Publ. Math. Debrecen
76/4 (2010), 465–479

On genuine q-Bernstein–Durrmeyer operators

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Abstract. In the present paper, we introduce genuine q-Bernstein–Durrmeyer operators and estimate the rate of convergence for continuous functions in terms of modulus of continuity. Furthermore we study some direct results for the genuine q-Bernstein–Durrmeyer operators.

1. Introduction

Let $q > 0$. For any $n \in N \cup \{0\}$, the q-integer $[n] = [n]_q$ is defined by

$$[n] := 1 + q + \cdots + q^{n-1}, \quad [0] := 0;$$

and the q-factorial $[n]! = [n]_q!$ by

$$[n]! := [1][2] \ldots [n], \quad [0]! := 1.$$

For integers $0 \le k \le n$, the q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

Define

$$(1-x)_q^n := \prod_{s=0}^{n-1} (1 - q^s x), \quad (1-x)_q^\infty := \prod_{s=0}^{\infty} (1 - q^s x),$$

Mathematics Subject Classification: 41A25, 41A35.
Key words and phrases: q-Durrmeyer operators; q-Bernstein polynomials; modulus of continuity; local and global approximation.
The research is supported by the Research Advisory Board of Eastern Mediterranean University under Project BAP-A-08-04.

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix} x^k(1-x)_q^{n-k}, \quad p_{n,n}(q;x) = x^n,$$

$$p_{\infty,k}(q;x) := \frac{x^k}{(1-q)^k[k]!}(1-x)_q^{\infty},$$

$$b_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{k(k-1)}{2}} x^k(1-x)^{n-k}}{(1-x+qx)\dots(1-x+q^{n-1}x)}.$$

The q-analogue of integration in the interval $[0, A]$ (see [10]) is defined by

$$\int_0^A f(t)d_qt := A(1-q)\sum_{n=0}^{\infty} f(Aq^n)\, q^n, \quad 0 < q < 1.$$

In the last two decades interesting generalizations of Bernstein polynomials were proposed by Lupaş [11]

$$R_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) b_{n,k}(q;x)$$

and by Phillips [17]

$$B_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) p_{n,k}(q;x).$$

The q-Bernstein polynomials quickly gained the popularity, see [6]–[9], [12]–[14], [16]–[24]. A comprehensive review of the results on this class along with extensive bibliography is given in [13]. To approximate continuous functions, V. Gupta and H. Wang [7] defined the q-Durrmeyer type operators as

$$M_{n,q}(f;x) := f(0)p_{n,0}(q;x) + [n+1]\sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x)\int_0^1 p_{n,k-1}(q;qt)f(t)d_qt,$$

and studied estimation of the rate of convergence for continuous functions in terms of modulus of continuity. In [8], the authors studied some direct local and global approximation theorems for the q-Durrmeyer operators $M_{n,q}$ for $0 < q < 1$. Some other analogues of the Bernstein–Durrmeyer operators related to the q-Bernstein basis functions $p_{n,k}(q;x)$ have been studied by M. M. Derriennic [2] and V. Gupta [6].

Motivation for this work are [6]–[8] and in this paper we introduce the following so called genuine q-Phillips–Durrmeyer and genuine q-Lupaş–Durrmeyer operators.

Definition 1. For $f \in C[0,1]$, we define the following q-Phillips–Durrmeyer operator:

$$
\begin{aligned}
U_{n,q}(f;x) &:= f(0)p_{n,0}(q;x) + f(1)p_{n,n}(q;x) \\
&\quad + [n-1]\sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q;x) \int_0^1 p_{n-2,k-1}(q;qt) f(t)\, d_q t, \qquad (1)
\end{aligned}
$$

where for $n = 1$ the sum is empty, i.e., equal to 0.

Definition 2. For $f \in C[0,1]$, we define the following q-Lupaş–Durrmeyer operator:

$$
\begin{aligned}
R^*_{n,q}(f;x) &:= f(0)b_{n,0}(q;x) + f(1)b_{n,n}(q;x) \\
&\quad + [n-1]\sum_{k=1}^{n-1} q^{1-k} b_{n,k}(q;x) \int_0^1 p_{n-2,k-1}(q;qt) f(t)\, d_q t, \qquad (2)
\end{aligned}
$$

where for $n = 1$ the sum is equal to 0.

