Publ. Math. Debrecen
78/1 (2011), 49–59
DOI: 10.5486/PMD.2011.4392

Cohomogeneity one Minkowski space $\mathbb{R}_1^n$

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Abstract. In this paper we study cohomogeneity one Minkowski space $\mathbb{R}_1^n$. Among other results, we prove that the orbit space is homeomorphic to $\mathbb{R}$ or $[0, \infty)$. We show that if there is a spacelike principal orbit, then each of the orbits is spacelike and principal. If $n = 3$ and there is a singular orbit, we characterize the orbits up to isometry, and the acting group up to conjugacy.

1. Introduction

Cohomogeneity one Rimannian manifolds have been studied by many mathematicians, see [1], [2], [3], [6], [21], [22], [23]. When the metric is indefinite there are not so much papers in the literature. With this paper we want to begin the study of cohomogeneity one pseudo-Riemannian manifolds. Here we take $M = \mathbb{R}_1^n$, i.e. a Minkowski space, and suppose that a connected closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}_1^n)$ acts properly on $\mathbb{R}_1^n$ with an orbit of codimension one.

The main result of this paper (found in § 3) is that if there is a spacelike principal orbit, then there is no singular orbit and each orbit is isometric to $\mathbb{R}^{n-1}$. When $n = 3$ and there is a singular orbit B, we prove that B is a timelike affine subspace of $\mathbb{R}_1^3$, each principal orbit is isometric to $\mathbb{R}_1^1 \times S^1(r)$, $r > 0$, and G is conjugate to $\mathbb{R} \times SO(2)$.

Mathematics Subject Classification: 53C30, 57S25.
Key words and phrases: cohomogeneity one, pseudo-Riemannian and Lorentzian manifolds.

2. Preliminaries

Let (M, g) be a complete pseudo-Riemannian manifold of dimension n and G a connected closed Lie subgroup of $\mathrm{Iso}_g(M)$ which acts properly on M. We say that M *is of cohomogeneity one* under the action of G, if G has an orbit of codimension one. For a general theory of (Riemannian) cohomogeneity one manifolds we refer to [2], [3], [4], [6], [21], [22]. Here we remind some of the indispensable backgrounds.

Definition 2.1 ([8, p. 53]). An action of a group G on a manifold M is said to be proper if the mapping $\varphi : G \times M \to M \times M$, $(g, x) \mapsto (g.x, x)$ is proper.

Here is some of the main properties of proper actions which we use.

The orbit space M/G of a proper action of G on M is Hausdorff and the orbits are closed submanifolds in M, and the stabilizer subgroups are compact [8, p. 149].

Lemma 2.2 ([8, p. 150]). *If there is a proper action of a Lie group G on a connected manifold M, then M possesses a G-invariant Riemannian structure which can be assumed complete.*

Throughout the paper we assume that the investigated group action is effective and proper, so we can pass to a Riemannian manifold (M, g'), in which the Riemannian metric g' on M is G-invariant. A result by Mostert (see [17]) for the compact case, and Berard Bergery [4] for the general case, says that the orbit space M/G, equipped with the quotient topology, is homeomorphic to $\mathbb{R}$, S^1, $[0, +\infty)$ or $[0, 1]$. When M is homotopy equivalent to $\mathbb{R}^n$, we prove in Lemma 3.3 that M/G is homeomorphic to $\mathbb{R}$ or $[0, +\infty)$.

Consider the projection map $M \to M/G$ to the orbit space. Given a point $x \in M$, we say that the orbit $G(x)$ is *principal* (resp. *singular*) if the corresponding image in the orbit space M/G is an internal (resp. boundary) point. A point x whose orbit is principal (resp. singular) will be called *regular* (resp. *singular*). All principal orbits are diffeomorphic to each other, each singular orbit is of dimension less than or equal to $n-1$, where $n = \dim M$. A singular orbit of dimension $n-1$ is called an *exceptional* orbit. Note that an exceptional orbit is never simply connected, and if M is simply connected then exceptional orbits do not exist. If M/G is homeomorphic to $\mathbb{R}$, then each orbit is principal and the orbits form a foliation on M. If M/G is homeomorphic to $[0, +\infty)$ and B is the singular orbit with $\dim B = n-m$, then M is homeomorphic to $G \times_H V$, where H is the isotropy subgroup of a singular point $y \in M$ and the manifold V is H-homeomorphic to an m-dimensional Euclidean space in which the subgroup H acts linearly and transitively on the unit sphere $S^{m-1} \subset V$ (see [2]). Since $G \times_H V$ is a V-bundle over G/H, M is a fibre bundle with typical fibre $\mathbb{R}^m$ over B. If $p : M \to B$ is the fibre bundle, then p restricted to a principal orbit is a fibre bundle with typical fibre S^{m-1} (see [6, p. 181] and [5]).

