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A new construction for abundant semigroups with multiplicative quasi-adequate transversals

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Abstract. In any abundant semigroup with a quasi-adequate transversal, we define two sets R and L and give some properties and characterizations associated with them. Then we give a structure theorem for abundant semigroups with multiplicative quasi-adequate transversals by means of two quasi-adequate semigroups R and L.

1. Introduction

The concept of an inverse transversal of a regular semigroup was first introduced by Blyth and McFadden in 1982 [1]. Since then, this class of regular semigroups has attracted several authors' attention and a series of important results have been obtained ([1], [13] and its references). If S is a regular semigroup, then an inverse transversal of S is an inverse subsemigroup S^o such that S^o meets $V(a)$ precisely once for each $a \in S$ (that is, $|V(a) \cap S^o| = 1$), where $V(a) = \{x \in S \mid axa = a \text{ and } xax = x\}$ denotes the set of inverses of a. Blyth and McFadden in [1] gave a structure theorem for regular semigroups with multiplicative inverse transversals. Orthodox transversals were introduced by Chen [2] as a generalization of inverse transversals, and a structure theorem for regular semigroups with quasi-ideal orthodox transversals was given. In [8] and [10], Kong also gave two structure theorems for this class of regular semigroups by means of a formal set (B, R) and a like-spined product (R, L) respectively.

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An analogue of an inverse transversal, which is termed an adequate transversal, was introduced for abundant semigroups by El-Qallali [4]. In [4], El-Qallali constructed abundant semigroups with multiplicative type-A transversals. Afterwards, Kong [11] considered some properties associated with adequate transversals and [9] gave a construction of abundant semigroups with quasi-ideal adequate transversals by a like-spined product (R, L). Quasi-adequate transversals, as a common generalization of orthodox transversals and adequate transversals, were introduced by Ni in [12]. And in [12], Ni gave a structure theorem for abundant semigroups with multiplicative quasi-adequate transversals by a QA-system. The aim of this paper is to get a construction for this class of abundant semigroups by the method used in [9] and [10], that is by a like-spined product (R, L). As a consequence, it will be rather difficult to describe the three relations $\mathcal{R}^*$, $\mathcal{L}^*$ and δ by two components $K(x)$ and L_a^*.

In order to overcome the above difficulty we introduce a new relation $\mathcal{K}$ by $(a, b) \in \mathcal{K}$ if $R_a^* = R_b^*$ and $\delta(a) = \delta(b)$ in quasi-adequate semigroups. By the set (R, L) and the new defined relation $\mathcal{K}$, a structure theorem for abundant semigroups with multiplicative quasi-adequate transversals is obtained in this paper.

On a semigroup S the relation $\mathcal{L}^*$ is defined by the rule that $a\mathcal{L}^*b$ if and only if the elements a, b of S are related by Green's relation $\mathcal{L}$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is dually defined. Evidently, $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence and $\mathcal{L} \subseteq \mathcal{L}^*$, $\mathcal{R} \subseteq \mathcal{R}^*$. If a, b are regular elements of S, then $a\mathcal{L}^*b$ ($a\mathcal{R}^*b$) if and only if $a\mathcal{L}b$ ($a\mathcal{R}b$), what is more, if S is a regular semigroup, then $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$. A semigroup in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains at least one idempotent is called *abundant*. An abundant semigroup S is called $quasi-adequate$ if its idempotents form a subsemigroup. An *adequate semigroup* is a quasi-adequate semigroup in which the idempotents commute. We list some basic results as follows which are frequently used in this paper. The following Lemma is due to Fountain [6] which providing an alternative description for $\mathcal{L}^*(\mathcal{R}^*)$.

