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On approximation properties of bi-parametric parabolic type potentials

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Abstract. In this paper we investigate the approximation properties of bi-parametric parabolic potentials type operators $A^\alpha_\beta f$ and $\mathcal{A}^\alpha_\beta f$ as the parameter $\alpha > 0$ tends to zero. For $\beta = 2$ these potentials coincide with the classical Jones-Sampson type parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$, respectively.

1. Introduction

Bi-parametric parabolic type potentials are defined as follows (see [1]):

$$\begin{aligned}(A^\alpha_\beta f)(x,t) &= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)} \int_0^\infty \int_{\mathbb{R}^n} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau) f(x-y,t-\tau)\,dyd\tau \\ &\equiv (h_\alpha * f)(x,t) \end{aligned} \tag{1.1}$$

and

$$\begin{aligned}(\mathcal{A}^\alpha_\beta f)(x,t) &= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)} \int_0^\infty \int_{\mathbb{R}^n} e^{-\tau} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau) f(x-y,t-\tau)\,dyd\tau \\ &\equiv \left(\widetilde{h_\alpha} * f\right)(x,t). \end{aligned} \tag{1.2}$$

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Here $\alpha > 0$, $\beta > 0$; $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}^1$; $f \in L_p(\mathbb{R}^{n+1})$, $1 \le p \le \infty$;

$$h_\alpha(y,\tau) = \frac{1}{\Gamma(\alpha/\beta)}\tau_+^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau), \quad \widetilde{h_\alpha}(y,\tau) = \frac{1}{\Gamma(\alpha/\beta)} e^{-\tau}\tau_+^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau);$$

$\tau_+^\theta = \tau^\theta$ if $\tau > 0$ and $\tau_+ = 0$, otherwise.

The kernel $w^{(\beta)}(y,\tau)$ is defined as the Fourier transform of $\exp(-\tau|\xi|^\beta)$

$$w^{(\beta)}(y,\tau) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{iy\cdot\xi - \tau|\xi|^\beta} d\xi, \quad y\cdot\xi = y_1\xi_1 + \cdots + y_n\xi_n, \quad \tau > 0. \tag{1.3}$$

The kernel $w^{(\beta)}(y,\tau)$ coincides with the classical Poisson kernel and Gauss–Weierstrass kernel for $\beta = 1$ and $\beta = 2$, respectively.

Also, note that for $\beta = 2$, the operators (1.1) and (1.2) become the classical Jones–Sampson parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$, respectively (see, [3], [6], [7], [9], [12], [14]).

In this paper it is investigated some approximation properties of the families of $A^\alpha_\beta f$ and $\mathcal{A}^\alpha_\beta f$ as the parameter $\alpha > 0$ tends to zero. It should be noted that the approximation properties of parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$ were investigated in [15]. The classical Riesz and Bessel kernels as approximations of the identity have been studied by T. Kurokawa [10]. See, also [4], [8] in which the relevant problems concerning to Riesz and Bessel type potentials have been studied in more general contexts.

We will need the following statements:

Lemma 1.1 ([1], [2], [5]; cf. [11, p. 44] for $n = 1$)**.** *Let $w^{(\beta)}(y,\tau)$ be defined as* (1.3)*. Then*

a) *For $y \in \mathbb{R}^n$ and $\tau > 0$,*

$$w^{(\beta)}(y,\tau) = \tau^{-n/\beta} w^{(\beta)}\big(\tau^{-1/\beta}y, 1\big); \tag{1.4}$$

b) *$w^{(\beta)}(y,\tau)$ is radial with respect to $y \in \mathbb{R}^n$ and positive provided that $0 < \beta \le 2$;*

c) *If $\beta > 0$ is an even integer, then $w^{(\beta)}(y,1)$ is infinitely smooth and rapidly decreasing. If $\beta \ne 2, 4, 6, \ldots$ then $w^{(\beta)}(y,1)$ has the following behavior when $|y| \to \infty$:*

$$w^{(\beta)}(y,1) = c_\beta |y|^{-n-\beta}(1 + o(1)); \tag{1.5}$$

d) *For all $\tau > 0$ and $\beta > 0$,*

$$\int\limits_{\mathbb{R}^n} w^{(\beta)}(y,\tau)dy = 1. \tag{1.6}$$

For $f \in L_p(\mathbb{R}^{n+1})$ we set

$$\|f\|_p = \left(\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |f(x,t)|^p dxdt \right)^{1/p}, \quad 1 \le p < \infty, \quad \|f\|_\infty = vrai \sup_{\mathbb{R}^{n+1}} |f(x,t)|.$$