Proposition 2.3 ([8, p. 152])**.** *Suppose that a Lie group G acts properly on a manifold M such that the orbit space M/G is connected. Then the union M_0 of all regular points is open and dense in M.*

Lemma 2.4 ([8, p. 137])**.** *Let G be a connected Lie group and H a Lie subgroup of G. If $\pi_n(G/H) = 0$ for each $n \geqslant 0$, where π_n is the n-th homotopy group, then the manifold G/H is diffeomorphic to $\mathbb{R}^m$, where $m = \dim G/H$.*

Throughout in the following $\mathbb{R}^n_p$ denotes the n-dimensional real vector space $\mathbb{R}^n$ with a scalar product of signature $(p, n-p)$ given by

$$\langle x, y\rangle = -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n} x_j y_j .$$

The set of all linear isometries $\mathbb{R}^n_p \to \mathbb{R}^n_p$ is a Lie group, which may be identified with the group $O(p, n-p)$ of all matrices $A \in GL(n, \mathbb{R})$ that preserve the scalar product defined above. The identity component of $O(p, n-p)$ is denoted by $SO_o(p, n-p)$. It is known that each maximal compact subgroup of $SO_o(p, n-p)$ is conjugate to $SO(p)\times SO(n-p)$ (see [11] or [16, p. 25]) where $SO(p)\times SO(n-p)$ is considered with respect to the standard decomposition $\mathbb{R}^n_p = \mathbb{R}^p_p \oplus \mathbb{R}^{n-p}$. We write $S^n_1(r)$ for the hypersurface $\{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = r^2\}$ and $H^n(r)$ for a connected component of the hypersurface $H^n_o(r) = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = -r^2\}$.

Definition 2.5 ([18]). Let $L^{n+1} = \mathbb{R}^{n+1}_1$ and let N be a connected spacelike hypersurface. N is said to be isoparametric if its shape operator S has constant eigenvalues (principal curvatures).

In [18] Nomizu showed that a complete connected spacelike isoparametric hypersurface in $\mathbb{R}^{n+1}_1$ has at most two distinct constant principal curvatures. If it has two distinct constant principal curvatures, then it is isometric to

$$H^k(r) \times E^{n-k} = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + x_2^2 + \dots + x_{k+1}^2 = -r^2\} .$$

If it is a complete, connected, totally umbilic spacelike hypersurface (has exactly one principal curvature) then it is isometric to either $\mathbb{R}^{n-1}$ or $H^n(r)$ (see [19, p. 117]).

Definition 2.6. Let $N \subset (M, g)$ be a pseudo-Riemannian submanifold of M. N is called extrinsically homogeneous, if there is a Lie group $G \subset \mathrm{Iso}_g(M)$ such that G acts on N transitively.

Suppose that $M = \mathbb{R}^n_1$ is of cohomogeneity one under the proper action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(M)$. Then each orbit $G(x)$ is extrinsically homogeneous, so isoparametric (see [20]). Furthermore, if $G(x)$ is a spacelike principal orbit, then it is a complete submanifold (see [10]).

Lemma 2.7 ([14])**.** *If a Lie group G is compact, or connected and semisimple, then any smooth representation of G by affine transformations of $\mathbb{R}^n$ admits a fixed point.*

3. Main results

We begin with the formulation of our main Theorem.

Theorem 3.1. *Let $\mathbb{R}^n_1$ be of cohomogeneity one under the proper action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^n_1)$. If there is a spacelike principal orbit, then each orbit is spacelike and isometric to $\mathbb{R}^{n-1}$. In particular there is no singular orbit.*

We prove the Theorem via proving some lemmas.