Lemma 1.1 ([6]). *Let S be a semigroup and $a, b \in S$. Then the following conditions are equivalent:*

(1) $a\mathcal{L}^*b$ $(a\mathcal{R}^*b)$;

(2) *For all $x, y \in S^1, ax = ay$ $(xa = ya)$ if and only if $bx = by$ $(xb = yb)$.*

As an easy but useful consequence of (2) we have

Corollary 1.2. *Let a be an element of a semigroup S and e be an idempotent of S. Then the following conditions are equivalent:*

(1) *$a\mathcal{L}^*e$ ($a\mathcal{R}^*e$);*

(2) *$a = ae$ ($ea = a$) and for all $x, y \in S^1$, $ax = ay$ ($xa = ya$) implies $ex = ey$ ($xe = ye$).*

Lemma 1.3 ([4])**.** *Let S be an abundant semigroup with the set of idempotents E and $x, y \in S$. If there exist $e, f \in E$ such that $x = eyf$ and $e\mathcal{L}y^+, f\mathcal{R}y^*$ for some $y^+, y^* \in E$, then $e\mathcal{R}^*x$ and $f\mathcal{L}^*x$.*

Let S be a quasi-adequate semigroup with the band of idempotents B. For $e \in B$, denote by $E(e)$ the $\mathcal{J}$-class of B containing e. It is known that $E(e)$ is a rectangular subband of B and $E(e) = V(e)$, the set of inverses of e in B (for detail, see [7]). Define a relation δ on S by: for $a, b \in S$,

$$a\delta b \Longleftrightarrow E(a^+)aE(a^*) = E(b^+)bE(b^*) \quad \text{for some } a^+, a^*, b^+, b^*.$$

It follows from [5] that δ is an equivalence relation which contained in any adequate congruence on S. In particular, if S is an orthodox semigroup, then δ is the least inverse congruence on S. Consequently, $\delta \cap (B \times B) = \mathcal{J}^B$ is the least semilattice congruence on B.

Lemma 1.4 ([5])**.** *Let S be a quasi-adequate semigroup with the band of idempotents B and $a, b \in S$. Then*

(1) *$\delta(a) = E(a^+)aE(a^*)$;*

(2) *$a\delta b \Longleftrightarrow b = eaf$ for some $e \in E(a^+)$, $f \in E(a^*)$;*

(3) *$\mathcal{H}^* \cap \delta = l$.*

For any quasi-adequate semigroup S, the result in Lemma 1.3 can be generalized.

Lemma 1.5 ([12])**.** *Let S be a quasi-adequate semigroup with the band of idempotents E and $x, y \in S$. If there exist $e, f \in E$ such that $x = eyf$ and $e \in E(y^+)$, $f \in E(y^*)$ for some $y^+, y^* \in E$, then $e\mathcal{R}^*x$ and $f\mathcal{L}^*x$.*

Let S be an abundant semigroup and U an abundant subsemigroup of S, U is called a $*$-subsemigroup of S if for any $a \in U$, there exist an idempotent $e \in L^*_a(S) \cap U$ and an idempotent $f \in R^*_a(S) \cap U$. As pointed out in [5], an abundant subsemigroup U of an abundant semigroup S is a $*$-subsemigroup of S if and only if $\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U)$ and $\mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U)$.

Let S^o be an abundant $*$-subsemigroup of S and E^o be the set of idempotents of S^o. S^o is called an *abundant transversal* [12] of S if for any $x \in S$, there exist $x^o \in S^o$, $i, \lambda \in E$ such that $x = ix^o\lambda$, where $i\mathcal{L}^*x^{o+}, \lambda\mathcal{R}^*x^{o*}$ for some $x^{o+}, x^{o*} \in E^o$. In this case, let

$$C_{S^o}(x) = \{x^o \in S^o \mid x = ix^o\lambda, i\mathcal{L}x^{o+}, \lambda\mathcal{R}x^{o*} \quad \text{for some } x^{o+}, x^{o*} \in E^o\},$$

$$I_x = \{i \in E \mid (\exists x^o \in C_{S^o}(x))\ x = ix^o\lambda, i\mathcal{L}x^{o+}, \lambda\mathcal{R}x^{o*} \quad \text{for some } x^{o+}, x^{o*} \in E^o\},$$