We also need the following classes of smooth functions:

$$\Lambda_\mu = \{ f \in L_\infty(\mathbb{R}^{n+1}) : \|f(x-y,t-\tau) - f(x,t)\|_\infty \le \mu(|y|^\beta + \tau) \} \tag{1.7}$$

where $\mu(r)$, $r \ge 0$ is a function of type of modulus of continuity. In the case of $\mu(r) = cr^\gamma$, $0 < \gamma \le 1$, the class (1.7) will be called as bi-parametric Lipschitz class, depending on the parameters γ and β.

Throughout the paper, the letters $c, c_1, c_2, \dots$ and $c(\delta,\beta)$, $c_1(\delta,\beta)$, $c_2(\delta,\beta)$, are used for constants ($c_i(\delta,\beta)$, $i = 1,2,\dots$ depends on the parameters δ and β). We write "$\varphi(\alpha) = O(\psi(\alpha))$, $\alpha \to 0^+$", if $|\varphi(\alpha)| \le c|\psi(\alpha)|$ as $\alpha \to 0^+$.

2. Formulation and proofs of the main results

Theorem 2.1. *Let $f \in L_p(\mathbb{R}^{n+1})$, $1 \le p \le \infty$, $0 < \beta \le 2$, and bi-parametric family of operators A^α_β be defined as* (1.1).

a) *If the limit*
$$\lim_{(y,\tau)\to(x,t)} f(y,\tau) = L, \quad -\infty \le L \le \infty,$$
exists at a point $(x,t) \in \mathbb{R}^{n+1}$, then $\lim_{\alpha\to 0^+} (A^\alpha_\beta f)(x,t) = L$. In particular, if f is continuous at a point (x,t), then
$$\lim_{\alpha\to 0^+} (A^\alpha_\beta f)(x,t) = f(x,t).$$

b) *Let $f \in L_p \cap C_0$, where $C_0 \equiv C_0(\mathbb{R}^{n+1})$ is the class of continuous functions f, for which $\lim_{|x|\to\infty} f(x) = 0$. Then the convergence $\lim_{\alpha\to 0^+} A^\alpha_\beta f = f$ is uniform on $\mathbb{R}^{n+1}$. If $f \in L_p \cap C$, the convergence is uniform on any compact $K \subset \mathbb{R}^{n+1}$.*

Proof. a) Firstly, it should be pointed out that the statement of the theorem is true also for complex-valued functions because of the linearity of operators A^α_β. Now, let $-\infty < L < \infty$. The positivity of $w^{(\beta)}(y,\tau)$ for $0 < \beta \le 2$ and the equalities

$$\int_{\mathbb{R}^n} w^{(\beta)}(y,\tau)dy = 1 \quad \text{and} \quad \int_0^\infty \tau^{\frac{\alpha}{\beta}-1} e^{-\tau} d\tau = \Gamma(\alpha/\beta),$$

yield that

$$|(A^{\alpha}_{\beta}f)(x,t)-L| \le \frac{1}{\Gamma(\alpha/\beta)}\int\limits_0^{\delta}\int\limits_{|y|<\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1}w^{(\beta)}(y,\tau)|f(x-y,t-\tau)-Le^{-\tau}|dy\ d\tau$$

$$+\frac{1}{\Gamma(\alpha/\beta)}\int\limits_0^{\delta}\int\limits_{|y|\ge\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1}w^{(\beta)}(y,\tau)|f(x-y,t-\tau)-Le^{-\tau}|dy\ d\tau$$

$$+\frac{1}{\Gamma(\alpha/\beta)}\int\limits_{\delta}^{\infty}\int\limits_{\mathbb{R}^n} \tau^{\frac{\alpha}{\beta}-1}w^{(\beta)}(y,\tau)|f(x-y,t-\tau)-Le^{-\tau}|dy\ d\tau$$

