

Rate of convergence for certain optimal stopping problems

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Dedicated to the 100th anniversary of the birthday of Béla Györfi

Abstract. We prove that the rate of convergence of a solution of the optimal stopping problem for a Lévy process on an interval $[0, T]$ to that on the interval $[0, \infty)$ is exponential as $T \rightarrow \infty$.

1. Introduction

The first paper to deal with a stopping time of a Lévy process in the context we consider below is MORDECKI [3] where an explicit expression is found for an optimal stopping time for a reward functions of either $(X_t - K)^+$ or $(K - X_t)^+$. MORDECKI [3] found that the optimal stopping time is of a threshold type.

A new approach appeared in [5] where the Appel polynomials are applied for optimal stopping problems of the discussed type. An analogue of MORDECKI's [3] result had been obtained in [5] for the discrete Markov chains and for the reward functions $g(x) = (x^n)^+$, $n \in \mathcal{N}$. It is proved in [5] that the rate of convergence of the solution of the optimal stopping problem on a finite interval converges to that on the infinite interval $[0, \infty]$. We shall concentrate on a generalization of this result for a broad class of Lévy processes.

A generalization of the result of [5] for general Lévy-type processes and the reward function $g(x) = (x^n)^+$, $n \in \mathcal{N}$, can be found in [2]. The most general

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result up to now is obtained in [4]. An explicit form of the optimal stopping moment for the optimal stopping problem for homogeneous Lévy processes and the reward function $g(x) = (x^\eta)^+$, $\eta > 0$, are found in [5]. The optimal stopping moment is constructed in [5] by using the Appel polynomials. However the rate of convergence is not discussed in [5], at all.

In the current paper we find the rate of convergence of a solution of the optimal stopping problem for a Lévy process on an interval $[0, T]$ to that on the interval $[0, \infty)$ as $T \rightarrow \infty$. It turns out that the rate of convergence is exponential.

2. Lévy–Itô decomposition

For convenience, we recall the well known Lévy–Itô decomposition for Lévy processes.

Theorem 2.1 (Lévy–Itô decomposition). *Let X_t be a Lévy process. Then there exists a triplet of stochastic processes $X_t^{(1)}$, $X_t^{(2)}$, and $X_t^{(3)}$ such that*

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)}, \quad (2.1)$$

where $X_t^{(1)}$ is a Brownian motion with drift, $X_t^{(2)}$ a compound Poisson process, $X_t^{(3)}$ a square integrable pure jump martingale.

The compound Poisson process $X_t^{(2)}$ in 2.1 is usually constructed from a simple Poisson process. We will assume that the intensity $\lambda(t)$ of the simple Poisson process is such that

$$\sum_{m=1}^{\infty} \frac{\lambda(2^m)}{2^m} < \infty. \quad (2.2)$$

Another useful assumption we use below for the process $X_t^{(3)}$ of 2.1 is that

$$\int_1^\infty \frac{\mathbf{E} |X_t^{(3)}|^\eta}{e^{\eta q t}} dt < \infty. \quad (2.3)$$

for some $\eta > 0$ and $q > 1$.

3. Main result

Theorem 3.1 (Main result). *Fix $\eta > 0$ and $q > 1$. Let $(X_t, t \geq 0)$ be a Lévy process such that*

$$\mathbf{E}(X_t^+)^{\eta} < \infty$$

and that X_t admits the Lévy–Itô decomposition (2.1) without drift. Let $X_0 = x$.

Assume that the square integrable pure jump process $X_t^{(3)}$ in representation (2.1) satisfies condition (2.3).

We further assume that the compound Poisson process $X_t^{(2)}$ in representation (2.1) is such that

$$X_t^{(2)} = \sum_{k \leq N_t} \xi_k \quad (3.1)$$

where the random variables ξ_k , $k \geq 1$, are nonnegative, independent, identically distributed, and such that

$$\mathbf{E}\xi_k^{\eta \vee 1} < \infty \quad \text{for some } \eta > 0.$$

The symbol N_t in representation (3.1) stands for a simple Poisson process with intensity $\lambda(t)$ such that the process N_t and the sequence $\{\xi_k\}$ are independent. Moreover we assume that the intensity λ satisfies condition (2.2).

