

On a problem of D. Brydak

By M. Czerni (Kraków)

1. Introduction

At the Third International Symposium on functional equations and inequalities at Noszvaj (Hungary) in September 1986, D. BRYDAK put the following slightly reformulated problem (see supplement in [2] p. 36):

Let $f : J \rightarrow J$, where $J = [0, \alpha)$, $\alpha > 0$ be strictly increasing and continuous in J . Moreover, let $0 < f(x) < x$ for $x \in (0, \alpha)$. Let $g : J \rightarrow \mathbb{R}_+$ be a continuous in J . Let the equation

$$(1) \quad \varphi[f(x)] = g(x)\varphi(x), \quad x \in J,$$

have a continuous solution, positive in $(0, \alpha)$ and depending on an arbitrary function. Let $\psi : J \rightarrow \mathbb{R}_+$ be a continuous solution of the inequality

$$(2) \quad \psi[f(x)] \leq g(x)\psi(x), \quad x \in J.$$

Does there always exist a solution $\varphi : J \rightarrow \mathbb{R}$ of equation (1) such that the limit

$$(3) \quad \lim_{x \rightarrow 0^+} \frac{\psi(x)}{\varphi(x)}$$

exists?

The answer to the above question is negative. In the present paper we are going to characterize such continuous, nonnegative solutions ψ of (2) that for every solution φ of (1) vanishing at zero only, the limit (3) does not exist.

At first we formulate assumptions about the given functions f and g as follows:

(**H**₁) Let $f : J \rightarrow J$ be strictly increasing and continuous in an interval $J = [0, \alpha)$. Moreover

$$(4) \quad 0 < f(x) < x \quad \text{for} \quad x \in (0, \alpha).$$

(**H**₂) Let $g : J \rightarrow \mathbb{R}$ be continuous in the interval J and $g(x) > 0$ for $x \in (0, \alpha)$.

In the sequel we shall consider the following classes of functions:

Definition 1. We denote by Ψ the family of all continuous, nonnegative solutions $\psi : J \rightarrow \mathbb{R}$ of the inequality (2) satisfying the condition

$$\psi(0) = 0.$$

We denote by Φ the family of all solutions $\varphi : J \rightarrow \mathbb{R}$ of equation (1) satisfying the conditions

$$\varphi(x) \neq 0 \quad \text{for} \quad x \in (0, \alpha), \quad \varphi(0) = 0.$$

It is necessary for further considerations to have the condition $\Phi \neq \emptyset$ fulfilled. For this reason (see [1]) we shall assume

(**H**₃) The sequence $\{G_n\}_{n \in \mathbb{N}}$ given by

$$(5) \quad G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)] \quad \text{for } x \in J, \quad n \in \mathbb{N}$$

where f^i is the i -th iterate of the function f , i.e. $f^0 = Id$, $f^{n+1} = f \circ f^n$ converges to zero almost uniformly in the interval J .

Remark. If hypothesis (**H**₃) is fulfilled, then equation (1) has a continuous solution in J depending on an arbitrary function and every continuous solution φ satisfies the condition $\varphi(0) = 0$ (see [4] p. 48). In particular (**H**₃) implies that equation (1) has a continuous solution, positive in $(0, \alpha)$.

Finally, we introduce the following subclass of Φ :

Definition 2. Let $\psi \in \Psi$ and $a \in \mathbb{R}$. We denote by Φ_a^ψ the family of all functions $\varphi \in \Phi$ such that the limit (3) exists and is equal to a .

Thus we may reformulate Brydak's problem as follows:

Is the formula

$$\bigwedge_{\psi \in \Psi} \bigvee_{a \in \mathbb{R}} \Phi_a^\psi \neq \emptyset$$

true?

2. Results

The following theorem contain results which are proved in [1] and will be needed in the sequel.

Theorem 1. *Let the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_3)$ be fulfilled and let $\psi \in \Psi$. Then there exists the limit*

$$\psi_0(x) := \lim_{n \rightarrow \infty} \frac{\psi[f^n(x)]}{G_n(x)} \quad \text{for } x \in (0, \alpha),$$

where $\{G_n\}$ is defined by formula (5) and the function

$$(6) \quad \varphi_0(x) = \begin{cases} \psi_0(x) & \text{for } x \in (0, \alpha) \\ 0 & \text{for } x = 0 \end{cases}$$

is a solution of equation (1) in J , upper semicontinuous in J , continuous at zero and fulfilling the inequalities

$$(7) \quad 0 \leq \varphi_0(x) \leq \psi(x).$$

Introduce the notation

$$\gamma_n(x) = \frac{\psi[f^n(x)]}{\varphi[f^n(x)]} \quad x \in J, \quad n \in \mathbb{N}.$$

Now, we formulate the following

Theorem 2. *Let the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_3)$ be fulfilled and let $\psi \in \Psi$ and $a \in \mathbb{R}$. If $\varphi \in \Phi_a^\psi$, then for every $x_0 \in (0, \alpha)$ the sequence (γ_n) converges to a , uniformly in the interval $(0, x_0]$. Moreover*

$$(8) \quad a\varphi(x) = \varphi_0(x) \quad \text{for } x \in J$$

where φ_0 is given by formula (6).