$$\equiv i_1(\alpha)+i_2(\alpha)+i_3(\alpha). \tag{2.1}$$

The choice of parameter $\delta>0$ is at our disposal. Given $\varepsilon>0$, we choose $\delta>0$ such that

$$|f(x-y,t-\tau)-L|<\varepsilon \quad \text{and} \quad |1-e^{-\tau}|<\varepsilon \tag{2.2}$$

for all $|y|<\delta^{1/\beta}$ and $0<\tau<\delta$. Then we have

$$i_1(\alpha) \le \frac{1}{\Gamma(\alpha/\beta)}\int\limits_0^{\delta}\int\limits_{|y|<\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1}w^{(\beta)}(y,\tau)|f(x-y,t-\tau)-L|dy\ d\tau$$

$$+\frac{|L|}{\Gamma(\alpha/\beta)}\int\limits_0^{\delta}\int\limits_{|y|<\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1}(1-e^{-\tau})w^{(\beta)}(y,\tau)dy\ d\tau$$

$$\le \varepsilon\frac{(1+|L|)}{\Gamma(\alpha/\beta)}\int\limits_0^{\delta}\tau^{\frac{\alpha}{\beta}-1}d\tau\int\limits_{\mathbb{R}^n}w^{(\beta)}(y,\tau)dy$$

$$\stackrel{(1.6)}{=} \frac{\varepsilon(1+|L|)}{\frac{\alpha}{\beta}\Gamma\left(\frac{\alpha}{\beta}\right)}\delta^{\alpha/\beta} = \varepsilon\frac{(1+|L|)\delta^{\alpha/\beta}}{\Gamma\left(1+\frac{\alpha}{\beta}\right)}. \tag{2.3}$$

Further,

$$i_2(\alpha) \le \frac{1}{\Gamma(\alpha/\beta)}\int\limits_0^{\delta}\int\limits_{|y|>\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1}w^{(\beta)}(y,\tau)|f(x-y,t-\tau)|dy\ d\tau$$

$$+\frac{|L|}{\Gamma(\alpha/\beta)}\int\limits_0^{\delta}\int\limits_{|y|>\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1}e^{-\tau}w^{(\beta)}(y,\tau)dy\ d\tau \equiv i_2'(\alpha)+i_2''(\alpha). \tag{2.4}$$

The application of Hölder's inequality gives

$$i_2'(\alpha) \le \frac{\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \int_{|y|>\delta^{1/\beta}} \left[\tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)\right]^{p'} dy d\tau \right)^{1/p'}$$

$$\overset{(1.4)}{=} \frac{\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \int_{|y|>\delta^{1/\beta}} \left[\tau^{\frac{\alpha}{\beta}-1} \tau^{-n/\beta} w^{(\beta)}(\tau^{-1/\beta} y, 1)\right]^{p'} dy\ d\tau \right)^{1/p'}$$

(we set $y = \tau^{\frac{1}{\beta}} z,\ dy = \tau^{\frac{n}{\beta}} dz$)

$$= \frac{\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \int_{|z|>(\frac{\delta}{\tau})^{1/\beta}} \tau^{(\frac{\alpha}{\beta}-1-\frac{n}{\beta})p'+\frac{n}{\beta}} (w^{(\beta)}(z,1))^{p'} dz d\tau \right)^{1/p'}$$

$$\overset{(1.5)}{\le} \frac{c_1(\delta,\beta)\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \tau^{(\frac{\alpha}{\beta}-1-\frac{n}{\beta})p'+\frac{n}{\beta}} d\tau \int_{|z|>(\frac{\delta}{\tau})^{1/\beta}} |z|^{-(n+\beta)p'} dz \right)^{1/p'}$$

$$= \frac{c_2(\delta,\beta)\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \tau^{(\frac{\alpha}{\beta}-1-\frac{n}{\beta})p'+\frac{n}{\beta}} \tau^{(\frac{n}{\beta}+1)p'-\frac{n}{\beta}} d\tau \right)^{1/p'}$$

