

Divisor functions arising from q -series

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Abstract. We consider some of the properties of divisor functions arising from q -series and theta functions. Using these we obtain several new identities involving divisor functions. On the other hand we prove a conjecture of Z. H. Sun concerning representations by the ternary quadratic form $x^2 + y^2 + 3z^2$, and also get an analogous result for $x^2 + y^2 + 2z^2$.

1. Introduction

Throughout this paper we use the standard notation

$$(a; q)_\infty := \prod_{n \geq 0} (1 - aq^n).$$

If there is no confusion, we write $(a)_\infty$ for $(a; q)_\infty$. In general, q denotes a fixed complex number of absolute value less than 1, so we may write $q = e^{\pi it}$ with $\text{Im } t > 0$.

For $N, m, r, s, k, l \in \mathbb{Z}$ with $N, m, k, l > 0$ satisfying $k \equiv l \pmod{2}$, $k \geq 3$ and $l \leq k - 2$, we define some necessary divisor functions for later use, which appear in many areas of number theory:

$$E_r(N; m) = \sum_{\substack{d|N \\ d \equiv r \pmod{m}}} 1 - \sum_{\substack{d|N \\ d \equiv -r \pmod{m}}} 1,$$

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$$\begin{aligned}
E_{r,\dots,s}(N; m) &= E_r(N; m) + \cdots + E_s(N; m), \\
\sigma_r(N; m) &= \sum_{\substack{d|N \\ d \equiv r \pmod{m}}} d, \\
\sigma_{r,\dots,s}(N; m) &= \sigma_r(N; m) + \cdots + \sigma_s(N; m), \\
\tau_{k,l}(N) &= \sum_{\substack{d|N \\ d \equiv (k-l)/2 \pmod{k}}} N/d + \sum_{\substack{d|N \\ d \equiv (k+l)/2 \pmod{k}}} N/d.
\end{aligned}$$

Let $r_Q(N)$ be the number of representations of a positive integer N by a positive definite quadratic form Q . Finding explicit formulas for $r_Q(N)$ is a classical problem in number theory (see for example [7]). In some special cases we can use divisor functions to evaluate $r_Q(N)$ rather than utilizing the standard approach via local densities. For example, Gauss (1801) showed that for $Q = x^2 + y^2$, $r_Q(N) = 4E_1(N; 4)$. Moreover, by the theory of theta functions we can further show that this is equivalent to

$$\frac{(q^2; q^2)_{\infty}^{10}}{(q)_{\infty}^4 (q^4; q^4)_{\infty}^4} = 1 + 4 \sum_{N=1}^{\infty} E_1(N; 4) q^N.$$

On the other hand, BERNDT found that many of the modular equations and theta function identities of RAMANUJAN yield elegant partition identities ([1], [2], [9], [10]).

In §2 we consider various identities, that is, those whose coefficients in q -series are given by divisor functions. As a special example of Theorem 4, we can derive the identity

$$\begin{aligned}
&\frac{(q^2; q^2)_{\infty}^2 (-q; q^2)_{\infty} (-q; q^2)_{\infty}}{(-q^2; q^2)_{\infty}^2 (q; q^2)_{\infty} (q; q^2)_{\infty}} + 2 \frac{(q^4; q^4)_{\infty}^2 (-q; q^4)_{\infty} (-q^3; q^4)_{\infty}}{(-q^4; q^4)_{\infty}^2 (q; q^4)_{\infty} (q^3; q^4)_{\infty}} \\
&= 3 + 8 \sum_{N=1}^{\infty} E_1(N; 8) q^N.
\end{aligned}$$

FARKAS showed in [4] and [5] that for $N, k, l \geq 1$ with $k \equiv l \pmod{2}$, $k \geq 3$ and $l \leq k - 2$,

$$\begin{aligned}
k \cdot \tau_{k,l}(N) &= 2 \cdot \sigma_{0, \frac{k-l}{2}, \frac{k+l}{2}}(N; k) + l \cdot E_{\frac{k-l}{2}}(N; k) \\
&\quad + k \cdot \sum_{j=1}^{N-1} E_{\frac{k-l}{2}}(j; k) E_{\frac{k-l}{2}}(N-j; k). \tag{1}
\end{aligned}$$

By applying Farkas' result to Theorem 4 we will obtain Theorem 5.

In §3 by means of divisor functions we prove SUN's conjecture ([11]) on the number of representations of the square of a prime by the ternary quadratic form $x^2 + y^2 + 3z^2$. An analogous result for $x^2 + y^2 + 2z^2$ is obtained by the same argument. We also obtain some new properties of divisor functions.

