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# On $f(p) + f(q) = f(p + q)$ for all odd primes $p$ and $q$

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**Abstract.** We characterize all nonvanishing multiplicative functions $f$ for which $f(p) + f(q) = f(p + q)$ for all odd primes $p$, $q$. As a corollary, a multiplicative function $f$ is the identity function if and only if $f(3) = 3$ and $f(p) + f(q) = f(p + q)$ for all odd primes $p$, $q$. Two questions posed by Claudia A. Spiro in 1992 are answered negatively. Two new conjectures are posed.

## 1. Introduction

Let $E$ be a subset of positive integers, and let $S$ be a set of arithmetic functions. If there is exactly one element $f(n)$ of $S$ which satisfies

$$f(m + n) = f(m) + f(n) \quad \text{for all } m, n \in E,$$

then we call $E$ an *additive uniqueness set of* $S$. Many authors have been interested in characterizing the identity function (see [1], [2], [3]).

C. A. Spiro [4] proved that the set of primes is an additive uniqueness set for $\{f$ multiplicative, $f(p_0) \neq 0$ for some prime $p_0\}$. At the end of the paper, C. A. Spiro asked if the set of all sufficiently large primes is an additive uniqueness set for $\{f$ multiplicative, $f(p_0) \neq 0$ for some prime $p_0\}$. Is there a subset of the primes having positive lower density in the set of primes, which is not an additive uniqueness set for $\{f$ multiplicative, nonvanishing$\}$?

Let $g_1(n) = n$ for all positive integers $n$ and $g_2(n) = 1$ for $2 \nmid n$ and $g_2(n) = 2$ for $2 \mid n$. It is easy to verify that both $g_1$ and $g_2$ are multiplicative, nonvanishing

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and $g_i(m+n) = g_i(m) + g_i(n)$ for arbitrarily odd numbers $m$, $n$. Hence the set of all positive odd numbers is not an additive uniqueness set for the set of nonvanishing multiplicative functions. Thus the set of all odd primes is not an additive uniqueness set for the set of nonvanishing multiplicative functions. This gives negative answers to the above Spiro's questions.

In this paper we characterize all nonvanishing multiplicative functions $f$ for which $f(p) + f(q) = f(p+q)$ for all odd primes $p$, $q$. That is,

**Theorem 1.** *Let $f$ be a multiplicative function such that there exists an odd prime $p_0$ at which $f$ does not vanish. If*

$$f(p) + f(q) = f(p+q) \quad \text{for all odd primes } p, q,$$

*then either $f = g_1$ or $f = g_2$.*

An immediate result is

**Corollary.** *A multiplicative function $f$ is the identity function if and only if $f(3) = 3$ and $f(p) + f(q) = f(p+q)$ for all odd primes $p$, $q$.*

*Remark.* For two numbers $a, b$, let $f(1) = 1$, $f(2) = a$, $f(4) = b$ and $f(n) = 0$ for all $n \neq 1, 2, 4$. Thus there are infinitely many multiplicative functions $f$ with $f(2) \neq 0$ such that $f(p) + f(q) = f(p+q)$ for all odd primes $p$, $q$. Conversely, we pose the following conjecture

*Conjecture 1.* If $f$ is a multiplicative function such that $f(2) \neq 0$, $f(3) = 0$ and $f(p) + f(q) = f(p+q)$ for all odd primes $p, q$, then $f(n) = 0$ for all $n \geq 5$.

*Remark.* If $f$ satisfies the conditions of Conjecture 1, then $f(p) = 0$ for all primes $p \geq 5$ (see the proof of Theorem 2. Thus by induction on $n$ we can prove that the Goldbach conjecture implies Conjecture 1. This implies that if Conjecture 1 is false, then the Goldbach conjecture is false.

## 2. Proof of Theorem 1

We prove the following Theorem 2. This form is convenient in the proof. Then by using Theorem 2 we give a proof of Theorem 1.

**Theorem 2.** *Let $f$, $g$ be two multiplicative functions such that there exists an odd prime $p_0$ at which $f$ does not vanish. If*

$$f(p) + f(q) = f(p+q), \quad g(p) + g(q) = g(p+q) \qquad \text{for all odd primes } p,\ q$$

*and $f(n) = g(n)$ for $n = 2, 3, 4$, then $f(n) = g(n)$ for all positive integers $n$.*

In the following $p, q$ always denote primes.

**Lemma 1.** *Let $f$, $g$ be as in Theorem 2. Then*

(i) *$f(n)=g(n)$ for all $n\le 18$;*

(ii) *if $f(3)\neq 0$, then either $f(n)=g_1(n)$ for $n=1,2,3,4$ or $f(n)=g_2(n)$ for $n=1,2,3,4$.*

PROOF. By $f(2)f(p_0)=f(p_0+p_0)=2f(p_0)$ and $f(p_0)\neq 0$ we have $f(2)=2$. By $f(2)=f(2)f(1)$ we have $f(1)=1$.