$$= \frac{c_2(\delta,\beta)\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \tau^{\frac{\alpha}{\beta}p'} d\tau \right)^{1/p'} \le \frac{c_3(\delta,\beta)\|f\|_p}{\Gamma(\alpha/\beta)} \le c_4(\delta,\beta)\|f\|_p \alpha, \tag{2.5}$$

as $\alpha \to 0^+$. Similarly,

$$i_2''(\alpha) \le \frac{|L|}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y|>\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau) dy d\tau$$

$$= \frac{|L|}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|z|>(\frac{\delta}{\tau})^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(z,1) dz d\tau \overset{(1.5)}{\le} c_5(\delta,\beta)|L|\alpha. \tag{2.6}$$

Now, using (2.5) and (2.6) in (2.4) we have

$$i_2(\alpha) \le c_6(\delta,\beta)(\|f\|_p + |L|)\alpha, \quad \text{as } \alpha \to 0^+. \tag{2.7}$$

Let us estimate $i_3(\alpha)$. By using Hölder inequality, we have

$$i_3(\alpha) \leq \frac{|L|}{\Gamma(\alpha/\beta)} \int\limits_{\delta}^{\infty} e^{-\tau} \tau^{\frac{\alpha}{\beta}-1} d\tau \int\limits_{\mathbb{R}^n} w^{(\beta)}(y,\tau) dy$$

$$+ \frac{\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int\limits_{\delta}^{\infty} \int\limits_{\mathbb{R}^n} \left(\tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)\right)^{p'} dy d\tau \right)^{1/p'} \equiv i_3'(\alpha) + i_3''(\alpha). \tag{2.8}$$

By (1.6) it yields that

$$i_3'(\alpha) \leq c_6(\delta,\beta)|L|\alpha \tag{2.9}$$

Using (1.4) and changing variables as $y = \tau^{\frac{1}{\beta}} z$, and then taking into account the estimate (1.5), we have

$$i_3''(\alpha) \leq \frac{c_7(\delta,\beta)\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int\limits_{\delta}^{\infty} \tau^{(\alpha-1-\frac{n}{\beta})p'+\frac{n}{\beta}} d\tau \right)^{1/p'} \leq c_8(\delta,\beta)\|f\|_p \alpha \tag{2.10}$$

for $\alpha < \frac{1}{p}\left(\frac{n}{\beta}+1\right)$

Therefore,

$$i_3(\alpha) \leq c_9(\delta,\beta)(\|f\|_p + |L|)\alpha, \quad \text{as } \alpha \to 0. \tag{2.11}$$

Now, substituting (2.3), (2.7) and (2.11) in (2.1), we obtain

$$|(A_\beta^\alpha f)(x,t) - L| \leq \varepsilon \frac{(1+|L|)\delta^{\alpha/\beta}}{\Gamma\left(1+\frac{\alpha}{\beta}\right)} + c(\delta,\beta)(\|f\|_p + 1)\alpha, \quad (\alpha \ll 1). \tag{2.12}$$

Since $\varepsilon > 0$ is arbitrary, the last estimate yields that

$$\lim_{\alpha\to 0} |(A_\beta^\alpha f)(x,t) - L| = 0.$$

Let now $\lim_{(y,\tau)\to(x,t)} f(y,\tau) = +\infty$ (the case of $L = -\infty$ is investigated in a similar way). For a given $M > 0$ there exists $\delta > 0$ such that $f(x-y,t-\tau) > M$, for $|y| < \delta^{1/\beta}$ and $0 < \tau < \delta$.