2. Special examples for divisor functions

From Fine's list of identities of the basic hypergeometric series ([6]), we list following ones for later use.

$$\frac{(q)_\infty^9}{(q^3; q^3)_\infty^3} = 1 - 9 \sum_{N=1}^{\infty} (\sigma_1^{\{2\}}(N; 3) - \sigma_2^{\{2\}}(N; 3)) q^N. \quad ([6], \text{ p. 85}) \quad (2)$$

$$\begin{aligned} \frac{(q)_\infty^3}{(q^3; q^3)_\infty} &= 1 - 3 \sum_{N=0}^{\infty} \{E_1(N; 3) - 3E_1(N/3; 3)\} q^N \\ &= 1 - 3 \sum_{n=0}^{\infty} E_1(3n+1; 3) q^{3n+1} + 6 \sum_{n=1}^{\infty} E_1(3n; 3) q^{3n} \\ &= 1 - 3 \sum_{n=0}^{\infty} E_1(3n+1; 3) q^{3n+1} + 6 \sum_{n=1}^{\infty} E_1(n; 3) q^{3n}. \end{aligned} \quad ([6], \text{ p. 79}) \quad (3)$$

$$\frac{(q^3; q^3)_\infty^3}{(q)_\infty} = \sum_{n=0}^{\infty} E_1(3n+1; 3) q^n. \quad ([6], \text{ p. 79}) \quad (4)$$

$$\frac{(q^2; q^2)_\infty^{10}}{(q)_\infty^4 (q^4; q^4)_\infty^4} = 1 + 4 \sum_{N=1}^{\infty} E_1(N; 4) q^N. \quad ([6], \text{ p. 78}) \quad (5)$$

$$\frac{(q^2; q^2)_\infty^{20}}{(q)_\infty^8 (q^4; q^4)_\infty^8} = 1 + 8 \sum_{N=1}^{\infty} q^N (2 + (-1)^N) \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega. \quad ([6], \text{ p. 78}) \quad (6)$$

$$\frac{q(q^4; q^4)_\infty^8}{(q^2; q^2)_\infty^4} = \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} \sigma(N) q^N. \quad ([6], \text{ p. 79}) \quad (7)$$

$$\frac{(q)_\infty^3 (q^2; q^2)_\infty^3}{(q^3; q^3)_\infty (q^6; q^6)_\infty} = 1 - 3 \sum_{N=1}^{\infty} \{\sigma_{1,-1}(N; 6) - 2\sigma_3(N; 6)\} q^N. \quad ([6], \text{ p. 84}) \quad (8)$$

$$\frac{q(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q)_\infty (q^2; q^2)_\infty} = \sum_{N=1}^{\infty} \left\{ \sigma_{1,-1}(N; 6) + \frac{2}{3} \sigma_3(N; 6) \right\} q^N. \quad ([6], \text{ p. 86}) \quad (9)$$

$$\frac{(q)_\infty^4 (q^3; q^3)_\infty^4}{(q^2; q^2)_\infty^2 (q^6; q^6)_\infty^2} = 1 - 4 \sum_{N=1}^{\infty} \{\sigma_{1,-1}(N; 6) - \sigma_{2,-2}(N; 6)\} q^N. \quad ([6], \text{ p. 85}) \quad (10)$$

$$\frac{q(q^2; q^2)_\infty^4 (q^6; q^6)_\infty^4}{(q)_\infty^2 (q^3; q^3)_\infty^2} = \sum_{N=1}^{\infty} \left\{ \sigma_{1,-1}(N; 6) + \frac{1}{2} \sigma_{2,-2}(N; 6) \right\} q^N. \quad ([6], \text{ p. 87}) \quad (11)$$

$$\frac{(q^2; q^2)_\infty^3 (q^3; q^3)_\infty^2 (q^6; q^6)_\infty}{(q)_\infty^2 (q^4; q^4)_\infty (q^{12}; q^{12})_\infty} = 1 + 2 \sum_{N=1}^{\infty} E_{1,5}(N; 12) q^N. \quad ([6], \text{ p. 80}) \quad (12)$$

$$\begin{aligned} & \frac{(q)_\infty (q^3; q^3)_\infty (q^4; q^4)_\infty^2 (q^6; q^6)_\infty^2}{(q^{12}; q^{12})_\infty^2} \\ &= 1 - \sum_{N=1}^{\infty} \{\sigma_{1,-1}(N; 12) - \sigma_{5,-5}(N; 12)\} q^N. \end{aligned} \quad ([6], \text{ p. 85}) \quad (13)$$

$$\begin{aligned} & \frac{(q^p; q^p)_\infty^2 (-q^r; q^p)_\infty (-q^{p-r}; q^p)_\infty}{(-q^p; q^p)_\infty^2 (q^r; q^p)_\infty (q^{p-r}; q^p)_\infty} \\ &= 1 + 2 \sum_{N=1}^{\infty} E_{r,p-r}(N; 2p) q^N, \quad 0 < r < p, \quad (r, p) = 1. \end{aligned} \quad ([6], \text{ p. 72}) \quad (14)$$

Now, we deduce from (8) and (9) that

$$\frac{(q)_\infty^3 (q^2; q^2)_\infty^3}{(q^3; q^3)_\infty (q^6; q^6)_\infty} + 3 \frac{q(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q)_\infty (q^2; q^2)_\infty} = 1 + 8 \sum_{N=1}^{\infty} \sigma_3(N; 6) q^N, \quad (15)$$

and

$$\frac{(q)_\infty^3 (q^2; q^2)_\infty^3}{(q^3; q^3)_\infty (q^6; q^6)_\infty} - 9 \frac{q(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q)_\infty (q^2; q^2)_\infty} = 1 - 12 \sum_{N=1}^{\infty} \sigma_{1,-1}(N; 6) q^N. \quad (16)$$

