, k)$, *or*
- $\alpha(\alpha(g, g)h, k) = \alpha(g, h)\alpha(k, k)$ *and* $\alpha(g, g)\alpha(h, k) = \alpha(g, h\alpha(k, k))$.

(2) *The loop $L = J(G, \beta)$ is RJ if and only if it is not associative and $\beta(g^2h, k) = g^2\beta(h, k)$ for all $g, h, k \in G$, that is, if and only if J1 holds identically in $L$.*

Proof. (1) With $\beta(g, h) = gh$, conditions (1) and (3) of Theorem 5.4 hold identically, as does the first alternative in (4). Thus the Jordan identity holds in $RJ$ if and only if (2) and (5) and, with $\beta(g, h) = gh$, these are precisely the conditions stated.

(2). With $\alpha(g, h) = gh$ and $\beta(g^2h, k) = g^2\beta(h, k)$, conditions (2) and (3) of Theorem 5.4 hold identically, as do the first of the alternatives in (4) and (5) (it is these that are equivalent to J1 holding identically – see Tables 4 and 5), so only condition (1) remains. □

*Remark 5.6.* Suppose $G$ has odd order. Then every element of $G$ is a square and every element of $G$ appears on the diagonal of the Latin square defined by $\alpha$. The condition $\beta(g^2h, k) = g^2\beta(h, k)$ that appears in Theorem 5.4 becomes $\beta(gh, k) = g\beta(h, k)$, so ($h = 1$) $\beta(g, k) = gk$ for all $g$ and $k$ and $L$ is in fact a $J(G, \alpha)$ loop. Similarly, the condition $\alpha(g^2h, k) = g^2\alpha(h, k)$ for all $g, h, k \in G$ that we see in Corollary 5.5 implies $\alpha(g, k) = g\alpha(k, 1)$ for all $g$, $k$. With $k = 1$,

this gives $\alpha(g,1) = g\alpha(1,1)$ for all $g$, so $\alpha(g,k) = g\alpha(k,1) = gk\alpha(1,1)$. Part (2) of Theorem 5.1 now tells us that our loop is a group. All this shows that our $J(G,\alpha,\beta)$ construction has a chance to produce an RJ loop only when $|G|$ is even. This is of interest because it is an open question whether or not there exist RJ loops of order $2k$, $k$ odd, other than those of exponent 2, and we do not know if our $J(G,\alpha,\beta)$ construction will produce a (nonassociative) Jordan loop when $|G|$ is odd.

The only known RJ loops are RJ because they satisfy condition J1 identically. It is unknown whether or not J1 is necessary for RJ, though this is true for loops of the form $J(G,\beta)$, by Corollary 5.5, but perhaps not for $J(G,\alpha)$ loops, as our next result suggests.

**Corollary 5.7.** *Condition J1 holds identically for a $J(G,\alpha)$ loop if and only if $\alpha(g^2h,k) = g^2\alpha(h,k)$ and $\alpha(\alpha(g,g)h,k) = \alpha(g,g)\alpha(h,k)$ for all $g,h,k \in G$.*

Proof. Condition J1 holds identically if and only if the last two columns of Table 4 are equal and the last two columns of Table 5 are equal for all $x,y,z \in G$. With $\beta(g,h) = gh$ identically, this occurs if and only if the two conditions on $\alpha$ given in the statement hold for all $g,h,k \in G$. □

## 6. Examples of RJ loops

We use the results of Section 5 to exhibit a number of RJ loops that are not of exponent 2.

**Theorem 6.1.** *Let $n$ be even and let $m \in \{0,1,2,\dots,n-1\}$, $m \neq n/2$. Define $f : \mathsf{Z}_n \times \mathsf{Z}_n \to \mathsf{Z}_n$ by*

$$f(i,j) = \begin{cases} i+j \pmod n & i \text{ or } j \text{ even} \\ i+j-2m \pmod n & i,\ j \text{ both odd} \end{cases}$$

*With $\alpha = \beta = f$, the loops $J(\mathsf{Z}_n,\alpha)$, $J(\mathsf{Z}_n,\beta)$ and $J(\mathsf{Z}_n,\alpha,\beta)$ are RJ.*

Proof. By Corollary 5.5, to ensure that $L = J(G,\alpha)$ is RJ, it is sufficient that $\alpha(g^2h,k) = g^2\alpha(h,k)$ and $\alpha(\alpha(g,g)h,k) = \alpha(g,g)\alpha(h,k)$ for all $g$, $h$, $k$. For $G = \mathsf{Z}_n$ and $\alpha = f$, these equations become $f(2i+j,k) = 2i + f(j,k)$ and $f(f(i,i)+j,k) = f(i,i) + f(j,k)$ for all $i,j,k \in \mathsf{Z}_n$. That $J(\mathsf{Z}_n,\alpha)$ is RJ now follows directly from Table 6.