Let $T > 0$ and let $\mathcal{M}$ and $\mathcal{M}_T$ denote the sets of all stopping times $\tau \in [0, \infty]$ and $\tau \in [0, T]$, respectively. Let $g(x)$ denote the function $(x^+)^{\eta}$ and let

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}(e^{-q\tau} g(X_{\tau}) \mathcal{I}_{\{\tau < \infty\}}), \quad V(x, T) = \sup_{\tau \in \mathcal{M}_T} \mathbf{E}(e^{-q\tau} g(X_{\tau})).$$

Then there exist a number $T_0 > 0$, an universal constant $c > 0$, and, for a given real number x , there exists a constant $C(x)$ such that

$$0 \leq V(x) - V(x, T) \leq C(x)e^{-cT} \quad (3.2)$$

for all $T > T_0$.

Remark 1. Theorem 3.1 is a generalization of Theorem 3 of the paper [5].

4. Auxiliary results

The proof of Theorem 3.1 is based on several auxiliary results.

Lemma 1. *Let $(X_t, t \geq 0)$ be a process such that it can be decomposed into a sum $X_t = P_t + Q_t + R_t$. Let η , q , $g(x)$, $V(x)$, and $V(x, T)$ be defined as in Theorem 3.1.*

Then the conclusion of Theorem 3.1 holds if

$$\mathbf{E} \left(\sup_{s \geq t} \frac{|P_s|}{s^\theta} \right)^\eta < \infty, \quad \mathbf{E} \left(\sup_{s \geq t} \frac{|Q_s|}{s^\theta} \right)^\eta < \infty, \quad \mathbf{E} \left(\sup_{s \geq t} \frac{|R_s|}{s^\theta} \right)^\eta < \infty.$$

for some $\theta > 0$ and all $t > 0$.

PROOF. Note that $V(x) \geq V(x, T)$, since $\mathcal{M}_T \subset \mathcal{M}$. Now let τ^* be a positive root of the Appel polynomial constructed from the random variable $M_{\tau, q} = \sup_{0 \leq t < \tau} X_t$, where τ is a random variable such that $P\{\tau > t\} = e^{-tq}$. Then

$$\begin{aligned} V(x, T) &= \sup_{\tau \in \mathcal{M}} \mathbf{E}(e^{-q\tau} g(X_\tau)) \geq \mathbf{E}(g(X_{\min(\tau^*, T)}) e^{-q \min(\tau^*, T)}) \\ &\geq \mathbf{E}(g(X_{\min(\tau^*, T)}) e^{-q\tau^*} \mathcal{I}_{\{\tau^* \leq T\}}), \end{aligned}$$

since $\tau^* \wedge T \in \mathcal{M}_T$.

Thus

$$V(x) - V(x, T) \leq \mathbf{E}(g(X_{\tau^*}) e^{-q\tau^*} \mathcal{I}_{\{T < \tau^* < \infty\}}).$$

Since the function $e^{-qs} s^{\theta\eta}$ is decreasing on the semiaxis being far enough of the origin,

$$\begin{aligned} V(x) - V(x, T) &\leq \mathbf{E}(g(X_{\tau^*}) e^{-q\tau^*} \mathcal{I}_{\{T < \tau^* < \infty\}}) \leq \mathbf{E} \left[\sup_{s \geq T} \frac{(P_s + Q_s + R_s)^\eta}{e^{qs}} \right] \\ &\leq 2^{2\eta} \left(\mathbf{E} \left[\sup_{s \geq T} \frac{|P_s|^\eta}{e^{qs}} \right] + \mathbf{E} \left[\sup_{s \geq T} \frac{|Q_s|^\eta}{e^{qs}} \right] + \mathbf{E} \left[\sup_{s \geq T} \frac{|R_s|^\eta}{e^{qs}} \right] \right) \\ &= 2^{2\eta} \left(\mathbf{E} \left[\sup_{s \geq T} \frac{|P_s|^\eta}{s^{\theta\eta}} \cdot \frac{s^{\theta\eta}}{e^{qs}} \right] + \mathbf{E} \left[\sup_{s \geq T} \frac{|Q_s|^\eta}{s^{\theta\eta}} \cdot \frac{s^{\theta\eta}}{e^{qs}} \right] + \mathbf{E} \left[\sup_{s \geq T} \frac{|R_s|^\eta}{s^{\theta\eta}} \cdot \frac{s^{\theta\eta}}{e^{qs}} \right] \right) \\ &\leq 2^{2q} \cdot c' \cdot \frac{T^{\theta\eta}}{e^{qT}} \left(\mathbf{E} \left[\sup_{s \geq T} \frac{|P_s|}{s^\theta} \right]^\eta + \mathbf{E} \left[\sup_{s \geq T} \frac{|Q_s|}{s^\theta} \right]^\eta + \mathbf{E} \left[\sup_{s \geq T} \frac{|R_s|}{s^\theta} \right]^\eta \right). \end{aligned}$$