In particular, if $N = 2^A 3^B p_1^{e_1} \dots p_r^{e_r} q_1^{f_1} \dots q_s^{f_s}$, where $p_i \equiv 1 \pmod{6}$ and $q_j \equiv -1 \pmod{6}$, then

$$\begin{aligned} & \frac{(q)_\infty^3 (q^2; q^2)_\infty^3}{(q^3; q^3)_\infty (q^6; q^6)_\infty} + 3 \frac{q(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q)_\infty (q^2; q^2)_\infty} = 1 + 8 \sum_{N=1}^{\infty} \sigma_3(N; 6) q^N \\ &= 1 + 8 \sum_{N=1}^{\infty} q^N \frac{3^B - 1}{3 - 1} \sigma(p_1^{e_1} \dots p_r^{e_r} q_1^{f_1} \dots q_s^{f_s}), \end{aligned} \quad (17)$$

and

$$\begin{aligned} & \frac{(q)_\infty^3 (q^2; q^2)_\infty^3}{(q^3; q^3)_\infty (q^6; q^6)_\infty} - 9 \frac{q(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q)_\infty (q^2; q^2)_\infty} = 1 - 12 \sum_{N=1}^{\infty} \sigma_{1,-1}(N; 6) q^N \\ &= 1 - 12 \sum_{N=1}^{\infty} q^N \sigma(p_1^{e_1} \dots p_r^{e_r} q_1^{f_1} \dots q_s^{f_s}). \end{aligned} \quad (18)$$

Similarly, by (10),(11),(13) and (16) we establish that

$$\begin{aligned} & \frac{(q)_\infty^3 (q^2; q^2)_\infty^3}{(q^3; q^3)_\infty (q^6; q^6)_\infty} - 9 \frac{q(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q)_\infty (q^2; q^2)_\infty} + 12 \frac{(q)_\infty (q^3; q^3)_\infty (q^4; q^4)_\infty^2 (q^6; q^6)_\infty^2}{(q^{12}; q^{12})_\infty^2} \\ &= 13 - 24 \sum_{N=1}^{\infty} \sigma_{1,-1}(N; 12) q^N, \\ & \frac{(q)_\infty^3 (q^2; q^2)_\infty^3}{(q^3; q^3)_\infty (q^6; q^6)_\infty} - 9 \frac{q(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q)_\infty (q^2; q^2)_\infty} - 12 \frac{(q)_\infty (q^3; q^3)_\infty (q^4; q^4)_\infty^2 (q^6; q^6)_\infty^2}{(q^{12}; q^{12})_\infty^2} \\ &= -11 - 24 \sum_{N=1}^{\infty} \sigma_{5,-5}(N; 12) q^N, \\ & \frac{(q)_\infty^4 (q^3; q^3)_\infty^4}{(q^2; q^2)_\infty^2 (q^6; q^6)_\infty^2} - 8 \frac{q(q^2; q^2)_\infty^2 (q^6; q^6)_\infty^2}{(q)_\infty^2 (q^3; q^3)_\infty^2} = 1 - 12 \sum_{N=1}^{\infty} \sigma_{1,-1}(N; 6) q^N, \end{aligned}$$

and

$$\begin{aligned} & \frac{(q)_\infty^3 (q^2; q^2)_\infty^3}{(q^3; q^3)_\infty (q^6; q^6)_\infty} - 9 \frac{q(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q)_\infty (q^2; q^2)_\infty} \\ &= \frac{(q)_\infty^4 (q^3; q^3)_\infty^4}{(q^2; q^2)_\infty^2 (q^6; q^6)_\infty^2} - 8 \frac{q(q^2; q^2)_\infty^2 (q^6; q^6)_\infty^2}{(q)_\infty^2 (q^3; q^3)_\infty^2}. \end{aligned}$$

It follows from (3) and (8) that

$$\begin{aligned} & 1 - 3 \sum_{N \geq 1} \{\sigma_{1,-1}(N; 6) - 2\sigma_3(N; 6)\} q^N \\ &= \frac{(q)_\infty^3 (q^2; q^2)_\infty^3}{(q^3; q^3)_\infty (q^6; q^6)_\infty} = \frac{(q)_\infty^3}{(q^3; q^3)_\infty} \cdot \frac{(q^2; q^2)_\infty^3}{(q^6; q^6)_\infty} \\ &= \left(1 - 3 \sum_{n \geq 0} E_1(3n+1; 3) q^{3n+1} + 6 \sum_{n \geq 1} E_1(3n; 3) q^{3n}\right) \\ &\quad \times \left(1 - 3 \sum_{n \geq 0} E_1(3n+1; 3) q^{6n+2} + 6 \sum_{n \geq 1} E_1(3n; 3) q^{6n}\right) \\ &= 1 - 3 \sum_{n \geq 0} E_1(6n+1; 3) q^{6n+1} - 3 \sum_{n \geq 0} E_1(6n+4; 3) q^{6n+4} + 6 \sum_{n \geq 0} E_1(6n; 3) q^{6n} \\ &\quad + 6 \sum_{n \geq 0} E_1(6n+3; 3) q^{6n+3} + 9 \sum_{n \geq 0, N \geq 0} E_1(6n+1; 3) E_1(3N+1; 3) q^{6n+6N+3} \\ &\quad + 9 \sum_{n \geq 0, N \geq 0} E_1(6n+4; 3) E_1(3N+1; 3) q^{6n+6N+6} \end{aligned}$$