Lemma 3.2. *If $\mathbb{R}^n_p$, $1 \leqslant p \leqslant n-1$, is of cohomogeneity one under the action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(M)$, then G is not compact.*

Proof. If G is compact then each (principal) orbit is compact, but there is no compact pseudo-Riemannian hypersurface in $\mathbb{R}^n_p$ (see [19, p. 125]), so G is not compact. □

Lemma 3.3. *If $M = \mathbb{R}^n_p$, $1 \leqslant p \leqslant n-1$, is of cohomogeneity one under the proper action of a connected Lie group G, then the orbit space M/G is a one dimensional Hausdorff space homeomorphic to $\mathbb{R}$ or $[0, +\infty)$. In particular, there is at most one singular orbit.*

Proof. Since the action of G on M is proper, by Lemma 2.2 M possesses a G-invariant Riemannian structure, hence we may assume that (M, g') is a Riemannian manifold and $G \subset \mathrm{Iso}_{g'}(M)$ acts on M by cohomogeneity one, therefore M/G is homeomorphic to one of the spaces (i) $\mathbb{R}$, (ii) S^1, (iii) $[0, +\infty)$, (iv) $[0, 1]$, (see [4]), and by Proposition 2.3 the set of regular points is dense in M. So using the same argument as in the proof of Proposition 3.3 of [22] we obtain that the case (iv) is impossible. We claim that case (ii) is also not possible. If $M/G \cong S^1$ then by [2] the projection $\pi : M \to S^1$ is a fibration with fibre G/K, where K is the stabilizer of a regular point. By Theorem 4.41 of [9, p. 379] there is a long exact sequence

$$\begin{aligned}\to \pi_m(G/K, x_0) \to \pi_m(M, x_0) \to \pi_m(S^1, b_0) \to \pi_{m-1}(G/K, x_0) \to \dots\\ \to \pi_0(M, x_0) \to 0\end{aligned}$$

of homotopy groups, where $x_0 \in \pi^{-1}(b_0)$ and $b_0 \in S^1$.

Hence $\pi_0(G/K, x_0) \cong \mathbb{Z}$ and this contradicts the connectedness of G/K. $\square$

Lemma 3.4. *Let $M = \mathbb{R}^n_p$, $1 \leqslant p \leqslant n-1$, be of cohomogeneity one under the proper action of a connected Lie group G. If there is a singular orbit $G(y)$, then it is diffeomorphic to $\mathbb{R}^k$ for some $1 \leqslant k \leqslant n-2$.*

Proof. Suppose that $G(y)$ is a singular orbit. If $\dim G(y) = 0$, then $G_y = G$, so by the properness of the action G must be compact, which contradicts Lemma 3.2. Thus $1 \leqslant \dim G(y) \leqslant n-2$. As there is at most one singular orbit by Lemma 3.3, and the action is proper, by Lemma 2.2 and [2] M is homeomorphic to $G \times_{G_y} V$ where V is an $(n-k)$-dimensional vector space. Hence M is a fibre bundle with base G/G_y. Thus M and $G(y)$ are of the same homotopy type, therefore $G(y)$ is diffeomorphic to $\mathbb{R}^k$ by Lemma 2.4. $\square$

Lemma 3.5. *Let $\mathbb{R}^n_1$ be of cohomogeneity one under the proper action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^n_1)$. Then*

(a) *If there is a spacelike principal orbit, there is no singular orbit.*

(b) *Each spacelike principal orbit is isometric to $\mathbb{R}^{n-1}$.*

Proof. (a) Suppose that $G(x_\circ)$ is a spacelike principal orbit, for some $x_\circ \in \mathbb{R}^n_1$. Since each orbit is extrinsically homogeneous, it is a complete isoparametric hypersurface (see [10] and [20]), so by [18] and [19, p. 117] it is isometric to one of the following spaces

(i) $\mathbb{R}^{n-1}$;

(ii) $H^{n-1}(r)$, a connected component of $H^{n-1}_\circ(r)$;

(iii) $H^k(r) \times \mathbb{R}^{n-k-1}$ where $0 < k < n$ and $r > 0$

where each of them, and so $G(x_\circ)$, is diffeomorphic to $\mathbb{R}^{n-1}$. We claim that there is no singular orbit. If $G(y)$ is a (unique) singular orbit, it is diffeomorphic to $\mathbb{R}^k$ by Lemma 3.4. Hence $G(x_\circ)$ must be a spherical fibre bundle over $G(y)$, (see [6, p. 181]), which is not obviously possible.

(b) Based on the proof of the case (a) it is enough to show that the cases $H^{n-1}(r)$ and $H^k(r) \times \mathbb{R}^{n-k-1}$ do not occur.

