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On (a, b)-balancing numbers

By TÜNDE KOVÁCS (Debrecen), KÁLMÁN LIPTAI (Eger)
and PÉTER OLAJOS (Miskolc)

Dedicated to Professors K. Győry and A. Sárközy on their 70th birthdays and Professors A. Pethő and J. Pintz on their 60th birthdays

Abstract. A positive integer n is called a balancing number if $1 + \ldots + (n-1) = (n+1) + \cdots + (n+r)$ for some positive integer r. Balancing numbers and their generalizations have been investigated by several authors, from many aspects. In this paper we introduce the concept of balancing numbers in arithmetic progressions, and prove several effective finiteness and explicit results about them. In the proofs of our results, among others, we combine Baker's method, the modular method developed by Wiles and others, a result of Bennett about the diophantine equation $|ax^n - by^n| = 1$, the Chabauty method and the theory of elliptic curves.

1. Introduction and main results

A positive integer n is called a balancing number (see [2] and [14]) if

$$1 + \cdots + (n-1) = (n+1) + \cdots + (n+r)$$

holds for some positive integer r. The sequence of balancing numbers is denoted by B_m $(m = 1, 2, \ldots)$. As one can easily check, we have $B_1 = 6$ and $B_2 = 35$.

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Note that by a result of Behera and Panda [2], we have

$$B_{m+1} = 6B_m - B_{m-1} \quad (m > 1).$$

In particular, there are infinitely many balancing numbers.

The literature of balancing numbers is very rich. In [26] and [27] Liptai proved that there are no Fibonacci and Lucas balancing numbers, respectively. Later, Szalay [39] derived the same results by a different method.

In [28] Liptai, Luca, Pintér and Szalay generalized the concept of balancing numbers in the following way. Let y, k, l be fixed positive integers with $y \geq 4$. A positive integer x with $x \leq y-2$ is called a (k,l)-power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

In [28] several effective and ineffective finiteness results were proved for (k,l)-power numerical centers.

Recently, the "balancing" property has been investigated in recurrence sequences (see [6]). In the present paper we extend the concept of balancing numbers to arithmetic progressions. Let $a > 0$ and $b \geq 0$ be coprime integers. If for some positive integers n and r we have

$$(a+b) + \cdots + (a(n-1)+b) = (a(n+1)+b) + \cdots + (a(n+r)+b)$$

then we say that $an+b$ is an (a,b)-balancing number. The sequence of (a,b)-balancing numbers is denoted by $B_m^{(a,b)}$ $(m = 1, 2, \dots)$. We mention that since $B_m^{(1,0)} = B_m$ for all m, we obtain a generalization of balancing numbers.

We prove several effective finiteness and explicit results concerning polynomial values in the sequences $B_m^{(a,b)}$. That is, we consider the equation

$$B_m^{(a,b)} = f(x) \tag{1}$$

in integers m and x with $m \geq 1$, where f is some polynomial with rational coefficients, taking only integral values at integers. From this point on, when we refer to equation (1) we always assume that a and b are arbitrary, but fixed coprime integers such that $a > 0$ and $b \geq 0$.

Our first result is the following.

Theorem 1. *Let $f(x)$ be a monic polynomial with integer coefficients, of degree ≥ 2. If a is odd, then for the solutions of (1) we have $\max(m, |x|) < c_0(f,a,b)$, where $c_0(f,a,b)$ is an effectively computable constant depending only on a, b and f.*

Our next result concerns the case where $f(x) = x^l$ with some $l \geq 2$. In this case solving equation (1) is equivalent to finding (a, b)-balancing numbers which are perfect powers.

Theorem 2. *If $a^2 - 4ab - 4b^2 = 1$, then there is no perfect power (a, b)-balancing number.*

Remark 1. One can easily check that the equation $a^2 - 4ab - 4b^2 = 1$ has infinitely many solutions in integers a, b with $a > 0$, $b \geq 0$. Hence Theorem 2 completely solves the proposed problem for infinitely many pairs (a, b).

The following theorem takes up the problem where the polynomial $f(x)$ in (1) has some combinatorial meaning. More precisely, we investigate binomial coefficients $\binom{x}{k}$, products of consecutive integers, power sums and alternating power sums. For positive integers k, x let

$$\Pi_k(x) = x(x+1)\dots(x+k-1),$$

$$S_k(x) = 1^k + 2^k + \dots + (x-1)^k,$$

$$T_k(x) = -1^k + 2^k - \dots + (-1)^{x-1}(x-1)^k.$$

We mention that $\Pi_k(x)$, $S_k(x)$ and $T_k(x)$ are polynomials in x, of degrees k, $k+1$ and k, respectively. Note that the coefficients of $\binom{x}{k}$, $S_k(x)$ and $T_k(x)$ are not integers. Further, in the case $f(x) = \Pi_k(x)$ Theorem 1 yields a finiteness result, however, only for the odd values of the parameter a.