$$\begin{aligned}
& -18 \sum_{n \geq 1, N \geq 0} E_1(6n; 3) E_1(3N+1; 3) q^{6n+6N+2} \\
& -18 \sum_{n \geq 0, N \geq 0} E_1(6n+3; 3) E_1(3N+1; 3) q^{6n+6N+5} \\
& +6 \sum_{n \geq 1} E_1(3n; 3) q^{6n} - 3 \sum_{n \geq 0} E_1(3n+1; 3) q^{6n+2} \\
& -18 \sum_{n \geq 0, N \geq 1} E_1(6n+1; 3) E_1(3N; 3) q^{6n+6N+1} \\
& -18 \sum_{n \geq 0, N \geq 1} E_1(6n+4; 3) E_1(3N; 3) q^{6n+6N+4} \\
& +36 \sum_{n \geq 1, N \geq 1} E_1(6n; 3) E_1(3N; 3) q^{6n+6N} + 36 \sum_{n \geq 0, N \geq 1} E_1(6n; 3) E_1(3N; 3) q^{6n+6N+3}.
\end{aligned}$$

From these results we obtain the following new identities involving divisor functions.

Lemma 1. *For M a positive integer, we have*

- (a) $\sigma_{1,5}(6M+1; 6) = \sigma(6M+1) = E_1(6M+1; 3) + 6 \sum_{n=0}^{M-1} E_1(6n+1; 3) E_1(M-n; 3).$
- (b) $\sigma_{1,5}(6M+2; 6) = E_1(3M+1; 3) + 6 \sum_{n=0}^{M-1} E_1(3n+1; 3) E_1(2(M-n); 3).$
- (c) $\sigma_{1,5}(6M+4; 6) = E_1(6M+4; 3) + 6 \sum_{n=0}^{M-1} E_1(6n+4; 3) E_1(M-n; 3).$
- (d) $\sigma_{1,5}(6M+5; 6) = \sigma(6M+5) = 6 \sum_{n=0}^M E_1(2n+1; 3) E_1(M-n; 3).$
- (e) $2\sigma_3(6M+3; 6) - \sigma_{1,5}(6M+3; 6) = 2E_1(2M+1; 3) + 3 \sum_{n=0}^M E_1(6n+1; 3) E_1(3(M-n)+1; 3) + 12 \sum_{n=0}^M E_1(2n+1; 3) E_1(M-n; 3).$
- (f) $2\sigma_3(6M; 6) - \sigma_{1,5}(6M; 6) = 2E_1(2M; 3) + 2E_1(M; 3) + 3 \sum_{n=0}^{M-1} E_1(6n+4; 3) E_1(3(M-n-1)+1; 3) + 12 \sum_{n=1}^{M-1} E_1(2n; 3) E_1(M-n; 3).$

Next, we derive from (4) and (9) that

$$\begin{aligned}
\frac{q(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q)_\infty (q^2; q^2)_\infty} &= \sum_{N=1}^{\infty} q^N (\sigma_{1,5}(N; 6) + \frac{2}{3} \sigma_3(N; 6)) \\
&= q \left(\sum_{n=0}^{\infty} E_1(3n+1; 3) q^n \right) \left(\sum_{m=0}^{\infty} E_1(3m+1; 3) q^{2m} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{N=0}^{\infty} q^{2N+1} \sum_{n=0}^N E_1(6n+1; 3) E_1(3(N-n)+1; 3) \\
&\quad + \sum_{N=1}^{\infty} q^{2N} \sum_{n=0}^{N-1} E_1(6n+4; 3) E_1(3(N-n-1)+1; 3),
\end{aligned}$$

and this implies the following lemma.

Lemma 2. *Let M be a positive integer.*

- (a) $\sigma_{1,5}(6M+1; 6) = \sigma(6M+1) = \sum_{n=0}^{3M} E_1(6n+1; 3) E_1(9M-3n+1; 3).$
- (b) $\sigma_{1,5}(6M+2; 6) = \sum_{n=0}^{3M} E_1(6n+4; 3) E_1(9M-3n+1; 3).$
- (c) $\sigma_{1,5}(6M+4; 6) = \sum_{n=0}^{3M+1} E_1(6n+4; 3) E_1(9M-3n+4; 3).$
- (d) $\sigma_{1,5}(6M+5; 6) = \sigma(6M+5) = \sum_{n=0}^{3M+2} E_1(6n+1; 3) E_1(9M-3n+7; 3).$
- (e) $\frac{2}{3} \sigma_3(6M+3; 6) + \sigma_{1,5}(6M+3; 6) = \sum_{n=0}^{3M+1} E_1(6n+1; 3) E_1(9M-3n+4; 3).$
- (f) $\frac{2}{3} \sigma_3(6M; 6) + \sigma_{1,5}(6M; 6) = \sum_{n=0}^{3M-1} E_1(6n+4; 3) E_1(9M-3n+1; 3).$
- (g) $\sigma_3(6M+3; 6) = \frac{3}{8} \left\{ 2E_1(2M+1; 3) + 3 \sum_{n=0}^M E_1(6n+1; 3) E_1(3(M-n)+1; 3) \right. \\ \left. + 12 \sum_{n=0}^M E_1(2n+1; 3) E_1(M-n; 3) + \sum_{n=0}^{3M+1} E_1(6n+1; 3) E_1(9M-3n+4; 3) \right\}.$
- (h) $\sigma_{1,5}(6M+3; 6) = \frac{1}{4} \left\{ -2E_1(2M+1; 3) - 3 \sum_{n=0}^M E_1(6n+1; 3) E_1(3(M-n)+1; 3) \right. \\ \left. - 12 \sum_{n=0}^M E_1(2n+1; 3) E_1(M-n; 3) + 3 \sum_{n=0}^{3M+1} E_1(6n+1; 3) E_1(9M-3n+4; 3) \right\}.$
- (i) $\sigma_3(6M; 6) = \frac{3}{8} \left\{ 2E_1(2M; 3) + 2E_1(M; 3) + 3 \sum_{n=0}^{M-1} E_1(6n+4; 3) \right. \\ \left. E_1(3(M-n-1)+1; 3) + 12 \sum_{n=1}^{M-1} E_1(2n; 3) E_1(M-n; 3) \right. \\ \left. + \sum_{n=0}^{3M-1} E_1(6n+4; 3) E_1(9M-3n+1; 3) \right\}.$
- (j) *For $M > 1$, we have $\sigma_{1,5}(6M; 6) = \frac{1}{4} \left\{ -2E_1(2M; 3) - 2E_1(M; 3) \right.$*

