

## A monotonicity property of $M$ -matrices

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**Abstract.** In this paper we show that if  $A \leq B$ ,  $A$  and  $B$  are  $M$ -matrices,  $A$  is nonsingular or irreducible singular and  $B$  is nonsingular, then  $L_A \leq L_B$ ,  $D_A \leq D_B$  and  $U_A \leq U_B$  also hold, where  $X = L_X D_X U_X$  denotes the  $LDU$ -factorization of the matrix  $X$ .

### 1. Introduction

For two matrices  $A$  and  $B$  ( $A, B \in \mathbb{R}^{m \times m}$ ) the inequality  $A \leq B$  holds if  $a_{ij} \leq b_{ij}$  ( $i, j = 1, \dots, m$ ). A matrix  $A$  ( $A \in \mathbb{R}^{m \times m}$ ) is said to be an  $M$ -matrix if  $A = sI - B$ , where  $s > 0$ ,  $B \geq 0$ ,  $I$  is the  $m \times m$  unit matrix and  $s \geq \varrho(B)$ , where  $\varrho(B)$  denotes the spectral radius of  $B$ . We shall prove the following monotonicity theorem on the  $LDU$ -factorization of two  $M$ -matrices  $A$  and  $B$  satisfying  $A \leq B$ .

**Theorem 1.** *Let  $A$  and  $B$  be two  $M$ -matrices such that  $A \leq B$  and assume that  $A$  is nonsingular or irreducible singular and  $B$  is nonsingular. Let  $A = L_A V_A$  and  $B = L_B V_B$  be the  $LU$ -factorizations of  $A$  and  $B$  such that both  $L_A$  and  $L_B$  are unit lower triangular. Then*

$$L_A \leq L_B, \quad V_A \leq V_B.$$

*In addition, if  $V_A = D_A U_A$  and  $V_B = D_B U_B$ , where  $U_A$  and  $U_B$  are unit upper triangular and  $D_A$  and  $D_B$  are diagonal matrices, then*

$$D_A \leq D_B, \quad U_A \leq U_B.$$

The result is related to the fact ([1],[2],[5]) that a nonsingular matrix  $A$  is an  $M$ -matrix if and only if there exist lower and upper triangular  $M$ -matrices  $R$  and  $S$ , respectively, such that  $A = RS$ .

Theorem 1 is not true for general matrices. Using the results of JAIN and SNYDER [3] we define  $A$  and  $B$  as follows:

$$A = \begin{bmatrix} 1 & -4 & 9 & -12 \\ -4 & 17 & -40 & 57 \\ 9 & -40 & 98 & -148 \\ -12 & 57 & -148 & 242 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -4 & 9 & -12 \\ -4 & 17 & -40 & 57 \\ 9 & -40 & 98 & -148 \\ -12 & 57 & -148 & 242 \end{bmatrix}.$$

These matrices are monotone,  $A \leq B$  and

$$L_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 9 & -4 & 1 & 0 \\ -12 & 9 & -4 & 1 \end{bmatrix} \quad L_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ \frac{9}{2} & -\frac{22}{9} & 1 & 0 \\ -6 & \frac{11}{3} & -\frac{240}{67} & 1 \end{bmatrix}.$$

$L_A$  and  $L_B$  are not comparable implying that Theorem 1 does not hold for monotone matrices. The reverse of Theorem 1 also fails to be true. Let

$$L_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -9 & -4 & 1 & 0 \\ -12 & -9 & -4 & 1 \end{bmatrix} \quad L_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix}.$$

The matrices  $L_A$  and  $L_B$  are  $M$ -matrices and  $L_A \leq L_B$ . The product matrix  $L_A L_A^T$  is not an  $M$ -matrix. Furthermore  $L_A L_A^T$  and  $L_B L_B^T$  are not comparable. We notice however that if  $L$  and  $R$  are lower triangular  $M$ -matrices such that  $LL^T \leq RR^T$  holds then  $L \leq R$  also holds. This result is due to SCHMIDT and PATZKE [4].