For these combinatorial choices of $f(x)$ our next statement yields a bound for the solutions of (1), without any assumptions for the parameters a and b.

Theorem 3. *Let $k \geq 2$ and $f(x)$ be one of the polynomials $\binom{x}{k}$, $\Pi_k(x)$, $S_{k-1}(x)$, $T_k(x)$. Then the solutions of equation (1) satisfy $\max(m, |x|) < c_1(a, b, k)$, where $c_1(a, b, k)$ is an effectively computable constant depending only on a, b and k.*

In our final result, under the assumption $a^2 - 4ab - 4b^2 = 1$, we provide the complete solution of (1) with the above choices of $f(x)$, for some small values of the parameter k. More precisely, we consider all cases where (1) can be reduced to an equation of genus 1. Further, we also solve a particular case of (1) which can be reduced to the resolution of a genus 2 equation.

Theorem 4. *Suppose that $a^2 - 4ab - 4b^2 = 1$. Let $f(x) \in \{\binom{x}{2}, \binom{x}{3}, \binom{x}{4}, \Pi_2(x), \Pi_3(x), \Pi_4(x), S_1(x), S_2(x), S_3(x), S_5(x)\}$. Then the solutions (m, x) of equation (1) are those contained in Table 1. For the corresponding parameter values we have $(a, b) = (1, 0)$ in all cases.*

$f(x)$	Solutions (m,x) of (1)
$\binom{x}{2}$	$(1,-3),(1,4)$
$\binom{x}{3}$	$(2,-5),(2,7)$
$\binom{x}{4}$	$(2,-4),(2,7)$
$\Pi_2(x)$	$(1,-3),(1,2)$
$\Pi_3(x)$	$(1,-3),(1,1)$
$\Pi_4(x)$	$\emptyset$
$S_1(x)$	$(1,-4),(1,3)$
$S_2(x)$	$(3,-8),(3,9),(5,-27),(5,28)$
$S_3(x)$	$\emptyset$
$S_5(x)$	$\emptyset$

Table 1

Remark 2. We considered some other related equations that lead to genus 2 equations. However, because of certain technical difficulties, we could not solve them by the Chabauty method. We checked that under the assumption $a^2-4ab-4b^2=1$ equation (1) has no "small" solutions (i.e. solutions with $|x|\le 10000$) in cases $f(x)\in\left\{\binom{x}{6},\binom{x}{8},\Pi_6(x),\Pi_8(x),S_7(x)\right\}$.

2. Proof of the theorems

For the proof of our theorems we need several lemmas. The first one is of principal importance, because it opens access to the application of deep methods.

Lemma 1. *For any $a>0$, $b\ge 0$ and $m\ge 1$*

$$y^2-8\left(B_m^{(a,b)}\right)^2=a^2-4ab-4b^2 \tag{2}$$

holds with some $y\in\mathbb{Z}$.

Proof. Using the definition of $B_m^{(a,b)}$ and writing $B_m^{(a,b)}=an+b$, a simple calculation shows that

$$ar^2+(a+2B_m^{(a,b)})r-(n-1)(B_m^{(a,b)}+b)=0.$$

The left hand side of this equality is a polynomial in r of degree two. Thus its discriminant must be a square in $\mathbb{Z}$. Since the discriminant in question is given by

$$8\left(B_m^{(a,b)}\right)^2+a^2-4ab-4b^2,$$

the statement follows. □

A polynomial $f(x)$ with complex coefficients is called *non-degenerate* if it has at least three zeros of odd multiplicities. For the proof of Theorem 1 we need the following two lemmas. The next result is due to Brindza [8].

Lemma 2. *Let B be a non-zero rational number and $g(x) \in \mathbb{Q}[x]$ a non-degenerate polynomial. Then for the integral solutions x, y of the hyperelliptic equation*

$$g(x) = By^2$$

we have $\max(|x|, |y|) < c_2(B, g)$, where $c_2(B, g)$ is an effectively computable constant depending only on B and g.