| $i$ | $j$ | $k$ | $f(2i+j,k)$ | $2i+f(j,k)$ | $f(f(i,i)+j,k)$ | $f(i,i)+f(j,k)$ |
|---|---|---|---|---|---|---|
| even | even | even | $2i+j+k$ | $2i+j+k$ | $2i+j+k$ | $2i+j+k$ |
| even | even | odd | $2i+j+k$ | $2i+j+k$ | $2i+j+k$ | $2i+j+k$ |
| even | odd | even | $2i+j+k$ | $2i+j+k$ | $2i+j+k$ | $2i+j+k$ |
| even | odd | odd | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ |
| odd | even | even | $2i+j+k$ | $2i+j+k$ | $2i+j+k-2m$ | $2i+j+k-2m$ |
| odd | even | odd | $2i+j+k$ | $2i+j+k$ | $2i+j+k-2m$ | $2i+j+k-2m$ |
| odd | odd | even | $2i+j+k$ | $2i+j+k$ | $2i+j+k-2m$ | $2i+j+k-2m$ |
| odd | odd | odd | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-4m$ | $2i+j+k-4m$ |

*Table 6.* All entries are modulo $n$

With reference again to Corollary 5.5, we see that $L = J(G, \beta)$ is RJ with $G = \mathsf{Z}_n$ and $\beta = f$ if and only if $f(2i + j, k) = 2i + f(j, k)$ for all $i$, $j$, $k$, a condition already verified.

Finally, with respect to $J(\mathsf{Z}_n, \alpha, \beta)$, we show that $\alpha = \beta = f$ satisfies conditions (1) to (3) of Theorem 5.4 and the first of the alternatives in conditions (4) and (5). Each of the latter is a statement of the form "$A$ and $B$" and, in each case, $A$ holds for all $g, h, k$ if and only if $B$ holds for all $g$, $h$, $k$. Also, with $\alpha = \beta$, conditions (1) and (2) are the same. In the present context, then, it is sufficient to verify that

$$f(f(f(i,i),j),k) = f(f(i,i),f(j,k)) = f(f(i,i)+j,k) = f(i,i)+f(j,k)$$

for all $i, j, k \in \mathsf{Z}_n$. When $i$ and either $j$ or $k$ are even, the four indicated elements of $G$ are equal because components are simply added together. Inspection of Table 7 shows that they are equal also in the remaining cases. □

| $i$ | $j$ | $k$ | $f(f(f(i,i),j),k)$ | $f(f(i,i),f(j,k))$ | $f(f(i,i)+j,k)$ | $f(i,i)+f(j,k)$ |
|---|---|---|---|---|---|---|
| even | odd | odd | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ |
| odd | even | even | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ |
| odd | even | odd | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ |
| odd | odd | even | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ | $2i+j+k-2m$ |
| odd | odd | odd | $2i+j+k-4m$ | $2i+j+k-4m$ | $2i+j+k-4m$ | $2i+j+k-4m$ |

*Table 7.* All entries are modulo $n$

We illustrate Theorem 6.1 in Tables 8, 9 and 10 with $G = \mathsf{Z}_8$ the group of integers mod 8 and $\alpha$ and $\beta$ modifications of the table for $G$ in locations $(i, j)$ when $i$ and $j$ are both odd as described in Theorem 6.1 with $m = 1$.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 8 |
| 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 10 | 11 | 12 | 13 | 14 | 15 | 8 | 9 |
| 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 11 | 12 | 13 | 14 | 15 | 8 | 9 | 10 |
| 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 12 | 13 | 14 | 15 | 8 | 9 | 10 | 11 |
| 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 13 | 14 | 15 | 8 | 9 | 10 | 11 | 12 |
| 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 | 14 | 15 | 8 | 9 | 10 | 11 | 12 | 13 |
| 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 15 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 8 | 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 |
| 10 | 11 | 12 | 13 | 14 | 15 | 8 | 9 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 |
| 11 | 12 | 13 | 14 | 15 | 8 | 9 | 10 | 3 | 2 | 5 | 4 | 7 | 6 | 1 | 0 |
| 12 | 13 | 14 | 15 | 8 | 9 | 10 | 11 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 13 | 14 | 15 | 8 | 9 | 10 | 11 | 12 | 5 | 4 | 7 | 6 | 1 | 0 | 3 | 2 |
| 14 | 15 | 8 | 9 | 10 | 11 | 12 | 13 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 |
| 15 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 7 | 6 | 1 | 0 | 3 | 2 | 5 | 4 |