$$\Lambda_x = \{\lambda \in E \mid (\exists x^o \in C_{S^o}(x))\ x = ix^o\lambda, i\mathcal{L}x^{o+}, \lambda\mathcal{R}x^{o*} \quad \text{for some } x^{o+}, x^{o*} \in E^o\},$$

$$I = \bigcup_{x \in S} I_x, \qquad \Lambda = \bigcup_{x \in S} \Lambda_x.$$

Let S be an abundant semigroup with the set of idempotents E and S^o a quasi-adequate $*$-subsemigroup of S with the set of idempotents E^o. S^o is called a $quasi-adequate\ transversal$ of S if

(QA1) $(\forall x \in S)\ \ C_{S^o}(x) \neq \emptyset$,

(QA2) $(\forall e \in E)\ \ (\forall g \in E^o)$,

$$C_{S^o}(e)C_{S^o}(g) \subseteq C_{S^o}(ge) \quad \text{and} \quad C_{S^o}(g)C_{S^o}(e) \subseteq C_{S^o}(eg).$$

A quasi-adequate transversal S^o is called a multiplicative quasi-adequate transversal of S if the following condition is satisfied

(M) $(\forall x, y \in S)\ \ \Lambda_x I_y \subseteq E^o$.

A subsemigroup S^o of S is called a *quasi-ideal* of S if $S^oSS^o \subseteq S^o$.

Lemma 1.6. [12] *Let S be an abundant semigroup with a multiplicative quasi-adequate transversal S^o. Then*

(1) *$IE^o \subseteq I$ and $E^o\Lambda \subseteq \Lambda$;*

(2) *I and Λ are subbands of S;*

(3) *$E^oI \subseteq E^o$ and $\Lambda E^o \subseteq E^o$;*

(4) *If $x \in E$, then $C_{S^o}(x) \subseteq E^o$.*

The following theorem will be used without further mention.

Lemma 1.7. (1) *Let e and f be $\mathcal{D}$-equivalent idempotents of a semigroup S. Then each element a of $R_e \cap L_f$ has a unique inverse a' in $R_f \cap L_e$, such that $aa' = e$ and $a'a = f$;*

(2) *Let a, b be elements of a semigroup S. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*

2. Some properties

The objective in this section is to introduce and investigate some elementary properties of the sets R and L and their sets of idempotents. It is known that R and L play an important role in the study of regular semigroups with both inverse transversals and orthodox transversals. For any result concerning R there is a dual result for L which we list but omit its proof.

Lemma 2.1. *Let S be an abundant semigroup with a quasi-adequate transversal S^o. Then*

(1) $I = \{e \in E : (\exists e^* \in E^o)\ e\mathcal{L}e^*\}$ *and* $\Lambda = \{f \in E : (\exists f^+ \in E^o)\ f\mathcal{R}f^+\}$;

(2) $I \cap \Lambda = E^o$.

Proof. (1). If $e \in I$, then there exists $x \in S$ such that $i_x = e$ and $i_x\mathcal{L}x^{o+}$ for some $x^{o+} \in E^o$. Conversely, if $e \in E$ and there exists $e^* \in E^o$ such that $e\mathcal{L}e^*$, then $e = ee^*e^*$ and this implies that $e = i_e \in I$. The result for Λ can be proved dually.

(2). It follows from (1) that $E^o \subseteq I \cap \Lambda$. Let $e \in I \cap \Lambda$, then there exist $e^+, e^* \in E^o$ such that $e^+\mathcal{R}e\mathcal{L}e^*$. So $e^+e^*, e^*e^+ \in E^o$ since S^o is quasi-adequate. It follows from Lemma 1.7 that $e = e^+e^* \in E^o$. □

Proposition 2.2. *Let S^o be a quasi-adequate transversal of an abundant semigroup S. Then $\mathcal{D}^{*S^o} = \mathcal{D}^{*S} \cap (S^o \times S^o)$.*