PROOF. Let us fix an $x_0 \in J \setminus \{0\}$ and suppose that (γ_n) does not converge to a , uniformly in $(0, x_0]$, i.e.

$$(9) \quad \bigvee_{\varepsilon > 0} \bigwedge_{n \in \mathbb{N}} \bigvee_{k_n \geq n} \bigvee_{x_n \in (0, x_0]} |\gamma_{k_n}(x_n) - a| \geq \varepsilon > 0.$$

Without loss of generality we may assume that the sequence $\{k_n\}$ is strictly increasing. But from hypothesis $(\mathbf{H}_1)$ we obtain

$$0 < f^{k_n}(x_n) \leq f^{k_n}(x_0)$$

and this implies that $\lim_{n \rightarrow \infty} f^{k_n}(x_n) = 0$, by virtue of (4). Thus the estimation in (9) proves that $\lim_{x \rightarrow 0} \frac{\psi(x)}{\varphi(x)}$ is not equal to a , contrary to our assumption on φ . This ends the proof of the first part of the theorem.

We have also

$$\begin{aligned} \varphi_0(x) = \psi_0(x) &= \lim_{n \rightarrow \infty} \frac{\psi[f^n(x)]\varphi(x)}{G_n(x)\varphi(x)} = \\ &= \left(\lim_{n \rightarrow \infty} \gamma_n(x) \right) \varphi(x) = a\varphi(x) \quad \text{for } x \neq 0 \end{aligned}$$

and

$$\varphi_0(0) = 0 = a\varphi(0).$$

Thus we obtain (8).

Theorem 3. *Let the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_3)$ be fulfilled and let $\psi \in \Psi$, $\varphi \in \Phi$ and $a \in \mathbb{R}$. If there exists an $x_0 \in J \setminus \{0\}$ such that the sequence (γ_n) converges to a , uniformly in the interval $[f(x_0), x_0]$, then $\varphi \in \Phi_a^\psi$.*

PROOF. It is sufficient to show that the limit (3) exists. Let us fix an $\varepsilon > 0$. Thus there exists a positive integer N that for every $n > N$ and $x \in [f(x_0), x_0]$ we have

$$(10) \quad |\gamma_n(x) - a| < \varepsilon.$$

Let us put $\delta := f^{N+1}(x_0)$. If we take $t \in (0, \delta)$, then $t = f^n(x)$ for some $n > N$ and $x \in [f(x_0), x_0]$ (see [4] p.21). Thus we obtain

$$\left| \frac{\psi(t)}{\phi(t)} - a \right| < \varepsilon$$

by virtue of (10) and this ends the proof of the theorem.

Now, we are going to give some simple result concerning the case where $\phi_0(x)$ is identically equal to zero.

Theorem 4. *Let the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_3)$ be fulfilled and let $\psi \in \Psi$ be such that $\varphi_0(x) = 0$ for $x \in J$. If $\varphi \in \Phi$ fulfils the condition*

$$\bigvee_{x_0 \in J \setminus \{0\}} \bigvee_{m > 0} \bigwedge_{x \in [f(x_0), x_0]} |\varphi(x)| > m,$$

then $\varphi \in \Phi_0^\psi$.

PROOF. Since the sequence $\left(\frac{\psi[f^n(x)]}{G_n(x)} \right)$ is decreasing for every $x \in J$ (see [1]), in view of the Dini's theorem $\lim_{n \rightarrow \infty} \frac{\psi[f^n(x)]}{G_n(x)} = 0$, uniformly in $[f(x_0), x_0]$. Consequently for every $\varepsilon > 0$ there exists such a positive integer N that for $n > N$ and $x \in [f(x_0), x_0]$ we have

$$|\gamma_n(x)| = \left| \frac{\psi[f^n(x)]}{G_n(x)\varphi(x)} \right| \leq \frac{1}{m} \left(\frac{\psi[f^n(x)]}{G_n(x)} \right) < \varepsilon$$

and by virtue of Theorem 3 this ends the proof of the theorem.