Classical genuine Bernstein–Durrmeyer operators were independently introduced by W. Chen [1] in 1987, and by T. N. T. Goodman & A. Sharma [5] later in 1991 and investigated by many authors, see for example [15], [4]. They possess many interesting properties, in particular they reproduce linear functions and thus interpolates every function $f \in C[0,1]$ at 0 and 1.

In the present paper, we study some approximation properties of the genuine q-Phillips–Durrmeyer operators $\{U_{n,q}(f)\}$ and q-Lupaş–Durrmeyer operators $\{R^*_{n,q}(f;x)\}$ defined by (1) and (2) respectively, for $0 < q < 1$. We estimate the rate of convergence for these operators and investigate the local and global direct approximation properties of $U_{n,q}$ and $R^*_{n,q}$.

2. Estimation of moments for $U_{n,q}$

In this section we obtain explicit formula for $U_{n,q}(t^m;x)$ for $m = 0,1,2$ and $U_{n,q}((t-x)^2;x)$.

Lemma 3. *We have*

$$
U_{n,q}(1;x) = 1, \quad U_{n,q}(t;x) = x,
$$

$$
U_{n,q}(t^2;x) = \frac{(1+q)x(1-x)}{[n+1]} + x^2,
$$

$$
U_{n,q}((t-x)^2;x) = \frac{(1+q)x(1-x)}{[n+1]} \le \frac{2}{[n+1]}x(1-x).
$$

PROOF. Note that for $s = 0, 1, \ldots,$ by the definition of q-Beta function (see [10]) we have

$$\int_0^1 t^s p_{n-2,k-1}(q;qt)d_q t = \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} q^{k-1} \int_0^1 t^{k+s-1}(1-qt)_q^{n-1-k} d_q t$$
$$= \frac{q^{k-1}[n-2]![k+s-1]!}{[k-1]![n+s-1]!}. \tag{3}$$

In order to prove the theorem we shall use the following identities (see [16]):

$$\sum_{k=0}^{n} p_{n,k}(q;x) = 1, \quad \sum_{k=0}^{n} p_{n,k}(q;x)\frac{[k]}{[n]} = x, \quad \sum_{k=0}^{n} p_{n,k}(q;x)\frac{[k]^2}{[n]^2} = x^2 + \frac{x(1-x)}{[n]}.$$

Using Definition 1 and (3) it is easily seen that $U_{n,q}(1;x) = 1$ and using the above identities we have $U_{n,q}(t;x) = x$ and

$$\begin{aligned} U_{n,q}(t^2;x) &= p_{n,n}(q;x) + [n-1]\sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q;x)\int_0^1 t^2 p_{n-2,k-1}(q;qt)d_q t \\ &= p_{n,n}(q;x) + \sum_{k=1}^{n-1} p_{n,k}(q;x)\frac{[k]+q[k]^2}{[n][n+1]} \\ &= p_{n,n}(q;x) + \frac{1}{[n+1]}\sum_{k=0}^{n-1}\frac{[k]}{[n]}p_{n,k}(q;x) + \frac{q[n]}{[n+1]}\sum_{k=0}^{n-1}\frac{[k]^2}{[n]^2}p_{n,k}(q;x) \\ &= p_{n,n}(q;x) + \frac{1}{[n+1]}(x - p_{n,n}(q;x)) \\ &\quad + \frac{q[n]}{[n+1]}\left(x^2 + \frac{x(1-x)}{[n]} - p_{n,n}(q;x)\right) \\ &= \frac{(1+q)x(1-x)}{[n+1]} + x^2. \end{aligned}$$

Lemma is proved. □

Lemma 4. *$U_{n,q}(t^m;x)$ is a polynomial of degree less than or equal to* $\min(m,n)$.