Proof. Let $a, b \in S^o$ and $a\mathcal{D}^{*S}b$, then $R^*_a \cap L^*_b \neq \emptyset$. Take $c \in R^*_a \cap L^*_b$. Since $a, b \in S^o$ and S^o is quasi-adequate, there exist $a^+, b^* \in E^o$ such that $a^+\mathcal{R}^*a\mathcal{R}^*c\mathcal{L}^*b\mathcal{L}^*b^*$. From the definition of a quasi-adequate transversal, $c = i_cc^o\lambda_c$, where $i_c\mathcal{L}c^{o+}, \lambda_c\mathcal{R}c^{o*}$ for some $c^{o+}, c^{o*} \in E^o$ and $i_c\mathcal{R}^*c\mathcal{L}^*\lambda_c$. Thus $a^+\mathcal{R}^*c\mathcal{R}^*i_c\mathcal{L}c^{o+}$ and so by Lemma 2.1, $i_c \in I \cap \Lambda = E^o$. Similarly, $c^{o*}\mathcal{R}\lambda_c\mathcal{L}^*c\mathcal{L}^*b$ and so $\lambda_c \in E^o$. Consequently,

$$c = i_cc^o\lambda_c \in E^o \cdot S^o \cdot E^o \subseteq S^o.$$

So $a\mathcal{D}^{*S^o}b$, and hence $\mathcal{D}^{*S^o} \supseteq \mathcal{D}^{*S} \cap (S^o \times S^o)$. The reverse inclusion is obvious. □

Proposition 2.3. *Let S^o be a quasi-adequate transversal of an abundant semigroup S. Then for every regular element x of S, x has an inverse x^o in S^o. In this case, $V_{S^o}(x^o) \subseteq C_{S^o}(x)$.*

Proof. For every regular element x, $x = i_xx^o\lambda_x$ for some $i_x \in I_x$, $x^o \in C_{S^o}(x)$, $\lambda_x \in \Lambda_x$, where $i_x\mathcal{L}x^{o+}$, $\lambda_x\mathcal{R}x^{o*}$ for some $x^{o+}, x^{o*} \in E^o$. Since x, i_x and λ_x are all regular, from $i_x\mathcal{R}^*x\mathcal{L}^*\lambda_x$ we deduce that $i_x\mathcal{R}x\mathcal{L}\lambda_x$, so by Lemma 1.7 x has an inverse x' in $R_{\lambda_x}\cap L_{i_x}$. Thus $x^{o*}\mathcal{R}\lambda_x\mathcal{R}x'\mathcal{L}i_x\mathcal{L}x^{o+}$ and so by Proposition 2.2, $x'\in S^o$. □

Proposition 2.4. *Suppose that S is an abundant semigroup with a quasi-adequate transversal S^o. Let*

$$R=\{x\in S:(\exists\lambda_x\in\Lambda_x)\ \lambda_x\in E^o\} \text{ and } L=\{a\in S:(\exists i_a\in I_a)\ i_a\in E^o\}.$$

Then

$$R=\{x\in S:(\exists l\in E^o)\ x\mathcal{L}^*l\} \text{ and } L=\{a\in S:(\exists h\in E^o)\ a\mathcal{R}^*h\}.$$

Consequently, $R\cap L=S^o, E(R)=I$ and $E(L)=\Lambda$.

Proof. It is clear that if $x\in R$, there exists $l=\lambda_x\in E^o$ such that $x\mathcal{L}^*\lambda_x$.

Conversely, for $x\in S$ if there exists $l\in E^o$ such that $x\mathcal{L}^*l$, then $\lambda_x\mathcal{L}^*x\mathcal{L}^*l$. Hence by Lemma 2.1, $\lambda_x\in I$. Therefore $\lambda_x\in I\cap\Lambda=E^o$.