If p is a prime and t is an integer, then by $p^\alpha \| t$ we mean that $p^\alpha | t$ but $p^{\alpha+1} \nmid t$. The following result of Brindza and Pintér [9] provides information on the structure of zeros of certain polynomials.

Lemma 3. *Let $P(X) = a_n X^n + \cdots + a_1 X + a_0$ be a polynomial with integral coefficients, for which a_0 is odd, $4|a_i$ $(i = 1, \ldots, n)$ and $2^3 \| a_n$. Then every zero of P is simple.*

Proof of Theorem 1. Using Lemma 1, from (1) we get the equation

$$8f^2(x) + a^2 - 4ab - 4b^2 = y^2.$$

It is easy to see that since a is odd, the left hand side of the above equation is a polynomial satisfying the conditions of Lemma 3. So, by Lemma 3 we know that the zeros of the left hand side are simple. Hence, by Lemma 2 the theorem follows. □

To prove Theorem 2, we need the following deep result of Bennett [3] about binomial Thue equations. Note that recently this result has been considerably generalized in certain sense (see e.g. the papers [4], [5] and the references given there). However, the following lemma is sufficient for our present purposes.

Lemma 4. *If A, B and n are integers with $AB \neq 0$ and $n \geq 3$, then the equation*

$$|Ax^n - By^n| = 1$$

has at most one solution in positive integers x, y.

Proof of Theorem 2. Using Lemma 1 and substituting $B_m^{(a,b)} = x^l$ into (2), by $a^2 - 4ab - 4b^2 = 1$ we obtain

$$y^2 - 8x^{2l} = 1, \tag{3}$$

with some $y \in \mathbb{Z}$. Rewrite (3) as

$$y^2 - 1 = 8x^t,$$

where $t = 2l$ with $t \geq 4$, as $l \geq 2$.

Obviously, y must be odd. Introducing the notation $y = 2k + 1$, we get

$$k(k+1) = 2x^t.$$

Thus we have $k = 2^\alpha x_1^t$ and $k + 1 = 2^\beta x_2^t$ with $\alpha\beta = 0$, $\alpha + \beta = 1$, where x_1, x_2 are some positive integers. This yields

$$|2^\beta x_2^t - 2^\alpha x_1^t| = 1. \tag{4}$$

Observe that $x_1 = x_2 = 1$ is a solution to (4). Hence by Lemma 4 there are no other solutions. Thus the only possible value for x is $x = 1$, which yields $B_m^{(a,b)} = 1$. Since this is impossible, the theorem follows. □

For the proof of Theorem 3, we need three more lemmas. The first result is due to Ping-Zhi [29].

Lemma 5. *Let k be an integer with $k \geq 5$ and B an algebraic number. Then apart form the cases where $k = 6$, $B = \frac{10 \pm 7\sqrt{7}}{1215}$, the polynomial $\binom{x}{k} + B$ is non-degenerate.*

Let $B_k(x)$ and $E_k(x)$ denote the kth Bernoulli and Euler polynomials, respectively (see e.g. [33]). The next lemmas are due to Pintér and Rakaczki [31] and Rakaczki [34], respectively.

Lemma 6. *If k is an integer with $k \geq 5$ and A, B are complex numbers with $B \neq 0$, then the polynomial $(B_k(x) + A)^2 + B$ is non-degenerate.*

Lemma 7. *If k is an integer with $k \geq 5$ and A, B are complex numbers with $B \neq 0$, then the polynomial $(E_k(x) + A)^2 + B$ is non-degenerate.*

Proof of Theorem 3. Assume first that $k \geq 5$. Using Lemma 1 and (1), we get the equation

$$y^2 = 8\left(f(x)\right)^2 - C(a,b), \tag{5}$$

where $C(a,b) = -(a^2 - 4ab - 4b^2)$. Observe that $C(a,b) \neq 0$. We consider the possible choices for $f(x)$ in turn.

Let $f(x) = \binom{x}{k}$. Factorizing the right hand side of (5), we obtain

$$y^2 = 8\left(f(x) + \sqrt{\frac{C(a,b)}{8}}\right)\left(f(x) - \sqrt{\frac{C(a,b)}{8}}\right). \tag{6}$$

Since $C(a,b) \neq 0$, the zeros of the factors on the right hand side of equation (6) are distinct. Moreover, as one can readily check, $\pm\sqrt{\frac{C(a,b)}{8}} \neq \frac{10\pm7\sqrt{7}}{1215}$, since $C(a,b) \in \mathbb{Z}$. Thus, by Lemmas 2 and 5 the theorem follows in this case.