Case $b-1$: If $G(x_\circ)$ is isometric to $H^{n-1}(r)$, for some $x_\circ \in \mathbb{R}^n_1$ and $r>0$, then without loss of generality we may assume that $G(x_\circ) = H^{n-1}(r)$ with $\langle x_\circ, x_\circ\rangle = -r^2$. We show that $G \subset SO_o(1, n-1) \times \{0\}$. Fix an arbitrary element $(A, a) \in G \subset SO_\circ(1, n-1) \ltimes \mathbb{R}^n$ and $y \in G(x_\circ)$. Then

$$Ay + a = (A, a).y \in G(x_\circ) = H^{n-1}(r)$$

Since a linear isometry $A : \mathbb{R}^n_1 \longrightarrow \mathbb{R}^n_1$ carries $H^{n-1}_\circ(r)$ on to itself and $H^{n-1}_\circ(r)$ is a pseudo-Riemannian submanifold, $A \mid H^{n-1}_\circ(r) \in \mathrm{Iso}(H^{n-1}_\circ(r))$. So

$$\langle Ay, Ay\rangle = -r^2, \quad \langle Ay + a, Ay + a\rangle = -r^2,$$

hence

$$\langle Ay, a\rangle = -\frac{1}{2}\langle a, a\rangle$$

so

$$\langle x, a\rangle = -\frac{1}{2}\langle a, a\rangle \quad \text{for all } x \in A(H^{n-1}(r)).$$

Thus for each $x \in A(H^{n-1}(r))$ and each curve γ in $A(H^{n-1}_\circ(r))$ with $\gamma(0) = x$ we have $\langle \gamma'(0), a\rangle = 0$, so $a \perp T_x A(H^{n-1}(r))$ for all $x \in A(H^{n-1}(r))$. But $A(H^{n-1}(r))$ is one of the connected components of $H^{n-1}_\circ(r)$, so $a = 0$. Therefore $G \subset SO_\circ(1, n-1)$ and this implies that G stabilizes the origin. In particular G must be compact, hence $G(x) = H^{n-1}(r)$ is compact, which is not true.

Case $b-2$: $G(x) = H^k(r) \times \mathbb{R}^{n-k-1} \subset \mathbb{R}^{k+1}_1 \oplus \mathbb{R}^{n-k-1}$. Fixing an arbitrary element $(A, a) \in G \subset SO_o(1, n-1) \ltimes \mathbb{R}^n$, let $A = (A_1, \ldots, A_n)$, where A_i is i-th row of A and $a = (a_1, \ldots, a_n)^T$. Let $p_1 : \mathbb{R}^{k+1}_1 \oplus \mathbb{R}^{n-k-1} \longrightarrow \mathbb{R}^{k+1}_1$ be the canonical projection, and denote by $\langle \mid \rangle$ the usual inner product in $\mathbb{R}^n$. Since

$$(A, a).x = Ax + a \in H^k(r) \times \mathbb{R}^{n-k-1}, \quad \forall\, x \in H^k(r) \times \mathbb{R}^{n-k-1},$$

it follows that

$$p_1(Ax + a) = \begin{bmatrix} \langle A_1 \mid x\rangle \\ \vdots \\ \langle A_{k+1} \mid x\rangle \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_{k+1} \end{bmatrix} \in H^k(r).$$

If

$$A_i = (A_{i_1}, A_{i_2}) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1} \quad \text{and} \quad x = (x_1, x_2) \in H^k(r) \times \mathbb{R}^{n-k-1}\ ,$$

then

$$\begin{bmatrix} \langle A_{1_1} \mid x_1\rangle \\ \vdots \\ \langle A_{k+1_1} \mid x_1\rangle \end{bmatrix} + \begin{bmatrix} \langle A_{1_2} \mid x_2\rangle \\ \vdots \\ \langle A_{k+1_2} \mid x_2\rangle \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_{k+1} \end{bmatrix} \in H^k(r).$$

If we fix $x_1 \in H^k(r)$ and choose $x_2 \in \mathbb{R}^{n-k-1}$ arbitrarily, then we get

$$A_{1_2} = \cdots = A_{k+1_2} = 0.$$

Hence

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \in SO_\circ(1,n),$$

where $B \in O(1,k)$ and $C \in O(n-k-1)$. Thus $A(H^k_\circ(r) \times \{0\}) = H^k_\circ(r) \times \{0\}$, i.e., $BH^k_\circ(r) = H^k_\circ(r)$ and

$$Bx_1 + \begin{bmatrix} a_1 \\ \vdots \\ a_{k+1} \end{bmatrix} \in H^k(r), \quad \forall\, x_1 \in H^k(r);$$

hence by *subcase* $b-1$ one gets that $a_1 = \cdots = a_{k+1} = 0$.