It is evident that if there exists $\lambda_x\in\Lambda_x$ such that $\lambda_x\in E^o$ ($i_a\in I_a$ such that $i_a\in E^o$), then $\Lambda_x\subseteq E^o$ ($I_a\subseteq E^o$). □

Proposition 2.5. *Let S be an abundant semigroup with a quasi-adequate transversal S^o. If S^o is a right ideal of S, then $\Lambda_x\subseteq E^o$ for every $x\in S$ and $E=I$.*

Dually, if S^o is a left ideal of S, then $I_a\subseteq E^o$ for every $a\in S$ and $E=\Lambda$.

Proof. By the definition of a quasi-adequate transversal, for every $x\in S$, $\lambda_x\in\Lambda_x$, $x=i_xx^o\lambda_x$ for some $i_x\in I_x,\ x^o\in C_{S^o}(x)$ and $i_x\mathcal{L}x^{o+},\lambda_x\mathcal{R}x^{o*}$ for some $x^{o+},x^{o*}\in E^o$. Since S^o is a right ideal of S, $\lambda_x=x^{o*}\lambda_x\in S^o$ and consequently $\Lambda_x\subseteq S^o\cap E=E^o$.

Let $h\in E$, then $h\in i_hh^o\lambda_h$ for some $i_h\in I_h$, $h^o\in C_{S^o}(h)$, $\lambda_h\in\Lambda_h$ where $\lambda_h\in E^o$ and $\lambda_h\mathcal{L}h$. Thus $h\in R\cap E=E(R)=I$ by Proposition 2.4. □

Proposition 2.6. *Let S^o be a quasi-adequate transversal of an abundant semigroup S. Then the following statements are equivalent:*

(1) *S^o is a quasi-ideal of S;*

(2) *$\Lambda I\subseteq S^o$;*

(3) *$SS^o\subseteq R, S^oS\subseteq L$;*

(4) *R is a left ideal and L is a right ideal of S.*

Proof. (1) and (2) are equivalent can be proved similarly as in [3].

(1) $\Longrightarrow$ (3). If (1) holds, then for any $y\in S$, $x^o\in S^o$, we have

$$yx^o=i_y\cdot y^o\lambda_y\cdot x^o\mathcal{L}^*y^{o+}y^o\lambda_yx^o=y^o\lambda_yx^o\mathcal{L}^*(y^o\lambda_yx^o)^*\in E^o,$$

since $y^o\lambda_yx^o\in S^oSS^o\subseteq S^o$. Hence $SS^o\subseteq R$. Dually $S^oS\subseteq L$.

(3) $\Longrightarrow$ (4). If (3) holds, then for any $x \in S$ and $y \in R$, there exists $h \in E^o$ such that $y\mathcal{L}^*h$, thus we have $xy = xyh \in SS^o \subseteq R$, whence $SR \subseteq R$; and dually $LS \subseteq L$.

(4) $\Longrightarrow$ (2). If (4) holds, then for any $f \in \Lambda$ and $e \in I$, there exist $h, l \in E^o$ such that $h\mathcal{R}f$ and $l\mathcal{L}e$. So we have

$$fe = fel \in SS^o \subseteq SR \subseteq R \quad \text{and} \quad fe = hfe \in S^oS \subseteq LS \subseteq L.$$

Consequently $fe \in R \cap L = S^o$ and we have (2). □

Remark 1. Since both a left ideal and a right ideal are subsemigroups, if one of the conditions in Proposition 2.6 is satisfied, then R and L are subsemigroups. From Proposition 2.6, it is evident that if S^o is a multiplicative quasi-adequate transversal of S, then S^o is a quasi-ideal of S. Obviously, if S is quasi-adequate then the quasi-adequate transversal S^o is multiplicative if and only if S^o is a quasi-ideal of S.

Proposition 2.7. *Suppose that S is an abundant semigroup with a multiplicative quasi-adequate transversal S^o. Let R and L be described as in Proposition 2.4. Then R and L are quasi-adequate semigroups with a common quasi-adequate transversal S^o which is a right ideal of R and a left ideal of L.*

Proof. From Remark1 it is evident that R is a subsemigroup of S.