Now assume that $f(x) = \Pi_k(x)$. In this case Lemma 1 and (1) give

$$y^2 = 8\left(\Pi_k(x)\right)^2 - C(a,b).$$

Since $\Pi_k(x) = k!\binom{x+k-1}{k}$, we get

$$y^2 = 8(k!)^2\left(\binom{x+k-1}{k} + \sqrt{\frac{C(a,b)}{8(k!)^2}}\right)\left(\binom{x+k-1}{k} - \sqrt{\frac{C(a,b)}{8(k!)^2}}\right).$$

Since $C(a,b) \neq 0$, the zeros of the factors on the right hand side are distinct again. Moreover, it is easy to see that $\pm\sqrt{\frac{C(a,b)}{8k!^2}} \neq \frac{10\pm7\sqrt{7}}{1215}$. Hence using Lemmas 2 and 5 the theorem follows also in this case.

Next let $f(x) = S_k(x)$. It is well-known that

$$S_{k-1}(x) = \frac{1}{k}\left(B_k(x) - B_k(0)\right).$$

Then by Lemma 1 and (1) again, we obtain that

$$y^2 = \frac{8}{k^2}\left(\left(B_k(x) - B_k(0)\right)^2 - \frac{k^2C(a,b)}{8}\right).$$

Applying Lemma 6 with $A = -B_k(0)$ and $B = -\frac{k^2C(a,b)}{8} \neq 0$, we see that the right hand side of this equation is non-degenerate. Thus, the theorem follows from Lemma 2.

Finally, let $f(x) = T_k(x)$. It is also well-known that for all $k \in \mathbb{N}$

$$T_k(x) = \frac{1}{2}\left(E_k(x) + (-1)^{x+1}E_k(0)\right)$$

is valid. Lemma 1 and (1) now yield

$$y^2 = 2\left(\left(E_k(x) + (-1)^{x+1}E_k(0)\right)^2 - \frac{C(a,b)}{2}\right).$$

Applying Lemma 7 with $A = (-1)^{x+1}E_k(0)$ and $B = -\frac{C(a,b)}{2} \neq 0$, we get that the right hand side of the above equation is non-degenerate. Again, the theorem follows from Lemma 2.

Consider now the cases when $2 \leq k \leq 4$. In all cases we get that the polynomial on the right hand side of (5) is non-degenerate because its discriminant is non-zero. We consider only one example, all the other cases can be handled similarly.

Let $f(x) = \binom{x}{2}$. In this case the discriminant of the polynomial on the right hand side of (5) is $D := -256C(a,b)^2(8C(a,b)-1)$. Since $C(a,b)$ is a non-zero integer, we get $D \neq 0$, indeed. Therefore, the polynomial on the right side of (5) is non-degenerate, and by Lemma 2 the theorem follows. □

As it was mentioned already, in our numerical results we consider all cases with the above choices of $f(x)$ and with $a^2 - 4ab - 4b^2 = 1$, where (1) can be reduced to an equation of genus 1. Such equations can be handled by a method developed by Stroeker, Tzanakis [36] and independently by Gebel, Pethő, Zimmer [16]. We mention that a similar approach has been used to solve several combinatorial Diophantine equations of different types, for example in [17], [18], [20], [21], [24], [25], [30], [32], [38], [42], [43]. Further, we also solve a particular case of (1) which can be reduced to a genus 2 equation. To solve this equation, we shall use the Chabauty method by the help of explicit techniques developed by Bruin. We note that the Chabauty method has already been successfully used to solve certain other combinatorial Diophantine equations, see e.g. the corresponding results in the papers [13], [19], [22], [23], [35], [40] and the references given there.

Proof of Theorem 4. Using Lemma 1 and the assumption $a^2 - 4ab - 4b^2 = 1$, equation (1) can be written as

$$y^2 = 8f(x)^2 + 1. \tag{7}$$

Actually, we solve equation (7) for all the cases of $f(x)$ listed in Theorem 4. We prove that the solutions are those contained in Table 2. Having the solutions of (7), the solutions of the original equation (1) can be determined with simple calculations.