Thus (3.2) holds for an arbitrary $c < q$, sufficiently large T_0 , and appropriate $C(x)$. $\square$

Lemma 2 (Wiener process). *Let $\eta > 0$ and let $X_t^{(1)} = W_t$, $t > 0$, be a Wiener process. If $\theta > \frac{1}{2}$ and $T > 0$, then*

$$\mathbf{E} \left(\sup_{t \geq T} \frac{|W_t|}{t^\theta} \right)^\eta < \infty.$$

PROOF. It is known that if X is a nonnegative random variable, then

$$\mathbf{E}X^\eta = \eta \int_0^\infty x^{\eta-1} \mathbf{P}(X \geq x) dx.$$

Without loss of generality assume that $T = 1$. Using the latter formula we get

$$\begin{aligned} \mathbf{E} \left(\sup_{s \geq 1} \frac{|W_s|}{s^\theta} \right)^\eta &= \eta \int_0^\infty x^{\eta-1} \mathbf{P} \left(\sup_{s \geq 1} \frac{|W_s|}{s^\theta} \geq x \right) dx \\ &\leq \eta \int_0^\infty x^{\eta-1} \sum_{m=0}^\infty \mathbf{P} \left(\sup_{2^m \leq s \leq 2^{m+1}} \frac{|W_s|}{s^\theta} \geq x \right) dx \\ &\leq \eta \sum_{m=0}^\infty \int_0^\infty x^{\eta-1} \mathbf{P} \left(\sup_{2^m \leq s \leq 2^{m+1}} |W_s| \geq x 2^{m\theta} \right) dx. \end{aligned}$$

For any $y > 0$,

$$\mathbf{P} \left(\sup_{2^m \leq s \leq 2^{m+1}} |W_s| \geq y \right) \leq \mathbf{P} \left(\sup_{s \leq 2^{m+1}} |W_s| \geq y \right) = 2\mathbf{P}(|W_{2^{m+1}}| \geq y).$$

Thus

$$\begin{aligned} \mathbf{E} \left(\sup_{s \geq 1} \frac{|W_s|}{s^\theta} \right)^\eta &\leq 2\eta \sum_{m=0}^\infty \int_0^\infty x^{\eta-1} \mathbf{P}(|W_{2^{m+1}}| \geq x 2^{m\theta}) dx \\ &= 2\eta \sum_{m=0}^\infty \int_0^\infty \left(\frac{y}{2^{m\theta}} \right)^{\eta-1} \mathbf{P}(|W_{2^{m+1}}| \geq y) \frac{dy}{2^{m\theta}} \\ &= 2\eta \sum_{m=0}^\infty \frac{1}{2^{m\theta\eta}} \int_0^\infty y^{\eta-1} \mathbf{P}(|W_{2^{m+1}}| \geq y) dy \\ &= 2\eta \sum_{m=0}^\infty \frac{1}{2^{m\theta\eta}} \mathbf{E}|W_{2^{m+1}}|^\eta. \end{aligned} \tag{4.1}$$