PROOF. Simple calculations shows that

$$U_{n,q}(t^m;x) = [n-1]\sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q;x)\int_0^1 p_{n-2,k-1}(q;qt)t^m d_q t + p_{n,n}(q;x)$$

$$
\begin{aligned}
&= [n-1]\sum_{k=1}^{n-1} p_{n,k}(q;x)\frac{[n-2]![k+m-1]!}{[k-1]![n+m-1]!} + p_{n,n}(q;x)\\
&= \frac{[n-1]!}{[n+m-1]!}\sum_{k=1}^{n-1} p_{n,k}(q;x)\frac{[k+m-1]!}{[k-1]!} + p_{n,n}(q;x)\\
&= \frac{[n-1]!}{[n+m-1]!}\sum_{k=1}^{n-1}[k][k+1]\dots[k+m-1]p_{n,k}(q;x) + p_{n,n}(q;x)\\
&= \frac{[n-1]!}{[n+m-1]!}\sum_{k=1}^{n}[k][k+1]\dots[k+m-1]p_{n,k}(q;x).
\end{aligned}
$$

Now using

$$
[k][k+1]\dots[k+m-1] = \prod_{s=0}^{m-1}\left(q^s[k]+[s]\right) = \sum_{s=1}^{m} c_s(m)[k]^s,
$$

where $c_s(m) > 0$, $s = 1,2,\dots,m$, are the constants independent of k, we get

$$
\begin{aligned}
U_{n,q}\left(t^m;x\right) &= \frac{[n-1]!}{[n+m-1]!}\sum_{k=1}^{n}\sum_{s=1}^{m} c_s(m)[k]^s p_{n,k}(q;x)\\
&= \frac{[n-1]!}{[n+m-1]!}\sum_{s=1}^{m} c_s(m)[n]^s B_{n,q}\left(t^s;x\right),
\end{aligned}
$$

where $B_{n,q}$ is the q-Bernstein operator. Since $B_{n,q}(t^s;x)$ is a polynomial of degree less than or equal to $\min(s,n)$ and $c_s(m) > 0$, $s = 1,2,\dots,m$, it follows that $U_{n,q}(t^m;x)$ is a polynomial of degree less than or equal to $\min(m,n)$. □

3. Convergence of genuine q-Phillips–Durrmeyer operators

Theorem 5. *Let $0 < q_n < 1$. Then the sequence $\{U_{n,q_n}(f)\}$ converges to f uniformly on $[0,1]$ for each $f \in C[0,1]$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof. The proof is standard, see for example [14], [6]. From the definition of $\{U_{n,q}(f)\}$ and Lemma 3 it follows that the operators U_{n,q_n} are positive linear operators on $C[0,1]$ and reproduce linear functions. The well-known Korovkin theorem implies that $U_{n,q_n}(f)$ converges to f uniformly on $[0,1]$ as $n \to \infty$ for any $f \in C[0,1]$ if and only if

$$
U_{n,q_n}\left(t^2;x\right) \to x^2 \tag{4}
$$

uniformly on $[0,1]$ as $n \to \infty$. If $q_n \to 1$, then $[n]_{q_n} \to \infty$ and hence (4) follows from Lemma 3. On the other hand, if we assume that for any $f \in C[0,1]$, $U_{n,q_n}(f)$ converges to f uniformly on $[0,1]$ as $n \to \infty$, then $q_n \to 1$. In fact, if the sequence $\{q_n\}$ does not tend to 1, then it must contain a subsequence $\{q_{n_k}\}$ such that $q_{n_k} \in (0,1)$, $q_{n_k} \to q_0 \in [0,1)$ as $k \to \infty$. Thus,

$$\frac{1}{[n_k+1]_{q_{n_k}}} = \frac{1-q_{n_k}}{1-(q_{n_k})^{n_k+1}} \to 1-q_0$$

as $k \to \infty$. Taking $n = n_k$, $q = q_{n_k}$in $U_{n,q_n}(t^2;x)$, by Lemma 3, we obtain

$$U_{n_k,q_{n_k}}\left(t^2;x\right) \to (1-q_0^2)x + q_0^2x^2 \neq x^2$$

as $k \to \infty$, which leads to a contradiction. Hence, $q_n \to 1$. This completes the proof of theorem. □