As it will be clear from the presentation, it is worth to split the resolution of (10) into three parts. Assume first that $f(x) \in \left\{\binom{x}{3}, \Pi_3(x), S_2(x)\right\}$. Then the right hand side of equation (7) can be transformed into a polynomial of degree 3. As the computations are similar in all cases, we consider only one example. Let $f(x) = S_2(x)$. Then (7) is given by

$$y^2 = 8(S_2(x))^2 + 1.$$

$f(x)$	Solutions (x, y) of (7)
$\binom{x}{2}$	$(-3, \pm 17), (-1, \pm 3), (0, \pm 1), (1, \pm 1), (2, \pm 3), (4, \pm 17)$
$\binom{x}{3}$	$(0, \pm 1), (1, \pm 1), (2, \pm 1), (-1, \pm 3), (3, \pm 3), (-5, \pm 99), (7, \pm 99)$
$\binom{x}{4}$	$(-4, \pm 99), (-1, \pm 3), (0, \pm 1), (1, \pm 1),$ $(2, \pm 1), (3, \pm 1), (4, \pm 3), (7, \pm 99)$
$\Pi_2(x)$	$(-3, \pm 17), (-1, \pm 1), (0, \pm 1), (2, \pm 17)$
$\Pi_3(x)$	$(-3, \pm 17), (-1, \pm 1), (0, \pm 1), (1, \pm 17)$
$\Pi_4(x)$	$(-3, \pm 1), (-2, \pm 1), (-1, \pm 1), (0, \pm 1)$
$S_1(x)$	$(-4, \pm 17), (-2, \pm 3), (-1, \pm 1), (0, \pm 1), (1, \pm 3), (3, \pm 17)$
$S_2(x)$	$(-27, \pm 19601), (-8, \pm 577), (-1, \pm 3), (0, \pm 1),$ $(1, \pm 1), (2, \pm 3), (9, \pm 577), (28, \pm 19601)$
$S_3(x)$	$(-1, \pm 3), (0, \pm 1), (1, \pm 1), (2, \pm 3)$
$S_5(x)$	$(-1, \pm 3), (0, \pm 1), (1, \pm 1), (2, \pm 3)$

Table 2

Using the well-known fact $S_2(x) = \frac{x(x-1)(2x-1)}{6}$, we get

$$y^2 = \frac{32x^6 - 96x^5 + 104x^4 - 48x^3 + 8x^2 + 36}{36}.$$

This leads to the elliptic equation

$$Y^2 = X^3 + 2X^2 + 576,$$

where $X = 8(x^2 - x)$, $Y = 24y$. One can compute the integer solutions of this equation with the procedure `IntegralPoints` of Magma [7]. Note that the procedure is based upon a method developed by Gebel, Pethő, Zimmer [16] and Stroeker, Tzanakis [36]. Following the substitutions backwards, we can determine the solutions x, y of the equation (7). The solutions are exactly the ones listed in Table 2. In all the other cases we get the solutions of (7) by a similar calculation.

Assume next that $f(x) \in \left\{\binom{x}{2}, \binom{x}{4}, \Pi_2(x), \Pi_4(x), S_1(x), S_3(x)\right\}$. Then the right hand side of equation (7) can be transformed into a polynomial of degree 4. Since the different choices of f can be handled similarly, we consider only one example, again. Let $f(x) = \Pi_4(x)$. Then (7) has the form

$$y^2 = 8(\Pi_4(x))^2 + 1.$$

Using $\Pi_4(x) = x(x+1)(x+2)(x+3)$, introducing the notation $X = x^2+3x$, this yields

$$y^2 = 8X^4 + 32X^3 + 32X^2 + 1.$$

This equation is of genus 1 and can be solved using the Magma procedure `IntegralQuarticPoints`. We note that this procedure is based upon results of Stroeker and Tzanakis [37] and Tzanakis [41]. Hence, we can find all integral solutions of equation (7), again. The solutions (x, y) are exactly the ones listed in Table 2. All the other cases are similar.

Finally, assume that $f(x) = S_5(x)$. In this case, equation (7) has the form

$$y^2 = 8(S_5(x))^2 + 1.$$

Hence, using the well-known assertion $S_5(x) = \frac{1}{12}(x-1)^2x^2(2x^2-2x-1)$, we get

$$Y^2 = 8X^6 - 8X^5 + 2X^4 + 36, \tag{8}$$

where $X = x^2 - x$ and $Y = 6y$. Equation (8) defines a curve of genus 2 over $\mathbb{Q}$. All its solutions can be determined by applying recent explicit Chabauty techniques due to Bruin. Here we only indicate the main steps of the method without explaining the background theory. For details we refer to the papers of Bruin [10], [11], [12], and the references given there.