Thus

$$G = \left\{ \left(\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix} \right) \mid B \in O(1,k),\ C \in O(n-k-1),\ b \in \mathbb{R}^{n-k-1} \right\}$$

This shows that $G(0) = \mathbb{R}^{n-k-1}$ is a singular orbit which contradicts to (a). $\square$

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Suppose that there is a spacelike principal orbit $G(x_\circ)$. We show that $G(x)$ is spacelike and principal for all $x \in \mathbb{R}^n_1$. Without loss of generality we may suppose that $x_\circ = 0$. Then $G(0)$ is isometric to $\mathbb{R}^{n-1}$ and there is no singular orbit by Lemma 3.5. Consider the foliation defined by

$$\{f_x = x + G(0) \mid x \in \mathbb{R}^n_1\}.$$

Here each leaf is a translation of $G(0)$. Denote the space of leaves by Δ, and consider the canonical projection map $\pi : \mathbb{R}^n_1 \to \Delta$. Then Δ with the quotient topology is a manifold diffeomorphic to $\mathbb{R}$. Each vector field X tangent to Δ has a unique lift $\bar{X}$ normal to the fibres on $M = \mathbb{R}^n_1$, hence one may define a scalar product $\langle\,|\,\rangle$ by $\langle X \mid X\rangle \circ \pi = \langle \bar{X}, \bar{X}\rangle$. Thus $(\Delta, \langle\,|\,\rangle)$ is isometric to $\mathbb{R}^1_1$, and π is a pseudo-Riemannian submersion. Now define the (isometric) action of G on Δ as follows

$$G \times \Delta \to \Delta, \quad (g, \pi(x)) \mapsto \pi(gx)$$

Since $G(\pi(0)) = \pi(0)$, G fixes Δ pointwise, i.e. G acts on Δ trivially. Thus $G(x)$ is contained in f_x for each $x \in \mathbb{R}^n_1$, which implies that $G(x)$ is spacelike, and hence isometric to $\mathbb{R}^{n-1}$ by Lemma 3.5. □

By Theorem 3.1, if there is a spacelike principal orbit, then each orbit is spacelike. One would like to get that this result holds, when there is a Lorentzian principal orbit. However the following example shows that this is expectation false. In fact, in the next example there is an open dense subset of M consisting of Lorentzian principal orbits, but there is another degenerate principal orbit!

Example 3.6. Let $M = \mathbb{R}^3_1$ and

$$G = \left\{ (A_t, b_{t,s}) \in SO_o(1,2) \ltimes \mathbb{R}^3 \mid A_t = \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix},\ b_{t,s} = \begin{bmatrix} s \\ s \\ t \end{bmatrix}\ s, t \in \mathbb{R} \right\}.$$

Then G is a subgroup of $\mathrm{Iso}(\mathbb{R}^3_1)$, the action of G on $\mathbb{R}^3_1$ is proper, and the orbit $G(0) = \{(s,s,t) \mid s, t \in \mathbb{R}\}$ is a two dimensional subspace of $\mathbb{R}^3_1$. Since $\{(1,1,0), (0,0,1)\}$ is a basis of this subspace, $v = (1,1,0)$ is null and normal to $u = (0,0,1)$, $G(0)$ is a degenerate subspace. If $a = (x_1, x_2, x_3)$ is an arbitrary point in $\mathbb{R}^3_1$ such that $x_1 \neq x_2$ then the shape operator of $G(a)$ at $A_t a + b_{t,s}$ is

$$S = -e^t \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

so the minimal polynomial of the shape operator is x^2 (the shape operator is not diagonalizable), hence $G(a)$ is a generalized cylinder of type 1 (see [13]). Thus for such an $a \in \mathbb{R}^3_1$ the orbit $G(a)$ is a Lorentzian orbit, but $G(0)$ is not Lorentzian.

As a final result, we characterize the orbits of $\mathbb{R}^3_1$ up to isometry, and the acting group up to conjugacy, when there is a singular orbit.

Theorem 3.7. *Let $\mathbb{R}^3_1$ be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^3_1)$. If there is a singular orbit B then:*

(a) *B is a one dimensional timelike affine subspace of $\mathbb{R}^3_1$.*

(b) *G is conjugate to*

$$S = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & SO(2) \end{bmatrix}, \begin{bmatrix} t \\ 0 \end{bmatrix} \right) \mid t \in \mathbb{R} \right\}$$

(c) *Each principal orbit D is isometric to $\mathbb{R}^1_1 \times S^1(r)$ for some $r > 0$.*

For the proof we need the following lemma.