## 2. Proof of theorem 1

We prove the result by induction. Assume that the  $k \times k$  matrices  $A$  and  $B$  are nonsingular and such that  $L_A \leq L_B$  and  $V_A \leq V_B$  hold. Let

$$A' = \begin{bmatrix} A & c \\ r^T & a \end{bmatrix} \quad B' = \begin{bmatrix} B & p \\ q^T & b \end{bmatrix} \quad (c, r, p, q \in R^k)$$

Assume that  $A'$  and  $B'$  are also nonsingular  $M$ -matrices. They have the  $LU$ -factorizations

$$A' = \begin{bmatrix} L_A & 0 \\ r^T V_A^{-1} & 1 \end{bmatrix} \begin{bmatrix} V_A & L_A^{-1}c \\ 0 & a - r^T A^{-1}c \end{bmatrix}$$

$$B' = \begin{bmatrix} L_B & 0 \\ q^T V_B^{-1} & 1 \end{bmatrix} \begin{bmatrix} V_B & L_B^{-1}p \\ 0 & b - q^T B^{-1}p \end{bmatrix}$$

By assumption  $L_A \leq L_B$ ,  $V_A \leq V_B$ ,  $c \leq p \leq 0$ ,  $r \leq q \leq 0$  and  $0 < a \leq b$ . The relation  $L_A^{-1} \geq L_B^{-1} \geq 0$  implies  $L_A^{-1}c \leq L_B^{-1}p \leq 0$ . The inequality

$V_A^{-1} \geq V_B^{-1} \geq 0$  implies  $r^T V_A^{-1} \leq q^T V_B^{-1} \leq 0$ . Notice that  $a - r^T A^{-1} c = \det(A')/\det(A) > 0$  and  $b - q^T B^{-1} p = \det(B')/\det(B) > 0$ . Finally  $a - r^T A^{-1} c \leq b - q^T B^{-1} p$  follows from  $A^{-1} \geq B^{-1} \geq 0$ . Thus we have proved the first part of the theorem for the nonsingular case. If  $A'$  is an irreducible singular matrix of order  $m$ , then by Theorem 4.16 of BERMAN and PLEMMONS [1, p. 156]  $A'$  has rank  $m - 1$ , each principal submatrix of  $A'$  other than  $A'$  itself is a nonsingular  $M$ -matrix,  $a = r^T A^{-1} c$  and  $A'$  has the  $LU$ -factorization

$$A' = \begin{bmatrix} L_A & 0 \\ r^T V_A^{-1} & 1 \end{bmatrix} \begin{bmatrix} V_A & L_A^{-1} c \\ 0 & 0 \end{bmatrix}$$

with singular upper triangular matrix  $V_{A'}$ . As  $0 \leq b - q^T B^{-1} p$  the theorem also holds in this case. Let  $V_A = D_A U_A$  and  $V_B = D_B U_B$ . As  $A^T$  and  $B^T$  are also  $M$ -matrices which satisfy  $A^T \leq B^T$  we have the  $LU$ -factorizations  $A^T = U_A^T (D_A L_A^T)$  and  $B^T = U_B^T (D_B L_B^T)$  with  $U_A^T \leq U_B^T$  and  $D_A L_A^T \leq D_B L_B^T$ . This implies  $U_A \leq U_B$  and  $0 \leq D_A \leq D_B$ . The same reasoning applies to  $A'$  and  $B'$  if they are nonsingular. If  $A'$  is irreducible singular, then

$$D_{A'} U_{A'} = \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_A & D_A^{-1} L_A^{-1} c \\ 0 & 0 \end{bmatrix}$$

$$D_{B'} U_{B'} = \begin{bmatrix} D_B & 0 \\ 0b - q^T B^{-1} p & 1 \end{bmatrix} \begin{bmatrix} U_B & D_B^{-1} L_B^{-1} p \\ 0 & 1 \end{bmatrix},$$

from which  $0 \leq D_{A'} \leq D_{B'}$  immediately follows. As  $D_A^{-1} L_A^{-1} c \leq D_B^{-1} L_B^{-1} p \leq 0$  the rest of the theorem also follows.

*Remark.* From Theorem 4.16 of BERMAN and PLEMMONS [1, p. 156] it also follows that Theorem 1 does not hold if  $B$  is irreducible singular and  $A \neq B$ .

## References

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