Definition 6. Let $q \in (0,1)$ be fixed. We define

$$U_{\infty,q}(f;x) = \begin{cases} f(0)\displaystyle\prod_{s=0}^{\infty}(1-q^sx) & \\ \quad +\dfrac{1}{1-q}\displaystyle\sum_{k=1}^{\infty} q^{1-k}p_{\infty,k}(q;x)\int_0^1 p_{\infty,k-1}(q;qt)f(t)d_qt & \text{if } x \in [0,1), \\ f(1) & \text{if } x = 1. \end{cases}$$

Define

$$A_{n,k}(f) = \begin{cases} f(0) & \text{if } k=0, \\ [n-1]q^{1-k}\int_0^1 p_{n-2,k-1}(q;qt)f(t)d_qt & \text{if } 1 \leq k \leq n-1, \\ f(1) & \text{if } k = n, \end{cases}$$

$$A_{\infty,k}(f) = \begin{cases} \dfrac{q^{1-k}}{1-q}\int_0^1 p_{\infty,k-1}(q;qt)f(t)d_qt & \text{if } k \geq 1, \\ f(0) & \text{if } k=0, \end{cases}$$

then $U_{n,q}(f;x)$ and $U_{\infty,q}(f;x)$ can be rewritten in the following form

$$U_{n,q}(f;x) = \sum_{k=0}^{n} A_{n,k}(f)p_{n,k}(q;x), \qquad x \in [0,1],$$

$$U_{\infty,q}(f;x) = \sum_{k=0}^{\infty} A_{\infty,k}(f)p_{\infty,k}(q;x), \quad x \in [0,1).$$

It is easily seen from

$$\int_0^1 t^s p_{\infty,k-1}(q;qt)d_qt = (1-q)^{s+1}\frac{q^{k-1}[k+s-1]!}{[k-1]!}$$

that

$$U_{\infty,q}(1;x)=1,\quad U_{\infty,q}(t;x)=x,\quad U_{\infty,q}\left(t^2;x\right)=\left(1-q^2\right)x(1-x)+x^2.$$

Lemma 7. *For $f\in C[0,1]$, we have $\|U_{n,q}f\|\le\|f\|$.*

Proof. Using Definition 1 and Lemma 3, we have

$$\begin{aligned}|U_{n,q}(f;x)| &\le |f(0)|p_{n,0}(q;x)+|f(1)|p_{n,n}(q;x)\\ &\qquad+[n-1]\sum_{k=1}^{n-1}q^{1-k}p_{n,k}(q;x)\int_0^1 p_{n-2,k-1}(q;qt)|f(t)|d_qt\\ &\le \|f\|U_{n,q}(1;x)=\|f\|.\end{aligned}$$

□

Lemma 8. *Let $f\in C[0,1]$. Then we have*

$$|A_{n,k}(f-f(1))|\le A_{n,k}(|f-f(1)|)\le\omega(f,q^{n-2})(1+q^{k-n+2}),\quad 0\le k\le n,$$

$$|A_{\infty,k}(f-f(1))|\le A_{\infty,k}(|f-f(1)|)\le\omega(f,q^{n-2})(1+q^{k-n+2}),\quad k\ge 0,\ n\ge 0.$$

Proof. For $1\le k\le n-1$ we have

$$\begin{aligned}|A_{n,k}(f)-A_{n,k}(1)f(1)| &\le [n-1]q^{1-k}\int_0^1 p_{n-2,k-1}(q;qt)|f(t)-f(1)|d_qt\\ &\le [n-1]q^{1-k}\int_0^1\omega(f,q^{n-2})\left(1+\frac{1-t}{q^{n-2}}\right)p_{n-2,k-1}(q;qt)d_qt\\ &=\omega(f,q^{n-2})\left(1+q^{-n+2}\left(1-\frac{[k]}{[n]}\right)\right)\\ &=\omega(f,q^{n-2})\left(1+\frac{q^k(1-q^{n-k})}{q^{n-2}(1-q^n)}\right)\le\omega(f,q^{n-2})(1+q^{k-n+2}).\end{aligned}$$

If $k=0$ or $k=n$ then

$$\begin{aligned}|A_{n,0}(f)-A_{n,0}(1)f(1)| &= |f(0)-f(1)|\le\omega(f,1)=\omega(f,q^{-n+2}q^{n-2})\\ &\le\omega(f,q^{n-2})(1+q^{-n+2}),\\ |A_{n,n}(f)-A_{n,n}(1)f(1)| &= 0.\end{aligned}$$

Similarly one can prove the second inequality. □

Theorem 9. *Let $0 < q < 1$ and $n \geq 3$. Then for each $f \in C[0,1]$ the sequence $\{U_{n,q}(f;x)\}$ converges to $f(x)$ uniformly on $[0,1]$. Furthermore,*

$$\|U_{n,q}(f) - U_{\infty,q}(f)\| \leq C_q \omega(f, q^{n-2}),$$

where $C_q = \frac{10}{1-q} + 4$.