$$\begin{aligned}
& -3 \sum_{n=0}^{M-1} E_1(6n+4; 3) E_1(3(M-n-1)+1; 3) - 12 \sum_{n=1}^{M-1} E_1(2n; 3) E_1(M-n; 3) \\
& + 3 \sum_{n=0}^{3M-1} E_1(6n+4; 3) E_1(9M-3n+1; 3) \Big\}.
\end{aligned}$$

Here, we observe that

$$\sum_{\substack{d|N \\ d \equiv 1 \pmod{6}}} 1 - \sum_{\substack{d|N \\ d \equiv 5 \pmod{6}}} 1 = \sum_{\substack{d|N \\ d \equiv 1 \pmod{12}}} 1 + \sum_{\substack{d|N \\ d \equiv 7 \pmod{12}}} 1 - \sum_{\substack{d|N \\ d \equiv 5 \pmod{12}}} 1 - \sum_{\substack{d|N \\ d \equiv 11 \pmod{12}}} 1.$$

Then we readily see that

$$\frac{(q^2; q^2)_\infty^6 (q^3; q^3)_\infty}{(q)_\infty^3 (q^6; q^6)_\infty^2} = 1 + 3 \sum_{N=1}^{\infty} E_1(N; 6) q^N = 1 + 3 \sum_{N=1}^{\infty} E_{1,7}(N; 12) q^N,$$

which gives the following lemma by (12).

Lemma 3.

$$\begin{aligned}
\text{(a)} \quad & \frac{3}{5} \frac{(q^2; q^2)_\infty^3 (q^3; q^3)_\infty^2 (q^6; q^6)_\infty}{(q)_\infty^2 (q^4; q^4)_\infty (q^{12}; q^{12})_\infty} + \frac{2}{5} \frac{(q^2; q^2)_\infty^6 (q^3; q^3)_\infty}{(q)_\infty^3 (q^6; q^6)_\infty^2} \\
& = 1 + \frac{12}{5} \sum_{N=1}^{\infty} E_1(N; 12) q^N. \\
\text{(b)} \quad & 3 \frac{(q^2; q^2)_\infty^3 (q^3; q^3)_\infty^2 (q^6; q^6)_\infty}{(q)_\infty^2 (q^4; q^4)_\infty (q^{12}; q^{12})_\infty} - 2 \frac{(q^2; q^2)_\infty^6 (q^3; q^3)_\infty}{(q)_\infty^3 (q^6; q^6)_\infty^2} \\
& = 1 + 12 \sum_{N=1}^{\infty} E_5(N; 12) q^N.
\end{aligned}$$

More generally we can derive the following identities.

Theorem 4. For $n \geq 1$ we have

$$\begin{aligned}
\text{(a)} \quad & \sum_{k=1}^n 2^{k-1} \frac{(q^{2^k}; q^{2^k})_\infty^2 (-q; q^{2^k})_\infty (-q^{2^k-1}; q^{2^k})_\infty}{(-q^{2^k}; q^{2^k})_\infty^2 (q; q^{2^k})_\infty (q^{2^k-1}; q^{2^k})_\infty} \\
& = (2^n - 1) + 2^{n+1} \sum_{N=1}^{\infty} E_1(N; 2^{n+1}) q^N. \\
\text{(b)} \quad & 2 \frac{(q^2; q^2)_\infty^6 (q^3; q^3)_\infty}{(q; q)_\infty^3 (q^6; q^6)_\infty^2} \\
& + 3 \sum_{k=1}^n 2^{k-1} \frac{(q^{3 \cdot 2^k}; q^{3 \cdot 2^k})_\infty^2 (-q; q^{3 \cdot 2^k})_\infty (-q^{3 \cdot 2^k-1}; q^{3 \cdot 2^k})_\infty}{(-q^{3 \cdot 2^k}; q^{3 \cdot 2^k})_\infty^2 (q; q^{3 \cdot 2^k})_\infty (q^{3 \cdot 2^k-1}; q^{3 \cdot 2^k})_\infty} \\
& = (3 \cdot 2^n - 1) + 3 \cdot 2^{n+1} \sum_{N=1}^{\infty} E_1(N; 3 \cdot 2^{n+1}) q^N.
\end{aligned}$$