We have

$$(A_\beta^\alpha f)(x,t) = \frac{1}{\Gamma(\alpha/\beta)} \int\limits_0^{\delta} \int\limits_{|y|<\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau) f(x-y,t-\tau) dy d\tau$$

$$+ \frac{1}{\Gamma(\alpha/\beta)} \int\limits_0^{\delta} \int\limits_{|y|>\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau) f(x-y,t-\tau) dy d\tau$$

$$+\frac{1}{\Gamma(\alpha/\beta)}\int_{\delta}^{\infty}\int_{\mathbb{R}^n}\tau^{\frac{\alpha}{\beta}-1}w^{(\beta)}(y,\tau)f(x-y,t-\tau)dyd\tau$$

$$\equiv j_1(\alpha)+j_2(\alpha)+j_3(\alpha).$$

Further,

$$j_1(\alpha)\geq\frac{M}{\Gamma(\alpha/\beta)}\int_{0}^{\delta}\int_{|y|<\delta^{1/\beta}}\tau^{\frac{\alpha}{\beta}-1}w^{(\beta)}(y,\tau)dyd\tau$$

we use (1.4) and set $y=\tau^{\frac{1}{\beta}}z$)

$$=\frac{M}{\Gamma(\alpha/\beta)}\int_{0}^{\delta}\int_{|z|<(\frac{\delta}{\tau})^{1/\beta}}\tau^{\frac{\alpha}{\beta}-1}w^{(\beta)}(z,1)dzd\tau\geq\frac{M}{\Gamma(\alpha/\beta)}\int_{0}^{\delta}\tau^{\frac{\alpha}{\beta}-1}d\tau\int_{|z|\leq 1}w^{(\beta)}(z,1)dz$$

$$=c\frac{M}{\frac{\alpha}{\beta}\Gamma\big(\frac{\alpha}{\beta}\big)}\delta^{\alpha/\beta}=c\frac{\delta^{\alpha/\beta}}{\Gamma\big(1+\frac{\alpha}{\beta}\big)}M. \tag{2.13}$$

It is not difficult to see that (cf. (2.7) and (2.11)

$$|j_2(\alpha)|\leq c_6(\delta,\beta)\big(\|f\|_p+1\big)\alpha,\qquad \|j_3(\alpha)|\leq c_9(\delta,\beta)\big(\|f\|_p+1\big)\alpha. \tag{2.14}$$

Using (2.13) and (2.14) we have

$$(A^{\alpha}_{\beta}f)(x,t)\geq c\frac{\delta^{\alpha/\beta}}{\Gamma\big(1+\frac{\alpha}{\beta}\big)}M-c_6(\delta,\beta)\big(\|f\|_p+1\big)\alpha-c_9(\delta,\beta)\big(\|f\|_p+1\big)\alpha,$$

and therefore,

$$\liminf_{\alpha\to 0^+}(A^{\alpha}_{\beta}f)(x,t)\geq cM.$$

Since $M>0$ is arbitrary, it follows $\lim_{\alpha\to 0^+}(A^{\alpha}_{\beta}f)(x,t)=\infty$.

b) Let $f\in L_p\cap C_0$. Given $\varepsilon>0$, we choose a parameter $\delta>0$ such that

$$\sup_{(x,t)\in\mathbb{R}^{n+1}}|f(x-y,t-\tau)-f(x,t)|<\varepsilon \quad\text{and}\quad (1-e^{-\tau})<\varepsilon \tag{2.15}$$

for all $|y|<\delta^{\frac{1}{\beta}}$ and $0<\tau<\delta$.

Now, setting $L=f(x,t)$ in (2.1), and using (2.3), (2.7), (2.11) and (2.15), we have

$$\|A^{\alpha}_{\beta}f-f\|_{\infty}\leq\varepsilon(1+\|f\|_{\infty})\frac{\delta^{\alpha/\beta}}{\Gamma\big(1+\frac{\alpha}{\beta}\big)}+c(\delta,\beta)(1+\|f\|_{\infty})\alpha.$$

The latter estimate implies $\lim_{\alpha\to 0^+}\|A^{\alpha}_{\beta}f-f\|_{\infty}=0$. □

Remark 2.2. By using the estimate

$$\|\mathcal{A}_\beta^\alpha f\|_p \le c\|f\|_p, \quad 1 \le p \le \infty \tag{2.16}$$

and equality (cf. (2.1))

$$\begin{aligned}
&\mathcal{A}_\beta^\alpha f(x,t) - L \\
&= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y|<\delta^{1/\beta}} e^{-\tau}\tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)(f(x-y,t-\tau)-L)dyd\tau \\
&+ \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y|>\delta^{1/\beta}} e^{-\tau}\tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)(f(x-y,t-\tau)-L)dyd\tau \\
&+ \frac{1}{\Gamma(\alpha/\beta)} \int_\delta^\infty \int_{\mathbb{R}^n} e^{-\tau}\tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)(f(x-y,t-\tau)-L)dyd\tau,
\end{aligned} \tag{2.17}$$

in complete analogy to Theorem 2.1 the following theorem can be proved.