Proof. The proof is similar to the one of Theorem 3 in [7]. For $x \in [0,1)$, by the definitions of $U_{n,q}(f;x)$ and $U_{\infty,q}(f;x)$, we know that

$$\begin{aligned}
|U_{n,q}(f;x) - U_{\infty,q}(f;x)| &\leq \sum_{k=0}^{n} |A_{n,k}(f - f(1)) - A_{\infty,k}(f - f(1))| p_{nk}(q;x) \\
&+ \sum_{k=0}^{n} |A_{\infty,k}(f - f(1))|\,|p_{nk}(q;x) - p_{\infty k}(q;x)| + \sum_{k=n+1}^{\infty} |A_{\infty,k}(f - f(1))| p_{\infty k}(q;x) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

From [7] we have the following estimation

$$|p_{n,k}(q;x) - p_{\infty,k}(q;x)| \leq \frac{q^{n-k}}{1-q}(p_{n,k}(q;x) + p_{\infty,k}(q;x)).$$

Using the above inequality and Lemma 8, for $1 \leq k \leq n-1$ we have

$$\begin{aligned}
&|A_{n,k}(f - f(1)) - A_{\infty,k}(f - f(1))| \\
&\quad \leq \int_0^1 q^{1-k} |f(t) - f(1)| \left| [n-1] p_{n-2,k-1}(q;qt) - \frac{1}{1-q} p_{\infty,k-1}(q;qt) \right| d_q t \\
&\quad \leq \int_0^1 q^{1-k} |f(t) - f(1)| \left| [n-1] - \frac{1}{1-q} \right| p_{\infty,k-1}(q;qt) d_q t \\
&\qquad + \int_0^1 q^{1-k} |f(t) - f(1)| [n-1] |p_{n-2,k-1}(q;qt) - p_{\infty,k-1}(q;qt)| d_q t \\
&\quad \leq q^{n-1} A_{\infty,k}(|f - f(1)|) + \frac{q^{n-k-1}}{1-q} A_{n,k}(|f - f(1)|) \\
&\qquad + q^{n-k-1}[n-1] A_{\infty,k}(|f - f(1)|) \\
&\quad \leq q^{n-1} \omega(f, q^{n-2})(1 + q^{k-n+2}) + 2\frac{q^{n-k-1}}{1-q} \omega(f, q^{n-2})(1 + q^{k-n+2}) \\
&\quad \leq \frac{6}{1-q} \omega(f, q^{n-2}).
\end{aligned}$$

On the other hand if $k = 0$ or $k = n$ then

$$|A_{n,0}(f - f(1)) - A_{\infty,0}(f - f(1))| = 0,$$

and

$$|A_{n,n}(f - f(1)) - A_{\infty,n}(f - f(1))|$$
$$= |A_{\infty,n}(f - f(1))| \le A_{\infty,n}(|f - f(1)|) \le (1 + q^2)\omega(f, q^{n-2}) \le 2\omega(f, q^{n-2}).$$

We start with estimation of I_1 and I_3. We have

$$I_1 \le \left(\frac{6}{1-q} + 2\right)\omega(f, q^{n-2})\sum_{k=0}^{n} p_{n,k}(q; x) = \left(\frac{6}{1-q} + 2\right)\omega(f, q^{n-2})$$

and

$$I_3 \le \omega(f, q^{n-2})\sum_{k=n+1}^{\infty}(1 + q^{k-n+2})p_{\infty,k}(q; x)$$
$$\le 2\omega(f, q^{n-2})\sum_{k=n+1}^{\infty} p_{\infty,k}(q; x) \le 2\omega(f, q^{n-2}).$$