Since S^o is also a quasi-adequate transversal of R and $R \subseteq S$, S^o is a multiplicative quasi-adequate transversal of R. Let $x \in R$ and $y^o \in S^o$, then $y^o x = y^o x \lambda_x \in S^o$ for some $\lambda_x \in E^o$ since S^o is a quasi-ideal of S. Thus S^o is a right ideal of R. Consequently by Proposition 2.4 and Lemma 1.6, $E(R) = I$ is a band, and thus R is quasi-adequate. The dual results for L can be proved similarly. □

3. The main theorem

The main objective in this section is to give a structure theorem for abundant semigroups with multiplicative quasi-adequate transversals. In what follows R denotes an abundant semigroup with a multiplicative quasi-adequate transversal S^o which is a right ideal of R. Then by Proposition 2.5, $\Lambda_x \subseteq E^o$ for every $x \in S$ and $E(R) = I$, it follows that R is a quasi-adequate semigroup with a right ideal quasi-adequate transversal S^o. For $a \in R$, the $\mathcal{R}^*$-class of R containing a will be denoted by R^*_a and the δ-class containing a will be denoted by $\delta(a)$. We define

$K(a) = K(b)$ if $R^*_a = R^*_b$ and $\delta(a) = \delta(b)$ for $a, b \in R$ and define a relation $\mathcal{K}$ on R by $(a, b) \in \mathcal{K}$ if $K(a) = K(b)$. Then $\mathcal{K}$ is an equivalence relation on R. L denotes an abundant semigroup with a multiplicative quasi-adequate transversal S^o which is a left ideal of L. Then L is quasi-adequate with $I_a \subseteq E^o$ for every $a \in S$ and $E(L) = \Lambda$.

Theorem 3.1. *Let R and L be quasi-adequate semigroups with a common quasi-adequate transversal S^o. Suppose that S^o is a right ideal of R and a left ideal of L. Let $L \times R \longrightarrow S^o$ described by $(a, x) \longmapsto a * x$ be a mapping such that for any $x, y \in R$ and for any $a, b \in L$:*

(1) *if $x \in E(R)$ and $a \in E(L)$, then $a * x \in E(S^o) = E^o$;*

(2) *$(a * x)y = a * (xy)$ and $b(a * x) = (ba) * x$;*

(3) *if $\{x, a\} \cap E^o \neq \emptyset$, then $a * x = ax$;*

(4) *For any $b_1, b_2 \in L^1$, $y_1, y_2 \in R^1$, if $x_1 \mathcal{R}^* x_2$ in R, then $y_1(b_1 * x_1) = y_2(b_2 * x_1)$ if and only if $y_1(b_1 * x_2) = y_2(b_2 * x_2)$; if $a_1 \mathcal{L}^* a_2$ in L, then $(a_1 * y_1)b_1 = (a_2 * y_2)b_2$ if and only if $(a_2 * y_1)b_1 = (a_2 * y_2)b_2$.*

Define a multiplication on the set

$$\Gamma \equiv R/\mathcal{K}| \times |L/\mathcal{L}^* = \{(K(x),\ L^*_a) \in R/\mathcal{K} \times L/\mathcal{L}^* : C_{S^o}(x) \cap C_{S^o}(a) \neq \emptyset\}$$

by

$$(K(x), L^*_a)(K(y), L^*_b) = (K(i_x(a * y)), L^*_{(a*y)\lambda_b}).$$

Then Γ is an abundant semigroup with a multiplicative quasi-adequate transversal which is isomorphic to S^o.

Conversely, every abundant semigroup with a multiplicative quasi-adequate transversal can be constructed in this way.