Lemma 3.8. *Let $\mathbb{R}^n_1$ be of cohomogeneity one under the proper action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^n_1)$, let B be a singular orbit, and $H = G_b$ the isotropy subgroup at a point $b \in B$. Then H is maximal compact subgroup in G.*

Proof. Suppose that H is not a maximal compact subgroup, and that $H \subsetneqq H'$, where H' is a compact Lie subgroup of G. There is a point $x_0 \in \mathbb{R}^n_1$ which is fixed under the action of H', so under H, by Lemma 2.7. We note that $G(x_0)$ is necessarily a singular orbit and x_0 does not belong to the orbit B, since otherwise H and H' would be conjugate, and hence equal. The unique geodesic γ through b and x_0 is fixed pointwise under the action of H, since b and x_0 are fixed. By the properness of the action M_0 is open and dense in M, so $\gamma(t_0)$ is a regular point for some $t_0 \in \mathbb{R}$ and is fixed under the action of H, hence $H \subset G_{\gamma(t_0)}$, which is a contradiction. □

Proof of Theorem 3.7. Since there is no exceptional orbit, by Lemma 3.4 we conclude that $\dim B = 1$. By Lemma 3.3, B is the only singular orbit, hence B is homotopic to M (see [22]), so B is diffeomorphic to $\mathbb{R}$ by Lemma 2.4.

We prove that B is a one dimensional affine subspace of $\mathbb{R}^3_1$. Fixing $y \in B$, as G_y is compact by the properness of the action and connected by Proposition 17 of [19, p. 309], $G_y(x)$ is a compact connected submanifold of $B \cong \mathbb{R}$ for each $x \in B$, so $G_y(x) = \{x\}$. Therefore B is left invariant by G_y pointwise and the geodesic γ through y and $x \in B$ is left invariant by G_y pointwise as well. By the uniqueness of the singular orbit B we get that $B = \gamma(\mathbb{R})$.

Suppose that

$$SO(1) \times SO(2) = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & SO(2) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \right\} \subset SO_o(1,2) \ltimes \mathbb{R}^3.$$

We claim that G_y is conjugate to $SO(1) \times SO(2)$. We know that G_y is a maximal compact subgroup in G by Lemma 3.8, and by [11, p. 275] any maximal compact subgroup of $SO_o(1,2) \ltimes \mathbb{R}^3$ is conjugate to $SO(1) \times SO(2)$, so G_y is conjugate to some subgroup H of $SO(1) \times SO(2)$. If $H \subsetneqq SO(1) \times SO(2)$, then $\dim H = 0$, since by Proposition 17 of [19, p. 309] H is connected, so $H = \{I\}$ and this is impossible since $G_x \subsetneqq H = \{I\}$ for each regular point x, a contradiction. Hence G_y is conjugate to $SO(1) \times SO(2)$. Since G_y leaves B pointwise invariant, it is a normal subgroup in G, and G is isomorphic to $G_y \times \mathbb{R}$, so G is Abelian and $\dim G = 2$, hence $G = Z_G(G_y)$ is conjugate to a Lie subgroup of

$$Z_{SO_o(1,2)\ltimes\mathbb{R}^3}(SO(1) \times SO(2)) = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & SO(2) \end{bmatrix}, \begin{bmatrix} t \\ 0 \end{bmatrix} \right) \mid t \in \mathbb{R} \right\} \quad ;$$

thus G is conjugate to S.

Now we prove that B is timelike. Without loss of generality we may suppose that

$$G = \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & SO(2) \end{pmatrix}, \begin{pmatrix} t \\ 0 \end{pmatrix} \right) \mid t \in \mathbb{R} \right\}.$$

Hence $B = G(0) = \mathbb{R}^1_1 \times \{0\}$ is timelike. Each element of $SO_o(1,2)$ preserves time- and space-orientation, thus the singular orbit is a timelike subspace.

The statement (c) is a consequence of (a) and (b). □

Acknowledgments. The authors would like to thank the Iranian Presidential Office for its support no 83211, and H. Abedi for useful discussions. The authors should thank the referee for his (her) English edition.

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(Received September 8, 2008; revised December 21, 2009)