Finally we estimate I_2 as follows:

$$I_2 \le \sum_{k=0}^{n}\omega(f, q^{n-2})(1 + q^{k-n+2})\frac{q^{n-k}}{1-q}(p_{n,k}(q; x) + p_{\infty,k}(q; x))$$
$$\le \frac{2}{1-q}\omega(f, q^{n-2})\sum_{k=0}^{n}(p_{n,k}(q; x) + p_{\infty,k}(q; x)) \le \frac{4}{1-q}\omega(f, q^{n-2}).$$

Thus we conclude that for $x \in [0, 1]$ (if $x = 1$ then $U_{n,q}(f; 1) - U_{\infty,q}(f; 1) = 0$)

$$|U_{n,q}(f; x) - U_{\infty,q}(f; x)| \le C_q\omega(f, q^{n-2}),$$

where $C_q = \frac{10}{1-q} + 4$. □

Theorem 10. *Let $0 < q < 1$ be fixed and let $f \in C[0, 1]$. Then $U_{\infty,q}(f; x) = f(x)$ for all $x \in [0, 1]$ if and only if f is linear.*

Proof. It immediately follows from Theorem 9 of [22] and the inequality

$$U_{\infty,q}(t^2; x) = (1 - q^2)x(1 - x) + x^2 > x^2, \quad \text{for all } x \in (0, 1). \qquad \square$$

At last, we discuss the approximating property of the operators $U_{\infty,q}$.

Theorem 11. *For any $f \in C[0, 1]$, $\{U_{\infty,q}(f)\}$ converges to f uniformly on $[0, 1]$ as $q \uparrow 1$.*

Proof. The proof is standard and follows from the Korovkin theorem, since the operators $U_{\infty,q}$ are positive linear operators on $C[0, 1]$, reproduce linear functions and

$$U_{\infty,q}(t^2; x) = (1 - q^2)x(1 - x) + x^2 \to x^2$$

uniformly on $[0, 1]$ as $q \uparrow 1$. □

4. Approximation properties of q-Phillips–Durrmeyer operators

We begin by considering the following K-functional:

$$K_2(f,\delta^2) := \inf\{\|f-g\| + \delta^2\|g''| : g \in C^2[0,1]\}, \quad \delta \geq 0,$$

where

$$C^2[0,1] := \{g : g, g', g'' \in C[0,1]\}.$$

Then, in view of a known result [3], there exists an absolute constant $C_0 > 0$ such that

$$K_2(f,\delta^2) \leq C_0\omega_2(f,\delta) \tag{5}$$

where

$$\omega_2(f,\delta) := \sup_{0<h\leq\delta}\ \sup_{x,x+2h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second modulus of smoothness of $f \in C[0,1]$.

Our first main result in this section is a local approximation property of $U_{n,q}$ stated below.

Theorem 12. *There exists an absolute constant $C > 0$ such that*

$$|U_{n,q}(f;x) - f(x)| \leq C\omega_2\left(f, \sqrt{\frac{x(1-x)}{[n+1]}}\right),$$

where $f \in C[0,1]$, $0 < q < 1$ and $x \in [0,1]$.

Proof. Using the Taylor formula

$$g(t) = g(x) + g'(x))(t-x) + \int_x^t (t-u)g''(u)du, \quad g \in C^2[0,1],$$

we obtain that

$$U_{n,q}(g;x) = g(x) + U_{n,q}\left(\int_x^t (t-u)g''(u)du; x\right), \quad g \in C^2[0,1].$$

Hence, by Lemma 3

$$|U_{n,q}(g;x) - g(x)| \leq U_{n,q}\left(\left|\int_x^t |t-u|.|g''(u)|du\right|; x\right)$$
$$\leq \|g''\|U_{n,q}((t-x)^2;x) \leq \|g''\|\frac{2}{[n+1]}x(1-x).$$

Now for $f \in C[0,1]$ and $g \in C^2[0,1]$ we obtain, in view of Lemma 7,

$$\begin{aligned}|U_{n,q}(f;x) - f(x)| &\leq |U_{n,q}(f-g;x)| + |U_{n,q}(g;x) - g(x)| + |f(x) - g(x)| \\ &\leq 2\|f-g\| + \|g''\| \frac{2}{[n+1]} x(1-x).\end{aligned}$$