Since W_t is a Gaussian random variable with zero mean and variance t , we have

$$\mathbf{E}|W_t|^\eta = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^\infty |x|^\eta e^{-x^2/2t} dx = \frac{t^{\eta/2}}{\sqrt{2\pi t}} \int_{-\infty}^\infty |x|^\eta e^{-x^2/2} dx = \kappa t^{\eta/2},$$

where

$$\kappa = \sqrt{\frac{2}{\pi}} \int_0^\infty x^\eta e^{-x^2/2} dx = 2^{\eta-\frac{1}{2}} \pi^{-\frac{1}{2}} \Gamma(\eta)$$

and where Γ is the gamma function. Thus

$$\sum_{m=0}^\infty \frac{1}{2^{m\theta\eta}} \mathbf{E}|W_{2^{m+1}}|^\eta = \kappa \sum_{m=0}^\infty \frac{2^{(m+1)/2}}{2^{m\theta\eta}} = \kappa 2^{\eta/2} \sum_{m=0}^\infty 2^{m\eta(\frac{1}{2}-\theta)} < \infty,$$

since $\theta > \frac{1}{2}$. This completes the proof. $\square$

Remark 2. The method used in the proof of Lemma 2 fits the case of the Wiener process with a drift $O(t^\theta)$, as well.

Lemma 3 (Simple Poisson process). *Let $\eta > 0$ and $q > 0$ and let $\Pi(t)$ be the simple Poisson process with intensity $\lambda(t)$. If*

$$\int_1^\infty e^{-\eta qt} \max\{\lambda(t), \lambda^\eta(t)\} dt < \infty,$$

then for every $T > 0$

$$\mathbf{E} \left(\sup_{t \geq T} \frac{\Pi(t)}{e^{qt}} \right)^\eta < \infty.$$

Lemma 3 implies the corresponding result for the difference of two Poisson process, that, in turn, allows one to consider the processes with both positive and negative jumps.

Corollary 1. *Let $\eta > 0$ and $p > 1/2$. Let $\Pi^1(t)$ and $\Pi^2(t)$ be two Poisson processes with intensities $\lambda_1(t) \rightarrow \infty$ and $\lambda_2(t) \rightarrow \infty$, respectively. If*

$$\int_1^\infty e^{-\eta qt} \lambda_i^\eta(t) dt < \infty, \quad i = 1, 2,$$

then for every $T > 0$

$$\mathbf{E} \left(\sup_{t \geq T} \left| \frac{\Pi^1(t) - \Pi^2(t)}{e^{qt}} \right| \right)^\eta < \infty.$$

In order to prove Lemma 3 we need both upper and lower bounds for moments of the Poisson distribution. The exact values of such moments can easily be evaluated for integer η , however this is not the case for non-integer η and thus we need to use the following estimates.

Lemma 4 (upper bound). *Let $\Pi \in Po(\lambda)$, $\lambda > 0$ and $\eta > 0$. Then there exists a constant $c > 0$, that does not depend on λ , such that*

$$\mathbf{E}(\Pi^\eta) \leq c\lambda^\eta$$

if $\lambda \geq 1$, and

$$\mathbf{E}(\Pi^\eta) \leq c\lambda$$

if $0 < \lambda < 1$.

Lemma 5 (lower bound). *Let $\Pi \in Po(\lambda)$, $\lambda > 0$ and $\eta > 0$. Then there exists a constant $c > 0$, that does not depend on λ , such that*

$$\mathbf{E}(\Pi^\eta) \geq c\lambda^\eta$$

if $\lambda \geq 1$, and

$$\mathbf{E}(\Pi^\eta) \geq c\lambda$$

if $0 < \lambda < 1$.

Although the constants in Lemmas 4 and 5 are denoted by the same symbol c , they are different, in fact.

First we show that Lemma 3 follows from Lemmas 4 and 5 and then prove Lemmas 4 and 5 themselves.

PROOF OF LEMMA 3. Without loss of generality assume that $T = 1$. We have

$$\begin{aligned} \mathbf{E} \left(\sup_{t \geq 1} \frac{\Pi(t)}{e^{qt}} \right)^\eta &\leq \sum_{k=1}^{\infty} \mathbf{E} \left(\sup_{k \leq t < k+1} \frac{\Pi(t)}{e^{qt}} \right)^\eta \\ &\leq \sum_{k=1}^{\infty} e^{-qk\eta} \mathbf{E}(\Pi^\eta(k+1)) \leq e^q \sum_{k=1}^{\infty} e^{-qk\eta} \mathbf{E}(\Pi^\eta(k)). \end{aligned}$$

Lemma 4 implies that

$$\mathbf{E} \left(\sup_{t \geq 1} \frac{\Pi(t)}{e^{qt}} \right)^\eta \leq ce^q \sum_{k=1}^{\infty} e^{-qk\eta} \max \{ \lambda(k), \lambda^\eta(k) \}.$$