PROOF. Note that $E_r(N; p) = E_{r, p+r}(N; 2p)$ and $E_{p+r}(N; 2p) = -E_{p-r}(N; 2p)$ imply $2E_r(N; 2p) = E_{r, p-r}(N; 2p) + E_r(N; p)$. Hence, we inductively get by the identity (14) that

$$\begin{aligned} & (2^n - 1) + 2^{n+1} \sum_{N=1}^{\infty} E_r(N; 2^n p) q^N \\ &= 2 \sum_{N=1}^{\infty} E_r(N; p) q^N + \sum_{k=0}^{n-1} 2^k \frac{(q^{2^k p}; q^{2^k p})_{\infty}^2 (-q^r; q^{2^k p})_{\infty} (-q^{2^k p-r}; q^{2^k p})_{\infty}}{(-q^{2^k p}; q^{2^k p})_{\infty}^2 (q^r; q^{2^k p})_{\infty} (q^{2^k p-r}; q^{2^k p})_{\infty}}. \end{aligned}$$

The first assertion follows from the above identity by observing $E_r(N; p) = 0$ for $p = 2, r = 1$.

As for the second, we know by [6, p. 80] that

$$\frac{(q^2; q^2)_{\infty}^6 (q^3; q^3)_{\infty}}{(q; q)_{\infty}^3 (q^6; q^6)_{\infty}^2} = 1 + 3 \sum_{N=1}^{\infty} E_1(N; 6) q^N.$$

Therefore, we have the assertion by the above identity with $p = 6, r = 1$. $\square$

By applying the Farkas' identity (1) to Theorem 4 we also have the following theorem.

Theorem 5. For $n \geq 1$ we have

$$\begin{aligned} & \sum_{1 \leq k, l \leq n} 2^{k+l-2} \\ & \frac{(q^{2^k}; q^{2^k})_{\infty}^2 (-q; q^{2^k})_{\infty} (-q^{2^k-1}; q^{2^k})_{\infty} (q^{2^l}; q^{2^l})_{\infty}^2 (-q; q^{2^l})_{\infty} (-q^{2^l-1}; q^{2^l})_{\infty}}{(-q^{2^k}; q^{2^k})_{\infty}^2 (q; q^{2^k})_{\infty} (q^{2^k-1}; q^{2^k})_{\infty} (-q^{2^l}; q^{2^l})_{\infty}^2 (q; q^{2^l})_{\infty} (q^{2^l-1}; q^{2^l})_{\infty}} \\ &= (2^n - 1)^2 + 2^{n+2} (2^n - 1) q + 2^{n+2} \sum_{N=2}^{\infty} (2^n \tau_{2^{n+1}, 2^{n+1}-2}(N) - \sigma_{0,1,-1}(N; 2^{n+1})) q^N. \end{aligned}$$

PROOF. The square of the right hand side of Theorem 4 (a) is equal to

$$\begin{aligned} & (2^n - 1)^2 + 2^{n+2} (2^n - 1) \sum_{N=1}^{\infty} E_1(N; 2^{n+1}) q^N + 2^{2n+2} \\ & \times \sum_{N=2}^{\infty} \sum_{j=1}^{N-1} E_1(j; 2^{n+1}) E_1(N-j; 2^{n+1}) q^N, \end{aligned}$$

and the term $\sum_{j=1}^{N-1} E_1(j; 2^{n+1}) E_1(N-j; 2^{n+1})$ can be replaced by using the identity (1). $\square$

3. Divisor functions and Sun's conjecture

We first recall some well-known facts (for instance, see [6], pp. 72-77) about a quantitative problem of quadratic forms over $\mathbb{Z}$ in terms of divisor functions.

Lemma 6. *Let N be a positive integer.*

- (1) *If $Q_1(x, y) = x^2 + 3y^2$, then $r_{Q_1}(N) = \begin{cases} 2E_1(N; 3) & \text{if } N \text{ is odd} \\ 6E_1(N; 3) & \text{if } N \text{ is even.} \end{cases}$*
- (2) *If $Q_2(x, y, z, w) = x^2 + y^2 + 3z^2 + 3w^2$, then $r_{Q_2}(N) = (-1)^{N-1} \cdot 4 \cdot \sum_{d|N} d \cdot \chi(d)$, where $\chi(d) = 1, -1$, and 0 according as $d \equiv \pm 1 \pmod{6}$, $d \equiv \pm 2 \pmod{6}$, and $d \equiv 0 \pmod{3}$, respectively.*
- (3) *If $Q_3(x, y) = x^2 + y^2$, then $r_{Q_3}(N) = 4E_1(N; 4)$.*
- (4) *If $Q'_1(x, y) = x^2 + 2y^2$, then $r_{Q'_1}(N) = 2E_{1,3}(N; 8)$.*
- (5) *If $Q'_2(x, y, z, w) = x^2 + y^2 + 2z^2 + 2w^2$, then $r_{Q'_2}(N) = 4k(\alpha) \sum_{d|m} d$, where $N = 2^\alpha m$ with $(2, m) = 1$ and $k(0) = 1$, $k(1) = 2$, $k(\alpha) = 6$ for $\alpha \geq 2$.*

Here we note that $E_1(N; 3) = \sum_{d|N} \left(\frac{d}{3}\right)$ is a multiplicative function of N , that is, for coprime positive integers N and M , $E_1(NM; 3) = E_1(N; 3)E_1(M; 3)$. Since $E_1(2^{2n}; 3) = 1$ and $E_1(2^{2n+1}; 3) = 0$ for any nonnegative integer n , we also get $E_1(4N; 3) = E_1(N; 3)$ by observing that E_1 is multiplicative. Now, we are ready to prove SUN's conjecture ([11], Conjecture 17) on the number of representations of the square of a prime by the ternary quadratic form $x^2 + y^2 + 3z^2$, and also to prove an analogous result for $x^2 + y^2 + 2z^2$. The method of our proof goes back to HURWITZ, and the work of HURWITZ can be found on [8, pp. 5–7] or [3, p. 311 of Volume 2].