Taking the infimum on the right hand side over all $g \in C^2[0,1]$, we obtain

$$|U_{n,q}(f;x) - f(x)| \leq 2K_2\left(f, \frac{1}{[n+1]} x(1-x)\right). \tag{6}$$

Now the desired inequality follows from (5) and (6). □

We next present the direct global approximation theorem for the operators $U_{n,q}$. In order to state the theorem we need the weighted K-functional of second order for $f \in C[0,1]$ defined by

$$K_{2,\phi}(f,\delta^2) := \inf\{\|f-g\| + \delta^2\|\phi^2 g''\| : g \in W^2(\phi)\}, \quad \delta \geq 0,\ \phi^2(x) = x(1-x)$$

where

$$W^2(\phi) := \{g \in C[0,1] : g' \in AC_{\text{loc}}[0,1], \quad \phi^2 g'' \in C[0,1]\},$$

and $g' \in AC_{\text{loc}}[0,1]$ means that g is differentiable and g' is absolutely continuous in every closed interval $[a,b] \subset [0,1]$. Moreover, the Ditzian–Totik modulus of second order is given by

$$\omega_2^{\phi}(f,\delta) := \sup_{0<h\leq\delta} \sup_{x\pm h\phi(x)\in[0,1]} |f(x-\phi(x)h) - 2f(x) + f(x+\phi(x)h)|.$$

It is well known that the K-functional $K_{2,\phi}(f,\delta^2)$ and the Ditzian–Totik modulus $\omega_2^{\phi}(f,\delta)$ are equivalent (see [3]).

Theorem 13. *There exists an absolute constant $C > 0$ such that*

$$\|U_{n,q}(f) - f\| \leq C\omega_2^{\phi}\left(f, \frac{1}{\sqrt{[n+1]}}\right),$$

where $f \in C[0,1]$, $0 < q < 1$.

Proof. From the Taylor expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)ds,$$

and Lemma 3, we see that

$$|U_{n,q}(g;x)-g(x)| \leq U_{n,q}\left(\left|\int_x^t |t-s|\,|g''(s)|ds\right|;x\right)$$
$$\leq \|\phi^2 g''\| U_{n,q}\left(\left|\int_x^t \frac{|t-s|}{\phi^2(s)}ds\right|;x\right).$$

Let $s=t+\tau(x-t)$, $\tau\in[0,1]$. Using the concavity of ϕ^2 we have

$$\frac{|t-s|}{\phi^2(s)} = \frac{\tau|x-t|}{\phi^2(t+\tau(x-t))} \leq \frac{\tau|x-t|}{\phi^2(t)+\tau(\phi^2(x)-\phi^2(t))} \leq \frac{|x-t|}{\phi^2(x)}.$$

Therefore

$$\left|\int_x^t \frac{|t-s|}{\phi^2(s)}ds\right| \leq \left|\int_x^t \frac{|x-t|}{\phi^2(x)}ds\right| = \frac{(t-x)^2}{\phi^2(x)}$$

and

$$|U_{n,q}(g;x)-g(x)| \leq \|\phi^2 g''\| \frac{1}{\phi^2(x)} U_{n,q}((t-x)^2;x).$$

Because the operator $U_{n,q}$ is bounded (see Lemma 7) we obtain for $f\in C[0,1]$, by Lemma 3 that

$$|U_{n,q}(f;x)-f(x)| \leq |U_{n,q}(f-g;x)| + |U_{n,q}(g;x)-g(x)| + |f(x)-g(x)|$$
$$\leq 2\|f-g\| + \|\phi^2 g''\| \frac{2}{[n+1]}.$$

Taking the infimum on the right hand side over all $g\in W^2(\phi)$ we obtain

$$\|U_{n,q}(f)-f\| \leq 2K_{2,\phi}\left(f,\frac{1}{[n+1]}\right).$$

Now, from the fact that $K_{2,\phi}(f,\delta^2)$ and $\omega_\phi^2(f,\delta)$ are equivalent we obtain the assertion. □

Corollary 14. *Assume that $q=q_n\to 1$ as $n\to\infty$. Then the sequence $\{U_{n,q}(f)\}$ converges to f uniformly on $[0,1]$ for each $f\in C[0,1]$.*