Since the Jacobian of the hyperelliptic curve determined by (8) has Mordell-Weil rank 3, the classical Chabauty-type method (see e.g. [15]) does not suffice to find the rational points on (8). To deal with this situation, we apply the elliptic Chabauty method, combined with Magma, following [12]. In the first step, we factorize the right-hand side of equation (8) over the number field $K = \mathbb{Q}(\alpha)$ where $\alpha = \sqrt{-2}$. For later use, we mention that $\{1, \alpha\}$ is an integral basis of K, and that the ring of integers O_K of K is a Euclidean ring. We obtain

$$Y^2 = (2\alpha X^3 - \alpha X^2 + 6)(-2\alpha X^3 + \alpha X^2 + 6). \tag{9}$$

This yields that

$$\delta Z^2 = 2\alpha X^3 - \alpha X^2 + 6 \tag{10}$$

is valid with some $\delta, Z \in O_K$, where δ is square-free in O_K. Observe that (9) and (10) imply that

$$\delta W^2 = -2\alpha X^3 + \alpha X^2 + 6$$

is also valid with some $W \in O_K$. Hence δ divides $(2\alpha X^3 - \alpha X^2 + 6) + (-2\alpha X^3 + \alpha X^2 + 6)$ in O_K, that is $\delta | 12$. Thus, using that the only units in O_K are ± 1,

$\alpha^2 = -2$, and 3 is a prime in O_K, we get that $\delta = \pm\alpha^{t_1}3^{t_2}$ with $t_1, t_2 \in \{0, 1\}$. Taking norms on both sides of (10), we obtain that $\delta \in \{-3, -1, 1, 3\}$. In the cases $\delta = \pm 1$, simple computations show that equation (10) has no solutions. We illustrate this only for $\delta = 1$. Write $Z = Z_1 + \alpha Z_2$ in (10) with $Z_1, Z_2 \in \mathbb{Z}$. Then comparing the coefficients of 1 and α on both sides of (10), we get $Z_1^2 - 2Z_2^2 = 6$. However, this is impossible modulo 16. The case of $\delta = -1$ can be excluded in a similar way.

Let now $\delta = 3$. Equation (10) defines a genus 1 curve over K that can be transformed into a Weierstrass-form elliptic curve E over K by the help of its point $P = (2, \alpha + 2)$. A minimal model of E is given by

$$ME: \quad v^2 = u^3 + 6u + (4\alpha - 1296).$$

Note that all these curves, together with the transformations among them can be handled by Magma. Now, as X, Y are known to be rational coordinates of the hyperelliptic curve defined by (8), one can apply the elliptic Chabauty method to solve (8) completely (following Bruin [12]). To have the method work, the rank of $ME(K)$ should be strictly less than the degree of K (which is 2). It turns out that the rank of $ME(K)$ is 1, so the elliptic Chabauty method is applicable. The procedure `PseudoMordellWeilGroup` of Magma is able to find a subgroup G of $ME(K)$ of finite odd index. Then using the procedure `Chabauty` with the prime 59, we get that $(X, Y) = (2, \pm 18)$ are the only solutions for equation (8) in this case. Substituting back, we obtain that the corresponding solutions to equation (7) are $(x, y) = (0, \pm 1), (1, \pm 1)$.

In case of $\delta = -3$ we can follow a similar argument. The rank of the corresponding elliptic curve is 1 again, so we can proceed as previously. The solutions for equation (8) can be found by using the prime 7 with `Aux:=19` in the procedure `Chabauty` of Magma. We obtain that all solutions of equation (8) are given by $(X, Y) = (0, \pm 6)$ in this case. Following the substitutions backwards, we get that the corresponding solutions to equation (7) are $(x, y) = (-1, \pm 3), (2, \pm 3)$.

From the solutions of equation (7), using (1) and $B_m^{(a,b)} = an + b$ with some integer $n > 0$, the parameters a, b, m can be found by simple calculations. Thus we obtain all solutions (m, x) of (1). They are exactly the ones listed in Table 1, all corresponding to the parameters $(a, b) = (1, 0)$. □

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TÜNDE KOVÁCS
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: tkovacs@math.klte.hu

KÁLMÁN LIPTAI
INSTITUTE OF MATHEMATICS AND INFORMATICS
ESZTERHÁZY KÁROLY COLLEGE
H-3300 EGER, ESZTERHÁZY TÉR 1
HUNGARY

E-mail: liptaik@ektf.hu

PÉTER OLAJOS
DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF MISKOLC
H-3515 MISKOLC-EGYETEMVÁROS
HUNGARY

E-mail: matolaj@uni-miskolc.hu

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