Theorem 2.3. *Let $f \in L_p(\mathbb{R}^{n+1})$, $1 \le p < \infty$, and the operator A_β^α be defined by* (1.2). *Then,*

a) *If the limit $\lim_{(y,\tau)\to(x,t)} f(y,\tau) = L$ exists, at a point $(x,t) \in R^{n+1}$, then $\lim_{\alpha\to 0^+}(A_\beta^\alpha f)(x,t) = L$. In particular, if f is continuous at a point (x,t), then $\lim_{\alpha\to 0^+}(A_\beta^\alpha f)(x,t) = f(x,t)$.*

b) *Let $f \in L_p \cap C_0$. Then the convergence $\lim_{\alpha\to 0^+} A_\beta^\alpha f = f$ is uniform on R^{n+1}. If $f \in L_p \cap C$, the convergence is uniform on any compact $K \subset R^{n+1}$.*

c) *If $f \in L_p(\mathbb{R}^{n+1})$, $1 \le p < \infty$, then $\lim_{\alpha\to 0^+} \|\mathcal{A}_\beta^\alpha f - f\|_p = 0$.*

The next theorem gives an estimation of an error of approximation of functions $f \in \Lambda_\mu$ by the families $A_\beta^\alpha f$ and $\mathcal{A}_\beta^\alpha f$ as $\alpha \to 0^+$.

Theorem 2.4. *Let the operator P_β^α be either A_β^α or A_β^α, $\alpha > 0$.*

a) *Suppose that $f \in L_p \cap \Lambda_\mu$, where $1 \le p < \infty$ and $\mu(s)$, $s \ge 0$, is a function of type of modulus of continuity which satisfies the condition*

$$\int_0^1 \frac{\mu(\tau)}{\tau} \ln\frac{1}{\tau}\, d\tau < \infty. \tag{2.18}$$

If $0 < \beta \le 2$, then

$$\|P_\beta^\alpha f - f\|_\infty = O(1)\alpha \quad \text{as } \alpha \to 0^+.$$

In particular, for Lipschitz functions (i.e., for $f \in \Lambda_\mu \cap L_p$, where $\mu(s) = cs^\gamma$, $0 < \gamma \leq 1$) we have

$$\|P^\alpha_\beta f - f\|_\infty = O(1)\alpha \quad \text{as } \alpha \to 0^+.$$

b) *Let $f \in L_p \cap \Lambda_\mu$, where $\mu(s) = cs^\lambda |\log s|^\gamma$, $0 < \lambda < 1$ and $\gamma \geq 0$. Then*

$$\|P^\alpha_\beta f - f\|_\infty = O(1)\alpha \quad \text{as } \alpha \to 0^+.$$

Proof. We only prove the case when $P^\alpha_\beta = A^\alpha_\beta$. The case of $P^\alpha_\beta = \mathcal{A}^\alpha_\beta$ may be proved by the same way.

a) Let $f \in L_p \cap \Lambda_\mu$, where Λ_μ is defined by (1.7). Setting $L = f(x,t)$ in (2.1), we get

$$|(A^\alpha_\beta f)(x,t) - f(x,t)| \leq i_1(\alpha) + i_2(\alpha) + i_3(\alpha). \tag{2.19}$$

In a complete analogy with (2.7) and (2.11), it follows that

$$i_2(\alpha) \leq c(\delta,\beta)(\|f\|_p + \|f\|_\infty)\alpha, \quad i_3(\alpha) \leq c(\delta,\beta)(\|f\|_p + \|f\|_\infty)\alpha. \tag{2.20}$$