Proposition 7. *Let $Q(x, y, z) = x^2 + y^2 + 3z^2$, and $Q'(x, y, z) = x^2 + y^2 + 2z^2$. Then we have*

- (1) $r_Q(p^2) = 4(p + 1 - \left(\frac{p}{3}\right))$ for any prime $p \geq 5$.
- (2) $r_{Q'}(p^2) = 4(p + 1 - \left(\frac{-2}{p}\right))$ for any prime $p \geq 3$.

PROOF. (1) Since p is odd, $p^2 \equiv 1 \pmod{8}$. Thus $p^2 = x^2 + y^2 + 3z^2$ implies that $(x, y, z) = (\text{odd}, \text{even}, \text{even})$ or $(\text{even}, \text{odd}, \text{even})$. With the notation as in Lemma 6 we derive that

$$r_Q(p^2) = 4 \sum_{\substack{1 \leq i \leq p \\ i = \text{odd}}} r_{Q_1}(p^2 - i^2) = 4 \left(\sum_{\substack{1 \leq i < p \\ i = \text{odd}}} r_{Q_1}(p^2 - i^2) + 1 \right)$$

$$\begin{aligned}
 &= 4 \left(6 \sum_{\substack{1 \leq i < p \\ i = \text{odd}}} E_1(p^2 - i^2; 3) + 1 \right) \quad \text{by Lemma 6-(1)} \\
 &= 4 \left(6 \sum_{\substack{1 \leq i < p \\ i = \text{odd}}} E_1\left(\frac{p-i}{2} \cdot \frac{p+i}{2}; 3\right) + 1 \right), \quad \text{because } E_1(4N; 3) = E_1(N; 3) \\
 &= 4 \left(6 \sum_{\substack{1 \leq i < p \\ i = \text{odd}}} E_1\left(\frac{p-i}{2}; 3\right) E_1\left(\frac{p+i}{2}; 3\right) + 1 \right), \quad \text{since } \left(\frac{p-i}{2}, \frac{p+i}{2}\right) = 1 \\
 &= 4 \left(6 \sum_{j=1}^{\frac{p-1}{2}} E_1(j; 3) E_1(p-j; 3) + 1 \right) = 4 \left(3 \sum_{j=1}^{p-1} E_1(j; 3) E_1(p-j; 3) + 1 \right) \\
 &= \sum_{j=1}^{p-1} 12 E_1(j; 3) E_1(p-j; 3) + 4 = \sum_{j=1}^{p-1} r_{Q_1}(j) r_{Q_1}(p-j) + 4 \\
 &= \sum_{j=0}^p r_{Q_1}(j) r_{Q_1}(p-j) + 4 - 2r_{Q_1}(p) = r_{Q_2}(p) + 4 - 2r_{Q_1}(p) \\
 &= 4 \left(p + 1 - \left(\frac{p}{3}\right) \right) \quad \text{by Lemma 6-(1) and (2).}
 \end{aligned}$$

(2) In a similar way we see that $E_{1,3}(N; 8) = \sum_{0 < d|N} \left(\frac{-2}{d}\right)$ is a multiplicative function of N and $E_{1,3}(2^\alpha; 8) = 1$. Then we claim that

$$\begin{aligned}
 r_{Q'}(p^2) &= 4 \left(\sum_{\substack{1 \leq i < p \\ i = \text{odd}}} r_{Q'_1}(p^2 - i^2) + 1 \right) \\
 &= 4 \left(2 \sum_{\substack{1 \leq i < p \\ i = \text{odd}}} E_{1,3}(p^2 - i^2; 8) + 1 \right) \quad \text{by Lemma 6-(4)} \\
 &= 4 \left(2 \sum_{\substack{1 \leq i < p \\ i = \text{odd}}} E_{1,3}\left(\frac{p-i}{2} \cdot \frac{p+i}{2}; 8\right) + 1 \right), \quad \text{because } E_{1,3}(2^\alpha N; 8) = E_{1,3}(N; 8) \\
 &= 4 \left(\sum_{j=1}^{p-1} E_{1,3}(j; 8) E_{1,3}(p-j; 8) + 1 \right) \\
 &= \sum_{j=0}^p r_{Q'_1}(j) r_{Q'_1}(p-j) + 4 - 2r_{Q'_1}(p) \\
 &= 4 \left(p + 1 - \left(\frac{-2}{p}\right) \right) \quad \text{by Lemma 6-(4) and (5).} \quad \square
 \end{aligned}$$

As an application we have the following corollary.

