

**Title:** Binary sequences generated by sequences  $\{n\alpha\}$ ,  $n = 1, 2, \dots$

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Let  $\alpha$  be an irrational number,  $I$  be a subinterval of the unit interval  $(0, 1)$ , and  $\{x\}$  denote the fractional part of  $x$ . In this paper we shall study arithmetical properties of the set  $A = \{n \in \mathbb{N}; \{n\alpha\} \in I\}$  and pseudorandom character of the sequence  $x_n$ ,  $n = 1, 2, \dots$ , where  $x_n = 1$  when  $\{n\alpha\} \in I$ , and  $x_n = -1$  otherwise. We prove, among others, that the gaps between successive elements of  $A$  are at most of three lengths,  $a$ ,  $b$  and  $a + b$  also in the case of an arbitrary interval  $I \subset (0, 1)$ , thereby extending the known Slater's results for intervals of the type  $I = (0, t)$  with  $t < 1/2$ . Further we exactly describe the set of positive integers which are not equal to a difference of two arbitrary elements from  $A$  and we prove that  $A$  contains infinite double-arithmetic progressions. Then we find a new lower estimate of the Mauduit–Sárközy well distribution measure of  $x_n$  for an arbitrary interval  $I$ . We also prove that the sequence  $x_n$  is Sturmian for every interval  $I$  of length  $\{\alpha\}$  or  $1 - \{\alpha\}$  in the sense that the number of 1's in any pair of finite subsegments of the same length occurring in  $x_n$  can differ by at most one. We prove (Theorem ??) that if  $|I| \leq 1/2$  then any subsequence of  $x_n$  of the form  $x_{n+kK}$ ,  $k = 1, 2, \dots$ , splits into consecutive blocks of 1's and blocks of  $-1$ 's whose lengths also differ by at most one. The proofs employ two geometric ideas: (i) a transposition of subintervals (cf. Lemma ??) of  $I$  to construct arithmetic progressions of the set  $A$ , (ii) properties (cf. Lemma ??) of line segments of the intersection of the graph of the sawtooth function  $x + \{k\alpha\}$  with  $I \times I$  to answer the question when two elements  $\{n\alpha\}$  and  $\{(n+k)\alpha\}$  simultaneously fall into  $I$ . This technique gives, for instance, a new proof of the mentioned Slater's three gap theorems.

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