Now we write $f(n)$ as a polynomial of $f(3)$ and $f(4)$.

Since

$$2f(5)=f(10)=f(3)+f(7),\qquad f(4)f(3)=f(12)=f(5)+f(7),$$

we have

$$2f(5)-f(3)=f(7)=f(4)f(3)-f(5).$$

So

$$f(5)=\frac{1}{3}f(3)(f(4)+1),\quad f(7)=2f(5)-f(3)=\frac{1}{3}f(3)(2f(4)-1),$$

$$f(10)=2f(5)=\frac{2}{3}f(3)(f(4)+1).$$

Thus

$$f(6)=2f(3),\quad f(8)=f(3)+f(5)=\frac{1}{3}f(3)(f(4)+4),\quad f(12)=f(3)f(4),$$

$$f(14)=2f(7)=\frac{2}{3}f(3)(2f(4)-1),\quad f(11)=f(14)-f(3)=\frac{1}{3}f(3)(4f(4)-5),$$

$$f(15)=f(3)f(5)=\frac{1}{3}(f(3))^2(f(4)+1),$$

$$f(16)=f(11)+f(5)=\frac{1}{3}f(3)(5f(4)-4),$$

$$f(13)=f(16)-f(3)=\frac{1}{3}f(3)(5f(4)-7),$$

$$f(18)=f(11)+f(7)=2f(3)(f(4)-1),$$

$$f(9)=f(18)/2=f(3)(f(4)-1).$$

By $f(20)=f(13)+f(7)=f(3)+f(17)$ we have

$$f(17)=f(13)+f(7)-f(3)=\frac{1}{3}f(3)(7f(4)-11).$$

Note that $g(2)=f(2)=2$, similarly we have $g(1)=1$ and for each $5\le n\le 18$, $g(n)$ is a polynomial of $g(3)$ and $g(4)$ which is the same polynomial as $f(n)$ being a polynomial of $f(3)$ and $f(4)$. Since $f(3)=g(3)$ and $f(4)=g(4)$, we have

$f(n) = g(n)$ for all $n \le 18$.

By $f(4)f(5) = f(20) = f(3) + f(17)$ we have

$$\frac{1}{3}f(3)f(4)(f(4)+1) = \frac{1}{3}f(3)(7f(4)-8).$$

If $f(3) \ne 0$, then $f(4) = 2$ or 4. By $f(3)f(8) = f(24) = f(11) + f(13)$ we have

$$\frac{1}{3}(f(3))^2(f(4)+4) = \frac{1}{3}f(3)(9f(4)-12).$$

Hence

$$f(3) = \frac{9f(4)-12}{f(4)+4} = \begin{cases} 1, & if\ f(4) = 2, \\ 3, & if\ f(4) = 4. \end{cases}$$

This completes the proof of Lemma 1. □

Motivated by Lemma 1 we state the following conjecture:

*Conjecture 2.* If $f$ is a multiplicative function such that $f(2) = 2$ and $f(p) + f(q) = f(p+q)$ for all odd primes $p$, $q$, then

$$f(2n) = \frac{1}{3}f(3)((n-3)f(4) + 12 - 2n),$$

$$f(2n-1) = \frac{1}{3}f(3)((n-2)f(4) + 7 - 2n), \quad n = 3, 4, \ldots .$$

**Lemma 2.** *Let $f$, $g$ be as in Theorem 2. If every even number $2m$ with $4 \le 2m \le 2N$ can be written as the sum of two primes, then $f(n) = g(n)$ for all $n \le N$.*

Proof is similar to [4, Lemma 3]. We omit the proof.

**Lemma 3.** *For every prime $p > 10^{10}$ there exist at least $3 \times 10^7$ primes $q < p$ such that $p + q \in H$, where*

$$H = \{n : v_p(n) \le 1 \text{ if } p > 1000;\ p^{v_p(n)+1} < 10^9 \text{ if } p < 1000\}$$

*and $v_p(n)$ denotes the exponent on $p$ in the prime factorization of $n$.*

Proof. Following the proof of [4, Lemma 5] we have the number $N(p)$ of primes $q < p$ with $p + q \in H$ satisfies

$$N(p) \ge 0.3\frac{p-1}{\log(p-1)} \ge 0.3 \times \frac{10^{10}}{10\log 10} > 3 \times 10^7.$$

This completes the proof of Lemma 3. □

**Lemma 4** ([4, Lemma 6]). *Almost every even positive integer is expressible as the sum of two primes.*

**Lemma 5** ([4, Lemma 7]). *For any positive integer $n$, put*

$$H_n = \{mn : m \in H,\ (m,n)=1\} \quad \text{if } 2 \mid n;$$
$$H_n = \{2mn : 2m \in H,\ (m,n)=1\} \quad \text{if } 2 \nmid n.$$