Since

$$\begin{aligned} \int_1^\infty \frac{\max \{ \lambda(t), \lambda^\eta(t) \}}{e^{qt\eta}} dt &= \sum_{k=1}^{\infty} \int_k^{k+1} \frac{\max \{ \lambda(t), \lambda^\eta(t) \}}{e^{qt\eta}} dt \\ &\geq e^{-q} \sum_{k=1}^{\infty} \frac{\max \{ \lambda(k), \lambda^\eta(k) \}}{e^{qk\eta}}. \end{aligned}$$

Lemma 3 is proved. $\square$

PROOF OF LEMMA 4. Set $p_k = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, \dots$. First let us consider the case $0 < \lambda < 1$:

$$\mathbf{E}(\Pi^\eta) = \sum_{k=1}^{\infty} k^\eta p_k \leq \sum_{k=1}^{\infty} k^{[\eta]+1} p_k = \mathbf{E}(\Pi^{[\eta]+1}). \quad (4.2)$$

Let f denote the moment generating function of the Poisson distribution with parameter λ and let $f^{(i)}$ denote its derivative of order i . Then $f(t) = e^{\lambda(t-1)} = e^{-\lambda} \cdot e^{\lambda t}$. Thus $f^{(i)}(t) = \lambda^i \cdot f(t)$, whence $f^{(i)}(1) = \lambda^i$, $i \geq 1$. Since the moment of any order j is a linear combination of derivatives $f'(1), f''(1), \dots, f^{(j)}(1)$, the expectation $\mathbf{E}(\Pi^{[\eta]+1})$ is a linear combination of $\lambda, \lambda^2, \dots, \lambda^{[\eta]+1}$. Using the triangular inequality, we get $\mathbf{E}(\Pi^{[\eta]+1}) \leq c\lambda$ for some constant $c > 0$ if $\lambda < 1$, that, taking into account (4.2) proves the second part of Lemma 4.

Now let $\lambda \geq 1$. As before, set $m = [\lambda]$. If $0 < \eta < 1$, then

$$\begin{aligned} \mathbf{E}(\Pi^\eta) &= \sum_{k=0}^{\infty} k^\eta p_k = \sum_{1 \leq k \leq m} k^\eta p_k + \sum_{k > m} k^\eta p_k \leq m^\lambda + \sum_{1 \leq k \leq m} p_k + \sum_{k > m} k^\eta p_k \\ &= m^\lambda + \sum_{1 \leq k \leq m} p_k + \lambda \sum_{k > m} \frac{p_{k-1}}{k^{1-\eta}} \leq m^\lambda \sum_{1 \leq k \leq m} p_k + \frac{\lambda}{m^{1-\eta}} \sum_{k > m} p_{k-1} \\ &= m^\lambda \left(\sum_{1 \leq k \leq m} p_k + \sum_{k > m-1} p_k \right) = m^\lambda (1 + P(\Pi = m)) \leq 2m^\eta. \end{aligned} \quad (4.3)$$

Since $m \leq \lambda$, the first part of Lemma 4 is proved for all $0 < \eta < 1$. If $\eta \geq 1$, then

$$\mathbf{E}(\Pi^\eta) = \sum_{k=1}^{\infty} k^\eta p_k = \lambda \sum_{k=1}^{\infty} k^{\eta-1} p_{k-1} = \lambda \sum_{k=0}^{\infty} (k+1)^{\eta-1} p_k \leq 2^{\eta-1} \lambda \sum_{k=0}^{\infty} k^{\eta-1} p_k.$$

Continuing these estimations, we obtain

$$\mathbf{E}(\Pi^\eta) \leq d \lambda^{[\eta]} \sum_{k=0}^{\infty} k^{\eta-[\eta]} p_{k-1}, \quad d = 2^{(\eta-1)+(\eta-2)+\dots+(\eta-[\eta])}.$$