Lemma 3.2. *The multiplication on Γ is well-defined.*

Proof. First it is easy to see that $(K(i_x(a * y)), L^*_{(a*y)\lambda_b}) \in \Gamma$, since

$$i_x(a * y) = i_x x^o(\lambda_a * i_y)y^o\lambda_y = i_x[x^o(\lambda_a * i_y)]^+ \cdot x^o(\lambda_a * i_y)y^o \cdot [(\lambda_a * i_y)y^o]^*\lambda_y$$

and

$$(a * y)\lambda_b = i_a \cdot x^o(\lambda_a * i_y)y^o\lambda_b = i_a[x^o(\lambda_a * i_y)]^+ \cdot x^o(\lambda_a * i_y)y^o \cdot [(\lambda_a * i_y)y^o]^*\lambda_b.$$

Let $i_x, i'_x \in I_x$, where $i_x \mathcal{L} x^{o+}$, $i'_x \mathcal{L} x^{o+'}$ for some $x^o \in C_{S^o}(x) \cap C_{S^o}(a)$. Then $R^*_{i_x(a*y)} = R^*_{i'_x(a*y)}$ and $\delta(i_x(a * y)) = \delta(i'_x(a * y))$, and hence the multiplication on Γ is not dependent on the choice of i_x. There is a dual result for λ_b.

We then prove that if $(K(x), L_a^*) \in \Gamma$, then $i_x \cdot a = x \cdot \lambda_a$. In fact, if $(K(x), L_a^*) \in \Gamma$, then there exists $x^o \in C_{S^o}(x) \cap C_{S^o}(a)$ such that $x = i_x x^o \lambda_x$, $i_x \mathcal{L} x^{o+}, \lambda_x \mathcal{R} x^{o*}$ for some $x^{o+}, x^{o*} \in E^o$ and $a = i_a x^o \lambda_a, i_a \mathcal{L} x^{o+'}, \lambda_a \mathcal{R} x^{o*'}$ for some $x^{o+'}, x^{o*'} \in E^o$. Thus $i_x a = i_x i_a x^o \lambda_a = i_x i_a \cdot x^o \cdot x^{o*} \lambda_a$ and $x\lambda_a = i_x x^o \lambda_x \lambda_a = i_x x^{o+'} \cdot x^o \cdot \lambda_x \lambda_a$. It is easy to check that $i_x i_a = i_x x^{o+'}$ and $x^{o*}\lambda_a = \lambda_x \lambda_a$ and so $i_x a = x\lambda_a$.

Next we prove that if $(K(x), L_a^*)$ and $(K(x'), L_{a'}^*)$ in Γ are such that $(K(x), L_a^*) = (K(x'), L_{a'}^*)$, then $i_x a = i_{x'} a'$. From $x \mathcal{R}^* x'$ and $x \delta x'$ we deduce that there exists $h \in E(x^*)$ such that $x' = xh$. Moreover, $h \mathcal{L}^* x'$. Thus $x\lambda_a = i_x x^o \lambda_x \lambda_a$ and $x' \lambda_{a'} = xh\lambda_{a'} = i_x x^o \lambda_x h \lambda_{a'}$. Since $x \in R$ we have $\lambda_x \in E^o$ and consequently $\lambda_x h \lambda_{a'} \in E^o I \Lambda \subseteq E^o \Lambda \subseteq \Lambda$ and $\lambda_x \lambda_a \in E^o \Lambda \subseteq \Lambda$. It is easy to check that $\lambda_x h \lambda_{a'}$ and $\lambda_x \lambda_a$ are in the same $\mathcal{H}^*-$ class and hence $\lambda_x h \lambda_{a'} = \lambda_x \lambda_a$. Therefore $x\lambda_a = x' \lambda_{a'}$ and consequently $i_x a = i_{x'} a'$.