*Then $H_n$ satisfies the following properties:*

(i) *Every element of $H_n$ is even.*

(ii) *The set $H_n$ has positive lower density.*

**Lemma 6** ([4, Lemma 8]). *If $n \in H$, then every positive divisor of $n$ must lie in $H$.*

**Lemma 7.** *Let $f$, $g$ be as in Theorem 2. Then $f(n)=g(n)$ for all $n \in H$.*

Proof. Since the Goldbach conjecture is true for $n \le 2\times 10^{10}$, we have $f(n)=g(n)$ for all $n \le 10^{10}$ from Lemma 2. Now let $n \in H$ with $n \ge 10^{10}$, and assume that $f(m)=g(m)$ for all $m<n$ with $m \in H$. If $n$ is not a prime power, then $n=mk$ with $m>1, k>1$ and $(m,k)=1$. By Lemma 6 we have $m,k \in H$. Thus $f(n)=f(m)f(k)=g(m)g(k)=g(n)$. If $n$ is a prime power, then, by $n \in H$ and $n \ge 10^{10}$, $n$ must be prime. By Lemma 3 there exists an odd prime $q<n$ with $n+q \in H$. If $n+q$ is a prime power, then, by $n+q \in H$ and $n+q \ge 10^{10}$, $n+q$ must be prime. This contradicts that both $n$ and $q$ are odd primes. If $n+q$ is not a prime power, then $n+q=m'k'$ with $m'>1$, $k'>1$ and $(m',k')=1$. By Lemma 6 we have $m',k' \in H$. Since $q<n$, we have $m'<n$ and $k'<n$. Thus $f(n)+f(q)=f(n+q)=f(m')f(k')=g(m')g(k')=g(n+q)=g(n)+g(q)$. Since $q$ is prime, we have $q \in H$. By $q<n$ we have $f(q)=g(q)$. Hence $f(n)=g(n)$. This completes the proof of Lemma 7. □

Proof of Theorem 2. Assume that $f(3)=0$. Define a multiplicative function $g_3$ by $g_3(2)=f(2)$, $g_3(3)=0$, $g_3(4)=f(4)$, $g_3(n)=0$ $(n \ge 5)$. Then $g_3(p)+g_3(q)=g_3(p+q)$ for all odd primes $p$, $q$. By Lemma 7 we have $f(n)=g_3(n)$ for all $n \in H$. Since $p_0 \in H$, we have $f(p_0)=g_3(p_0)=0$, a contradiction. Hence $f(3) \ne 0$. By Lemma 1 there exists $i \in \{1,2\}$ such that $f(n)=g_i(n)$ for $n=1,2,3,4$. By Lemma 7 we have $f(n)=g_i(n)$ for all $n \in H$. If Theorem 2 is false, let $n$ be the minimal counter-example. For each $k \in H_n$, let $k=nm$, we have $m \in H$, $(n,m)=1$, $2|k$ and then $f(m)=g_i(m)$. Since every

prime lies in $H$, we have $f(p)=g_i(p)$ for all primes $p$. Thus, if $k$ is the sum of two odd primes, then $f(k)=g_i(k)$. By $g_i(m)\neq 0$ and

$$g_i(n)g_i(m)=g_i(k)=f(k)=f(n)f(m)=f(n)g_i(m)$$

we have $f(n)=g_i(n)$, a contradiction. Hence no element of $H_n$ is the sum of two odd primes. By Lemma 4, $H_n$ has zero density. This contradicts Lemma 5. Since $g(n)=f(n)=g_i(n)$ for $n=1,2,3,4$, the same arguments give $g(n)=g_i(n)$ for all $n$. This completes the proof of Theorem 2. □

Proof of Theorem 1. By the initial part of the proof of Theorem 2 we have $f(3)\neq 0$. By Lemma 1 there exists $i\in\{1,2\}$ such that $f(n)=g_i(n)$ for $n=1,2,3,4$. By Theorem 2 we have $f=g_1$ or $f=g_2$. This completes the proof of Theorem 1. □

## References

[1] P. V. Chung and B. M. Phong, Additive uniqueness sets for multiplicative functions, *Publ. Math. Debrecen* **55** (1999), 237–243.

[2] J. M. De Koninck, I. Kátai and B. M. Phong, A new characteristic of the identity function, *J. Number Theory* **63** (1997), 325–338.

[3] B. M. Phong, A characterization of the identity function with an equation of Hosszú type, *Publ. Math. Debrecen* **69** (2006), 219–226.

[4] Claudia A. Spiro, Additive uniqueness sets for arithmetic functions, *J. Number Theory* **42** (1992), 232–246.

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