If $\eta \in \mathcal{N}$, then this inequality coincides with the statement of the first part of Lemma 4. If $\eta \notin \mathcal{N}$, we use lemma 4 for the case of $0 < \eta < 1$ and get

$$\mathbf{E}(\Pi^\eta) \leq d \lambda^{[\eta]} \cdot 2 \lambda^{\eta-[\eta]},$$

which completes the proof of Lemma 4. Thus Lemma 4 is proved. $\square$

PROOF OF LEMMA 5. Set $p_k = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, \dots$. First consider the case of $0 < \lambda < 1$:

$$\mathbf{E}(\Pi^\eta) = \sum_{k=0}^{\infty} k^\eta p_k > p_1 = e^{-\lambda} \lambda \geq \frac{\lambda}{e},$$

that proves the second part of Lemma 5. Now let $\lambda \geq 1$. Set $m = [\lambda]$. Starting from the case $0 < \eta < 1$:

$$\begin{aligned} \mathbf{E}(\Pi^\eta) &= \sum_{k=0}^{\infty} k^\eta p_k = \sum_{1 \leq k \leq m} k^\eta p_k + \sum_{k > m} k^\eta p_k = \lambda \sum_{1 \leq k \leq m} \frac{p_{k-1}}{k^{1-\eta}} + \sum_{k > m} k^\eta p_k \\ &\geq \frac{\lambda}{m^{1-\eta}} \sum_{1 \leq k \leq m} p_{k-1} + m^\eta \sum_{k > m} p_k = \frac{\lambda}{m^{1-\eta}} \sum_{0 \leq k \leq m-1} p_k + m^\eta \sum_{k > m} p_k \end{aligned}$$

$$\begin{aligned}
&\geq m^\eta \left(\sum_{0 \leq k \leq m-1} p_k + \sum_{k > m} p_k \right) = m^\eta (1 - P(\Pi = m)) \\
&\geq 2^{-\eta} \lambda^\eta (1 - \mathbf{P}(\Pi = m)).
\end{aligned} \tag{4.4}$$

We show that there exists a constant $c > 0$ such that

$$1 - \mathbf{P}(\Pi = m) \geq c \tag{4.5}$$

if $\lambda \geq 1$.

Using Stirling's formula:

$$\mathbf{P}(\Pi = m) = e^{-\lambda} \frac{\lambda^m}{m!} = e^{-\lambda} \frac{\lambda^m}{\sqrt{2\pi m} \cdot m^m \cdot e^{-m+\theta_m}},$$

where $0 < \theta_m < \frac{1}{12m}$. Since

$$\left(\frac{\lambda}{m}\right)^m \leq \left(\frac{m+1}{m}\right)^m = \left(1 + \frac{1}{m}\right)^m \leq e, \quad e^{-\lambda+m+\theta_m} \leq 1,$$

we have

$$\mathbf{P}(\Pi = m) \leq \frac{e}{\sqrt{2\pi m}} \leq \frac{e}{\sqrt{4\pi}} < 1, \quad m \geq 2$$

If $1 \leq \lambda \leq 2$, then

$$\mathbf{P}(\Pi = m) = \mathbf{P}(\Pi = 1) = e^{-\lambda} \lambda < 1.$$

This implies (4.5). Inequality (4.5) proves Lemma 5 for $0 < \eta < 1$. In order to complete the proof of Lemma 5, consider the case of $\eta \geq 1$:

$$\begin{aligned}
\mathbf{E}(\Pi^\eta) &= \sum_{k=1}^{\infty} k^\eta p_k = \lambda \sum_{k=1}^{\infty} k^{\eta-1} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\
&= \lambda \sum_{k=0}^{\infty} (k+1)^{\eta-1} \frac{\lambda^k}{k!} e^{-\lambda} \geq \lambda \sum_{k=0}^{\infty} k^{\eta-1} \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \mathbf{E}(\Pi^{\eta-1}).
\end{aligned}$$

Continuing with these estimates, we obtain

$$\mathbf{E}(\Pi^\eta) \geq \lambda^{[\eta]} \sum_{k=0}^{\infty} k^{\eta-[\eta]} \frac{\lambda^k}{k!} e^{-\lambda}.$$

This inequality coincides with the second part of Lemma 5 if $\eta \in \mathcal{N}$. For $\eta \notin \mathcal{N}$, we use Lemma 5 for $\eta < 1$:

$$E(\Pi^\eta) \geq \lambda^{[\eta]} \cdot c \lambda^{\eta-[\eta]}.$$

Note that constant c is the same as in the case of $0 < \eta < 1$, that is, it does not depend on η . Thus, Lemma 5 is proved. $\square$