Corollary 8. *Let $[r]$ be the greatest integer less than or equal to $r \in \mathbb{R}$.*

(1) *For any prime $p \geq 5$,*

$$\sum_{k=1}^{\frac{p-1}{2}} (3E_1(p^2 - (2k-1)^2; 3) + E_1(p^2 - 4k^2; 3)) = \begin{cases} p-2, & p \equiv 1 \pmod{3} \\ p+1, & p \equiv 2 \pmod{3}, \end{cases}$$

$$\sum_{k=1}^{\lfloor \frac{p}{\sqrt{3}} \rfloor} E_1(p^2 - 3k^2; 4) = \begin{cases} \frac{p-3}{2}, & p \equiv 1 \pmod{12} \\ \frac{p-1}{2}, & p \equiv \pm 5 \pmod{12} \\ \frac{p+1}{2}, & p \equiv 11 \pmod{12}. \end{cases}$$

(2) *For any prime $p \geq 3$,*

$$\sum_{k=1}^{p-1} E_{1,3}(p^2 - k^2; 8) = \begin{cases} p-2, & p \equiv 1, 3 \pmod{8} \\ p+1, & p \equiv 5, 7 \pmod{8}, \end{cases}$$

$$\sum_{k=1}^{\lfloor \frac{p}{\sqrt{2}} \rfloor} E_1(p^2 - 2k^2; 4) = \begin{cases} \frac{p-3}{2}, & p \equiv 1 \pmod{8} \\ \frac{p-1}{2}, & p \equiv \pm 3 \pmod{8} \\ \frac{p+1}{2}, & p \equiv -1 \pmod{8}. \end{cases}$$

PROOF. (1) It follows from by Lemma 6-(1) that

$$\begin{aligned} r_Q(p^2) &= \sum_{i=-p}^p r_{Q_1}(p^2 - i^2) = r_{Q_1}(p^2) + 2 \sum_{i=1}^{p-1} r_{Q_1}(p^2 - i^2) + 2r_{Q_1}(0) \\ &= 2E_1(p^2; 3) + 2 + 2 \left\{ 6 \sum_{\substack{i=1 \\ \text{odd}}}^{p-2} E_1(p^2 - i^2; 3) + 2 \sum_{\substack{i=1 \\ \text{even}}}^{p-1} E_1(p^2 - i^2; 3) \right\}. \end{aligned}$$

On the other hand, we know by Proposition 7-(1) that $r_Q(p^2) = 4(p+1 - (\frac{p}{3}))$.

Hence we have that

$$\begin{aligned} &\sum_{k=1}^{\frac{p-1}{2}} \{3E_1(p^2 - (2k-1)^2; 3) + E_1(p^2 - 4k^2; 3)\} \\ &= \left(p+1 - \left(\frac{p}{3} \right) \right) - \frac{1}{2}E_1(p^2; 3) - \frac{1}{2} = p - \frac{1}{2} \left(1 + 3 \left(\frac{p}{3} \right) \right), \end{aligned}$$

from which we derive the first statement.

Next, we obtain by Lemma 6-(3) and Proposition 7-(1) that

$$\begin{aligned} r_Q(p^2) &= \sum_{i=-[\frac{p}{\sqrt{3}}]+1}^{[\frac{p}{\sqrt{3}}]} r_{Q_3}(p^2 - 3i^2) = 4E_1(p^2; 4) + 8 \sum_{i=1}^{[\frac{p}{\sqrt{3}}]} E_1(p^2 - 3i^2; 4) \\ &= 4 \left(p + 1 - \left(\frac{p}{3} \right) \right). \end{aligned}$$

Since $E_1(N; 4) = \sum_{0 < d|N} \left(\frac{-1}{d} \right)$,

$$\begin{aligned} \sum_{k=1}^{[\frac{p}{\sqrt{3}}]} E_1(p^2 - 3k^2; 4) &= \frac{1}{2} \left(p + 1 - \left(\frac{p}{3} \right) - E_1(p^2; 4) \right) \\ &= \frac{1}{2} \left(p - 1 - \left(\frac{p}{3} \right) - \left(\frac{-1}{p} \right) \right). \end{aligned}$$

This enables us to conclude the second statement.

(2) We first observe by Lemma 6-(4) and Proposition 7-(2) that

$$\begin{aligned} r_{Q'}(p^2) &= \sum_{i=-p}^p r_{Q'_1}(p^2 - i^2) = r_{Q'_1}(p^2) + 2 \sum_{i=1}^{p-1} r_{Q'_1}(p^2 - i^2) + 2r_{Q'_1}(0) \\ &= 2E_{1,3}(p^2; 8) + 2 + 4 \sum_{i=1}^{p-1} E_{1,3}(p^2 - i^2; 8) = 4 \left(p + 1 - \left(\frac{-2}{p} \right) \right), \end{aligned}$$

and by Lemma 6-(3) and Proposition 7-(2) that

$$\begin{aligned} r_{Q'}(p^2) &= \sum_{i=-[\frac{p}{\sqrt{2}}]+1}^{[\frac{p}{\sqrt{2}}]} r_{Q_3}(p^2 - 2i^2) = 4E_1(p^2; 4) + 8 \sum_{i=1}^{[\frac{p}{\sqrt{2}}]} E_1(p^2 - 2i^2; 4) \\ &= 4 \left(p + 1 - \left(\frac{-2}{p} \right) \right). \end{aligned}$$

Therefore, we deduce the conclusions by similar arguments as in (1). $\square$

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