Remark 15. In [8] it is proved that for the operator $M_{n,q}(f;x)$ the following local and global inequalities hold

$$|M_{n,q}(f;x)-f(x)| \leq C\omega_2\left(f,\frac{1}{\sqrt{[n+2]}}\left(x(1-x)+\frac{1}{[n+2]}\right)\right) + \omega\left(f,\frac{2x}{[n+2]}\right),$$

$$\|M_{n,q}(f)-f\| \leq C\omega_2^\phi\left(f,\frac{1}{\sqrt{[n+2]}}\right) + \overrightarrow{\omega}_{2x}\left(f,\frac{1}{[n+2]}\right),$$

where $f\in C[0,1]$ and $0<q<1$.

Since the operator $U_{n,q}(f;x)$ preserves linear functions, first modulus of continuities do not appear in our estimations.

5. q-Lupaş–Durrmeyer operators

It was proved in [11] and [14] that $R_{n,q}(f,x)$ reproduce linear functions and $R_{n,q}(t^2,x)$ was explicitly evaluated:

$$R_{n,q}(t^2,x)=\frac{qx^2}{1-x+qx}+\frac{x(1-x)}{[n](1-x+qx)}.$$

Lemma 16. *Let $0<q<1$. Then for all $x\in[0,1]$ we have*

$$R^*_{n,q}(1;x)=1,\quad R^*_{n,q}(t;x)=x,$$

$$R^*_{n,q}(t^2;x)=\frac{x}{[n+1]}+\frac{q^2[n]x^2}{[n+1](1-x+qx)}+\frac{qx(1-x)}{[n+1](1-x+qx)},$$

$$R^*_{n,q}\big((t-x)^2;x\big)\le U_{n,q}\big((t-x)^2;x\big)=\frac{1+q}{[n+1]}x(1-x).$$

Proof. We prove only the last inequality. It is clear that

$$\begin{aligned}R_{n,q}((t-x)^2;x)&=\frac{qx^2}{1-x+qx}+\frac{x(1-x)}{[n](1-x+qx)}-x^2\\&=\frac{x(1-x)-(1-q)[n]x^2(1-x)}{[n](1-x+qx)}\\&=\frac{x(1-x)(1-x+q^nx)}{[n](1-x+qx)}\le\frac{x(1-x)}{[n]}=B_{n,q}\big((t-x)^2,x\big).\end{aligned}\tag{7}$$

Using the inequality (7) we get desired estimation.

$$\begin{aligned}R^*_{n,q}((t-x)^2;x)&=R^*_{n,q}(t^2;x)-2xR^*_{n,q}(t;x)+x^2\\&=\frac{x}{[n+1]}+\frac{q^2[n]x^2}{[n+1](1-x+qx)}+\frac{qx(1-x)}{[n+1](1-x+qx)}-x^2\\&=\frac{x}{[n+1]}+\frac{q[n]}{[n+1]}\left(\frac{qx^2}{1-x+qx}+\frac{x(1-x)}{[n](1-x+qx)}\right)-x^2\\&\le\frac{x}{[n+1]}+\frac{q[n]}{[n+1]}\left(\frac{x(1-x)}{[n]}+x^2\right)-x^2=\frac{1+q}{[n+1]}x(1-x).\qquad\square\end{aligned}$$

Using Lemma 16 and mimic the proofs of Theorems 12 and 13 we may easily obtain local and global approximation results for $R^*_{n,q}(f)$.

Theorem 17. *There exists an absolute constant $C>0$ such that*

$$|R^*_{n,q}(f;x)-f(x)|\le C\omega_2\left(f,\sqrt{\frac{x(1-x)}{[n+1]}}\right),$$

where $f\in C[0,1]$, $0<q<1$ and $x\in[0,1]$.

Theorem 18. *There exists an absolute constant $C>0$ such that*

$$\|R_{n,q}^{*}(f)-f\| \leq C\omega_2^{\phi}\left(f,\frac{1}{\sqrt{[n+1]}}\right),$$

where $f\in C[0,1]$, $0<q<1$.

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(Received April 2, 2009; revised October 18, 2009)