Let us estimate $i_1(\alpha)$. We have

$$\begin{aligned} i_1(\alpha) &\leq \frac{1}{\Gamma(\alpha/\beta)} \int\limits_0^\delta \int\limits_{|y|<\delta^{\frac{1}{\beta}}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)|f(x-y,t-\tau) - e^{-\tau} f(x,t)|dyd\tau \\ &\leq \frac{1}{\Gamma(\alpha/\beta)} \int\limits_0^\delta \int\limits_{|y|<\delta^{\frac{1}{\beta}}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)\|f(x-y,t-\tau) - f(x,t)\|_\infty dyd\tau \\ &\quad + \frac{\|f\|_\infty}{\Gamma(\alpha/\beta)} \int\limits_0^\delta \tau^{\frac{\alpha}{\beta}-1}(1-e^{-\tau})\Big(\int\limits_{|y|<\delta^{\frac{1}{\beta}}} w^{(\beta)}(y,\tau)dy\Big)d\tau \\ &\equiv i'_1(\alpha) + i''_2(\alpha). \end{aligned} \tag{2.21}$$

Since,

$$\int\limits_{|y|<\delta^{\frac{1}{\beta}}} w^{(\beta)}(y,\tau)dy \leq \int\limits_{\mathbb{R}^n} w^{(\beta)}(y,\tau)dy = 1,$$

it follows that

$$i''_1(\alpha) \leq c(\delta,\beta)\frac{\|f\|_\infty}{\Gamma(\alpha/\beta)} = c_1(\delta,\beta)\|f\|_\infty \alpha. \tag{2.22}$$

On the other hand,

$$i_1'(\alpha) \le \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y|<\delta^{\frac{1}{\beta}}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)\mu(|y|^\beta+\tau)dyd\tau$$

$$= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y|<\delta^{\frac{1}{\beta}}} \tau^{\frac{\alpha}{\beta}-1}\tau^{-n/\beta} w^{(\beta)}(\tau^{-\frac{1}{\beta}}y,1)\mu(|y|^\beta+\tau)dyd\tau$$

(we set $y=\tau^{\frac{1}{\beta}}z,\ dy=\tau^{\frac{n}{\beta}}dz$)

$$= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|z|<(\frac{\delta}{\tau})^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(z,1)\mu(\tau(|z|^\beta+1))dzd\tau.$$

Since $\mu(s)$ is a function of type of modulus of continuity, we have

$$\mu(\tau(|z|^\beta+1)) \le (|z|^\beta+2)\mu(\tau).$$

Therefore, for $0<\delta<1$ we have

$$i_1'(\alpha) \le \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1}\mu(\tau)\left(\int_{|z|<(\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z,1)(|z|^\beta+2)dz\right)d\tau.$$

Since

$$\int_{|z|\le 1} w^{(\beta)}(z,1)(|z|^\beta+2)dz \equiv c_1(\delta,\beta)<\infty,$$

it follows that

$$\int_{|z|<(\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z,1)(|z|^\beta+2)dz = \int_{|z|\le 1} w^{(\beta)}(z,1)(|z|^\beta+2)dz$$

$$+\int_{1<|z|<(\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z,1)(|z|^\beta+2)dz = c_1(\delta,\beta)+\int_{1<|z|<(\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z,1)(|z|^\beta+2)dz,$$

and therefore,

$$i_1'(\alpha) \le \frac{c_1(\delta,\beta)}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1}\mu(\tau)d\tau$$

$$+\frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1}\mu(\tau)\left(\int_{1<|z|<(\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z,1)(|z|^\beta+2)dz\right)d\tau. \tag{2.23}$$

Further,

$$\int\limits_{1<|z|<(\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z,1)(|z|^\beta+2)dz \overset{(1.5)}{\le} c_2(\delta,\beta) \int\limits_{1<|z|<(\frac{\delta}{\tau})^{1/\beta}} |z|^{-n-\beta}(|z|^\beta+2)dz$$

$$\le c_3(\delta,\beta) \int\limits_{1<|z|<(\frac{\delta}{\tau})^{1/\beta}} |z|^{-n}dz = c_4(\delta,\beta)\ln\left(\frac{\delta}{\tau}\right), \quad (0<\tau<\delta).$$