Lemma 6 (compound Poisson process). *Let $X_t^{(2)}$ be a compound Poisson process represented in the form of (3.1) where N_t is a simple Poisson process whose intensity satisfies (2.2). We also assume that the random variables ξ_k , $k \geq 1$, are independent, identically distributed, and such that*

$$\mathbf{E}\xi_k^{\eta \vee 1} < \infty$$

for some $\eta > 0$. We further assume that the process N_t and the sequence ξ_k , $k \geq 1$, are independent. Then

$$\mathbf{E} \left(\sup_{s \geq t} \frac{|X_s^{(2)}|}{s} \right)^\eta < \infty.$$

for all $t > 0$.

PROOF. We provide the proof for the case of $\eta = 1$. Other cases are proved similarly. Put $\mu = \mathbf{E}\xi_1$. Then

$$\begin{aligned} \mathbf{E} \left(\sup_{t \geq 1} \frac{1}{t} \sum_{k \leq N_t} \xi_k \right) &= \int_0^\infty \mathbf{P} \left(\sup_{t \geq 1} \frac{1}{t} \sum_{k \leq N_t} \xi_k \geq x \right) dx \\ &\leq \int_0^\infty \sum_{m=0}^\infty \mathbf{P} \left(\sup_{2^m \leq t \leq 2^{m+1}} \frac{1}{t} \sum_{k \leq N_t} \xi_k \geq x \right) dx \\ &\leq \int_0^\infty \sum_{m=0}^\infty \mathbf{P} \left(\frac{1}{2^m} \sum_{k \leq N_{2^{m+1}}} \xi_k \geq x \right) dx \\ &= \int_0^\infty \sum_{m=0}^\infty \sum_{l=0}^\infty \mathbf{P}(N_{2^{m+1}} = l) \mathbf{P} \left(\frac{1}{2^m} \sum_{k \leq l} \xi_k \geq x \right) dx \\ &= \sum_{m=0}^\infty \sum_{l=0}^\infty \mathbf{P}(N_{2^{m+1}} = l) \mathbf{P} \left(\frac{1}{2^m} \sum_{k \leq l} \xi_k \geq x \right) dx \\ &= \sum_{m=0}^\infty \sum_{l=0}^\infty \mathbf{P}(N_{2^{m+1}} = l) \mathbf{E} \left[\frac{1}{2^m} S_l \right] \\ &= \mu \sum_{m=0}^\infty \sum_{l=0}^\infty \mathbf{P}(N_{2^{m+1}} = l) \frac{l}{2^m} \\ &= \mu \sum_{m=0}^\infty \frac{1}{2^m} \sum_{l=0}^\infty l \mathbf{P}(N_{2^{m+1}} = l) = \mu \sum_{m=0}^\infty \frac{1}{2^m} \mathbf{E} N_{2^{m+1}} \\ &= \mu \sum_{m=0}^\infty \frac{\lambda(2^{m+1})}{2^m} \mathbf{E} N_{2^{m+1}} = 2\mu \sum_{m=1}^\infty \frac{\lambda(2^m)}{2^m} < \infty. \quad \square \end{aligned}$$

Lemma 7 (martingale). *Let Y_t be a stochastic process such that $|Y_t|$ is a right continuous submartingale. Let $q > 0$, $\eta > 1$, $T > 0$. If (2.3) holds, then*

$$\mathbf{E} \left(\sup_{t \geq T} \frac{|Y_t|}{e^{qt}} \right)^\eta < \infty$$

for all $T > 0$.

Remark 3. Our assumption that $|Y_t|$ is a right continuous submartingale is weaker than the assumption that Y_t is a right continuous submartingale and, moreover, that Y_t is a martingale.