Finally, we shall prove

Theorem 5. *Let the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_3)$ be fulfilled, $\psi \in \Psi$ and $a \in \mathbb{R} \setminus \{0\}$. If $\varphi \in \Phi_a^\psi$, then the solution ϕ_0 defined by (6) is continuous in J .*

PROOF. Since the limit (3) exists and is not equal to zero then $\psi(x) \neq 0$ for every x in a vicinity of zero. Thus, and because of $(\mathbf{H}_1)$ there exists an $x_0 \in J \setminus \{0\}$ such that

$$M := |a|^{-1} \max_{x \in [f(x_0), x_0]} \psi(x) > 0.$$

Let us fix an $\varepsilon > 0$. By virtue of Theorem 2, for $\frac{\varepsilon}{M}$ there exists such an $N > 0$ that for $n > N$ and $x \in (0, x_0]$ we have

$$(11) \quad |\gamma_n(x) - a| < \frac{\varepsilon}{M}.$$

Hence, in view of (8) and (11) we obtain

$$\begin{aligned} \left| \frac{\psi[f^n(x)]}{G_n(x)} - \varphi_0(x) \right| &= \left| \frac{\psi[f^n(x)]\varphi(x)}{G_n(x)\varphi(x)} - \varphi_0(x) \right| = |\gamma_n(x)\varphi(x) - \varphi_0(x)| = \\ &= \frac{\varphi_0(x)}{|a|} |\gamma_n(x) - a| \leq M \frac{\varepsilon}{M} = \varepsilon \quad \text{for } x \neq 0. \end{aligned}$$

Moreover, let us note that the estimation

$$0 \leq \frac{\psi[f^n(x)]}{G_n(x)} \leq \psi(x)$$

implies that for every $n \in \mathbb{N}$

$$\lim_{x \rightarrow 0} \frac{\psi[f^n(x)]}{G_n(x)} = 0.$$

Thus the decreasing sequence

$$\eta_n(x) := \begin{cases} \frac{\psi[f^n(x)]}{G_n(x)} & \text{for } x \in (0, x_0] \\ 0 & \text{for } x = 0 \end{cases}$$

of continuous functions tends uniformly to φ_0 on $[0, x_0]$ and this proves that φ_0 is continuous in $[0, x_0]$. Consequently (see [4] p. 70) φ_0 is continuous in J . This ends the proof of the theorem.

3. Concluding remarks

Let $\psi \in \Psi$. There are four possible cases.

Case 1: φ_0 is continuous and $\varphi_0 > 0$ on $J \setminus \{0\}$.

In this case $\varphi_0 \in \Phi_1^\psi$. (see [1], Theorem 3.9).

Case 2: $\varphi_0(x) = 0$ for $x \in J$.

In this case $\Phi_0^\psi \neq \emptyset$, by virtue of Theorem 4. Indeed, it is sufficient to take an arbitrary function $\bar{\varphi}$ defined in $[f(x_0), x_0]$ and fulfilling the conditions

$$\begin{aligned}\bar{\varphi}[f(x_0) &= g(x_0)\bar{\varphi}(x_0), \\ |\bar{\varphi}(x)| &> m, \quad x \in [f(x_0), x_0].\end{aligned}$$

Thus we can construct its extension φ , using equation (1), successively in the intervals $[f^{k+1}(x_0), f^k(x_0)]$ for any integer values of k . (see [4] p. 32). If we put additionally $\varphi(0) = 0$ then $\varphi \in \Phi_0^\psi$.

Case 3: φ_0 is continuous, it is not identically equal to zero but there exists such an $\bar{x} \in J \setminus \{0\}$ that $\varphi_0(\bar{x}) = 0$.

In this case $\Phi_a^\psi = \emptyset$ for every $a \in \mathbb{R}$, by virtue of (8).

Case 4: φ_0 is not continuous.

In this case $\Phi_a^\psi = \emptyset$ for every $a \in \mathbb{R}$, by virtue of Theorem 5. An example of such a function $\psi \in \Psi$ that ϕ_0 is not continuous is found in [3].

Consequently, the answer to Brydak's problem is negative (case 3 and 4).

References

- [1] D. BRYDAK, On functional inequalities in a single variable, *Dissertationes Math. CLX*, 1979.
- [2] D. BRYDAK, Problem 2, Report of the Third International Symposium on Functional Equations and Inequalities, Noszvaj 1986, *Publ. Math. Debrecen* **38** (1991), 36.
- [3] D. BRYDAK and B. CHOCZEWSKI, On non-homogeneous linear functional inequality, *Zeszyty Naukowe U.J. Prace Mat.* **18** (1976), 89–92.
- [4] M. KUCZMA, Functional equations in a single variable, *Polish Scientific Publishers, Warszawa*, 1986.

MAREK CZERNI
DEPT. OF MATHEMATICS
PEDAGOGICAL UNIVERSITY OF CRACOW
PODCHORĄŻYCH 2
PL-30-084 KRAKÓW
POLAND

(Received July 27, 1992; revised November 29, 1993)