Using this in (2.23), for $\alpha \to 0^+$ we have

$$i_1'(\alpha) \le \frac{c_1(\delta,\beta)}{\Gamma(\alpha/\beta)} \int\limits_0^\delta \tau^{\frac{\alpha}{\beta}-1}\mu(\tau)d\tau + \frac{c_4(\delta,\beta)}{\Gamma(\alpha/\beta)} \int\limits_0^\delta \tau^{\frac{\alpha}{\beta}-1}\mu(\tau)\ln\frac{\delta}{\tau}d\tau$$

$$\le \frac{c_5(\delta,\beta)}{\Gamma(\alpha/\beta)} \int\limits_0^\delta \frac{\mu(\tau)}{\tau}\ln\frac{\delta}{\tau}d\tau \overset{(2.18)}{\le} c_6(\delta,\beta)\alpha. \tag{2.24}$$

The estimates (2.21), (2.22) and (2.24) yield that

$$i_1(\alpha) \le c_7(\delta,\beta)(\|f\|_\infty+1)\alpha. \tag{2.25}$$

Finally, from (2.20) and (2.25) it follows that

$$\|A_\beta^\alpha f - f\|_\infty \le c_8(\delta,\beta)(\|f\|_p+\|f\|_\infty+1)\alpha, \quad \text{as } \alpha\to 0^+.$$

b) By taking into account (2.21), we have

$$i_1'(\alpha) \equiv \frac{1}{\Gamma(\alpha/\beta)} \int\limits_0^\delta \int\limits_{|y|<\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)\|f(x-y,t-\tau)-f(x,t)\|_\infty dy d\tau$$

$$\le \frac{c}{\Gamma(\alpha/\beta)} \int\limits_0^\delta \int\limits_{|y|<\delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y,\tau)(|y|^\beta+\tau)^\lambda |\log(|y|^\beta+\tau)|^\gamma dy d\tau$$

(we assume $0<\delta<\dfrac{1}{e}$ and set $y=\tau^{\frac{1}{\beta}}z,\ dy=\tau^{\frac{n}{\beta}}dz$)

$$\le \frac{c}{\Gamma(\alpha/\beta)} \int\limits_0^\delta \tau^{\frac{\alpha}{\beta}-1+\lambda}|\log\tau|^\gamma \int\limits_{\mathbb{R}^n} w^{(\beta)}(z,1)(|z|^\beta+1)^\lambda \left(1+\frac{\log(|z|^\beta+1)}{|\log\tau|}\right)^\gamma dz d\tau.$$

Since $0<\tau<\delta<\frac{1}{e}$, we have $\frac{1}{|\log\tau|}<1$ and therefore, by (1.5)

$$i_1'(\alpha)\le\frac{c_1(\delta,\beta)}{\Gamma(\alpha/\beta)}\int\limits_0^\delta \tau^{\frac{\alpha}{\beta}-1+\lambda}|\log\tau|^\gamma d\tau$$

$$\times\int\limits_{|z|\ge1}|z|^{-n-\beta}(|z|^\beta+1)^\lambda(1+\log(|z|^\beta+1))^\gamma dz$$

$$\le\frac{c_2(\delta,\beta)}{\Gamma(\alpha/\beta)}\int\limits_0^\delta\tau^{-1+\lambda}|\log\tau|^\gamma d\tau\int\limits_{|z|\ge1}|z|^{-n-\beta+\beta\lambda}(\log|z|)^\gamma dz. \qquad (2.26)$$

Since $\lambda>0$, the first integral is finite for all $\gamma\ge0$.

Let us estimate the second integral. We have

$$\int\limits_{|z|\ge1}|z|^{-n-\beta+\beta\lambda}(\log|z|)^\gamma dz=c_3(\delta,\beta)\int\limits_1^\infty r^{\beta(\lambda-1)-1}(\log r)^\gamma dr.$$

Since $0<\lambda<1$, the latter integral is finite for all $\gamma\ge0$. Now it follows from (2.26) that

$$i_1'(\alpha)\le c_4(\delta,\beta)\alpha,\quad\text{as }\alpha\to0^+. \qquad (2.27)$$

Finally, the desired result follows from (2.19), (2.20), (2.21) and (2.27). □

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