The following two properties are well known for submartingales. Namely, if Y_t is submartingale and $\mathbf{E}|Y_t|^\eta < \infty$ for some $\eta > 1$, then

$$\mathbf{E}|Y_t|^\eta \text{ is nondecreasing in } t. \quad (4.6)$$

Lemma 8 ([1], p. 140, Theorem 6.2.16). *Let Y_t , $t \geq 0$, be a right continuous submartingale. Let A be a certain subset of real numbers and let $Y^*(\omega) = \sup_{t \in A} Y_t(\omega)$. If $p > 1$, then $Y^* \in \mathbf{L}_p$ if and only if*

$$\sup_{t \in A} \|Y_t\|_{\mathbf{L}_p} < \infty.$$

In particular, if $\frac{1}{r} = 1 - \frac{1}{p}$, then

$$\|Y^*\|_{\mathbf{L}_p} \leq r \sup_{t \in A} \|Y_t\|_{\mathbf{L}_p}.$$

In fact, we only need the following particular case of Lemma 8 corresponding to the case of $A = [k, k+1]$ and for $|X_t^{(3)}|$ instead of Y_t :

$$\mathbf{E} \left(\sup_{k \leq t \leq k+1} |X_t^{(3)}| \right)^\eta \leq \left(1 - \frac{1}{\eta} \right)^{-\eta} \mathbf{E} |X_{k+1}^{(3)}|^\eta. \quad (4.7)$$

PROOF OF LEMMA 7. Without loss of generality we assume that $T = 1$. It follows from (4.7) that

$$\begin{aligned} \mathbf{E} \left(\sup_{t \geq 1} \frac{|Y_t|}{e^{qt}} \right)^\eta &\leq \sum_{k=1}^{\infty} \mathbf{E} \left(\sup_{k \leq t \leq k+1} \frac{|Y_t|}{e^{qt}} \right)^\eta \leq \sum_{k=1}^{\infty} e^{-qk\eta} \mathbf{E} \left(\sup_{k \leq t \leq k+1} |Y_t| \right)^\eta \\ &\leq \left(1 - \frac{1}{\eta} \right)^{-\eta} \sum_{k=1}^{\infty} e^{-qk\eta} \mathbf{E} |Y_{k+1}|^\eta \\ &\leq \left(1 - \frac{1}{\eta} \right)^{-\eta} e^{2q\eta} \sum_{k=1}^{\infty} e^{-q(k+1)\eta} \mathbf{E} |Y_k|^\eta \end{aligned}$$

$$\begin{aligned}
&\leq \left(1 - \frac{1}{\eta}\right)^{-\eta} e^{2q\eta} \sum_{k=1}^{\infty} \int_k^{k+1} \frac{\mathbf{E}|Y_t|^\eta}{e^{qt\eta}} dt \\
&= \left(1 - \frac{1}{\eta}\right)^{-\eta} e^{2q\eta} \int_1^{\infty} \frac{\mathbf{E}|Y_t|^\eta}{e^{qt\eta}} dt < \infty. \quad \square
\end{aligned}$$

Remark 4. Lemma 7 can also be proved for the case of $\eta = 1$. However the condition for this case is as follows

$$\int_1^{\infty} \frac{\mathbf{E}|Y_t| \ln^+ |Y_t|}{e^{qt}} dt < \infty$$

where $\ln^+ z = \ln(1 + z)$ for $z \geq 0$. The idea of the proof remains the same, but another Doob's inequality applies.

5. Proof of Theorem 3.1

First we write down the Lévy–Itô decomposition (2.1). Then we put $P_s = X_s^{(1)}$, $Q_s = X_s^{(2)}$, and $R_s = X_s^{(3)}$. The assumptions of Lemma 1 hold for P_s , Q_s , and R_s by Lemmas 2, 6, and 7, respectively. Therefore Theorem 3.1 follows from Lemma 1.

References

- [1] R. J. ELLIOTT and P. E. KOPP, *Mathematics of Financial Markets*, 2nd edition, *Springer, Berlin*, 2005.
- [2] A. E. KYPRIANOU and B. A. SURYA, On the Novikov-Shiryaev optimal stopping problems in continuous time, *Electron. Comm. Probab.* **10** (2005), 146–151.
- [3] E. MORDECKI, Optimal stopping and perpetual options for Lévy processes, *Finance Stoch.* **6**, no. 4 (2002), 473–493.
- [4] A. NOVIKOV and A. SHIRYAEV, On a solution of the optimal stopping problem for processes with independent increments, *Stochastics* **79**, no. 3–4 (2007), 393–406.
- [5] A. A. NOVIKOV and A. N. SHIRYAEV, On an effective solution of the optimal stopping problem for random walks, *Theory Probab. Appl.* **49**, no. 2 (2004), 344–354.

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