*Table 8.* An RJ loop $J(\mathbf{Z}_8, \alpha)$ with $\alpha(i,j) = i + j \pmod 8$ unless $i, j$ are both odd, in which case $\alpha(i,j) = i + j - 2 \pmod 8$.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 9 | 8 | 11 | 10 | 13 | 12 | 15 | 14 |
| 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 10 | 11 | 12 | 13 | 14 | 15 | 8 | 9 |
| 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 11 | 10 | 13 | 12 | 15 | 14 | 9 | 8 |
| 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 12 | 13 | 14 | 15 | 8 | 9 | 10 | 11 |
| 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 13 | 12 | 15 | 14 | 9 | 8 | 11 | 10 |
| 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 | 14 | 15 | 8 | 9 | 10 | 11 | 12 | 13 |
| 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 15 | 14 | 9 | 8 | 11 | 10 | 13 | 12 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 8 | 11 | 10 | 13 | 12 | 15 | 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 |
| 10 | 11 | 12 | 13 | 14 | 15 | 8 | 9 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 |
| 11 | 10 | 13 | 12 | 15 | 14 | 9 | 8 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 |
| 12 | 13 | 14 | 15 | 8 | 9 | 10 | 11 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 13 | 12 | 15 | 14 | 9 | 8 | 11 | 10 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 |
| 14 | 15 | 8 | 9 | 10 | 11 | 12 | 13 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 |
| 15 | 14 | 9 | 8 | 11 | 10 | 13 | 12 | 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

*Table 9.* An RJ loop $J(\mathbf{Z}_8, \beta)$ with $\beta(i,j) = i + j \pmod 8$ unless $i, j$ are both odd, in which case $\beta(i,j) = i + j - 2 \pmod 8$.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 9 | 8 | 11 | 10 | 13 | 12 | 15 | 14 |
| 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 10 | 11 | 12 | 13 | 14 | 15 | 8 | 9 |
| 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 11 | 10 | 13 | 12 | 15 | 14 | 9 | 8 |
| 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 12 | 13 | 14 | 15 | 8 | 9 | 10 | 11 |
| 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 13 | 12 | 15 | 14 | 9 | 8 | 11 | 10 |
| 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 | 14 | 15 | 8 | 9 | 10 | 11 | 12 | 13 |
| 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 15 | 14 | 9 | 8 | 11 | 10 | 13 | 12 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 8 | 11 | 10 | 13 | 12 | 15 | 14 | 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 |
| 10 | 11 | 12 | 13 | 14 | 15 | 8 | 9 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 |
| 11 | 10 | 13 | 12 | 15 | 14 | 9 | 8 | 3 | 2 | 5 | 4 | 7 | 6 | 1 | 0 |
| 12 | 13 | 14 | 15 | 8 | 9 | 10 | 11 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 13 | 12 | 15 | 14 | 9 | 8 | 11 | 10 | 5 | 4 | 7 | 6 | 1 | 0 | 3 | 2 |
| 14 | 15 | 8 | 9 | 10 | 11 | 12 | 13 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 |
| 15 | 14 | 9 | 8 | 11 | 10 | 13 | 12 | 7 | 6 | 1 | 0 | 3 | 2 | 5 | 4 |

*Table 10.* An RJ loop $J(\mathsf{Z}_8, \alpha, \beta)$ with $\alpha = \beta(i,j) = i + j \pmod 8$ unless $i, j$ are both odd, in which case $\alpha = \beta(i,j) = i + j - 2 \pmod 8$.

## 7. Concluding remarks

As mentioned, we do not know if conditions J2 and J3 are redundant; that is, we do not know if a loop is RJ if and only if condition J1 holds identically. Note that J1 simply says that squares are in the left and right nuclei of $L$ (these nuclei are the same in any commutative loop).

In what follows, suppose that $L$ is an RJ loop and that an RJ loop satisfies J1 identically. We find answers to several questions that intrigue us.

(1) A Jordan loop algebra $FL$ over any field $F$ is power associative.
This follows from the theorem of Kokoris stated in a footnote within Section 4 when $|F| > 2$ and also when $F$ is the field of 2 elements because in characteristic 2, squares of elements of $L$ being nuclear implies the same for squares of elements in $FL$ and this easily gives power associativity of $FL$.

(2) If $L$ has odd order, then $L$ is a group.
This follows directly from Corollary 4.3.

(3) If $L$ has finite order, then $|a| \mid |L|$ for any $a \in L$.
To understand this, we use the fact $a^2 \in N_\lambda$ which is a group of order a

divisor of $|L|$ – see Lemma 2.1. Now $a^{2|N_\lambda|} = 1$. Also $L/N_\lambda$ is an RJ loop. If it has odd order, it is a group, and of exponent 2, a contradiction. So $|L/N_\lambda|$ is even so $2|N_\lambda| \,\big|\, |L|$, so $|a| \,\big|\, |L|$.

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