Finally we prove that the multiplication on Γ is not dependent on the choice of x, a, y and b. Let

$$(K(x), L_a^*) = (K(x'), L_{a'}^*) \quad \text{and} \quad (K(y), L_b^*) = (K(y'), L_{b'}^*).$$

We have

$$(K(x), L_a^*)(K(y), L_b^*) = (K(i_x(a * y)), L^*_{(a*y)\lambda_b})$$

and

$$(K(x'), L_{a'}^*)(K(y'), L_{b'}^*) = (K(i_{x'}(a' * y')), L^*_{(a'*y')\lambda_{b'}}).$$

We now prove that $\delta(i_x(a*y)) = \delta(i_{x'}(a'*y'))$. Since $\delta(x) = \delta(x')$ and $\delta(y) = \delta(y')$, from Lemma 1.4, we have $x = kx'l$ for some $k \in E(x'^+)$, $l \in E(x'^*)$ and $y = py'q$ for some $p \in E(y'^+), q \in E(y'^*)$. Again by Lemma 1.4, $k\mathcal{R}^* x$ and $p\mathcal{R}^* y$ and so $k\mathcal{R}^* x'$ and $p\mathcal{R}^* y$. Thus $x = x'l$ and $y = y'q$. Consequently, similar as the above proof, we can show that

$$i_x(a*y) = i_{x'}(a'*y) = i_{x'}(a'*y')q = i_{x'}(a'*y')[i_{x'(a'*y')}]^* q$$

and

$$[i_{x'}(a'*y')]^* q \in E((i_{x'}(a'*y'))^*)$$

It follows from Lemma 1.4 that $\delta(i_x(a*y)) = \delta(i_{x'}(a'*y'))$.

We then show that $i_x(a*y)\mathcal{R}^* i_{x'}(a'*y')$. From the proof of $\delta(i_x(a*y)) = \delta(i_{x'}(a'*y'))$ we have $i_x(a*y) = i_{x'}(a'*y')q$. Similarly, we have $i_{x'}(a'*y') = i_x(a*y)q'$ from some $q' \in E(y^*)$. Thus $i_x(a*y)\mathcal{R}^* i_{x'}(a'*y')$. Dually, we can show that $(a*y)\lambda_b \mathcal{L}^* (a'*y')\lambda_{b'}$. □

Lemma 3.3. *The set Γ is a semigroup.*

Proof. Let $e, f, g \in \Gamma$, where $e = (K(x), L_a^*)$, $f = (K(x_1), L_{a_1}^*)$, $g = (K(x_2), L_{a_2}^*)$. Then

$$\begin{aligned}(ef)g &= (K(i_x(a * x_1)), L^*_{(a*x_1)\lambda_{a_1}})(K(x_2), L^*_{a_2}) \\ &= (K(i_{i_x(a*x_1)}(((a*x_1)\lambda_{a_1}) * x_2)), L^*_{(((a*x_1)\lambda_{a_1})*x_2)\lambda_{a_2}}) \\ &= (K(i_x(a*x_1)^+(a*x_1)(\lambda_{a_1} * x_2)), L^*_{(a*x_1)(\lambda_{a_1}*x_2)\lambda_{a_2}}) \\ &= (K(i_x(a*x_1)(\lambda_{a_1} * x_2)), L^*_{(a*x_1)(\lambda_{a_1}*x_2)\lambda_{a_2}}).\end{aligned}$$

On the other hand,

$$\begin{aligned}e(fg) &= (K(x), L_a^*)\ (K(i_{x_1}(a_1 * x_2)), L^*_{(a_1*x_2)\lambda_{a_2}}) \\ &= (K(i_x(a * (i_{x_1}(a_1 * x_2))), L^*_{(a*(i_{x_1}(a_1*x_2)))\lambda_{a_2}}) \\ &= (K(i_x(a * (x_1(\lambda_{a_1} * x_2))), L^*_{(a*(x_1(\lambda_{a_1}*x_2)))\lambda_{a_2}}) \quad (i_{x_1}a_1 = x_1\lambda_{a_1}) \\ &= (K(i_x(a*x_1)(\lambda_{a_1} * x_2)), L^*_{(a*x_1)(\lambda_{a_1}*x_2)\lambda_{a_2}}).\end{aligned}$$

Therefore $(ef)g = e(fg)$. □

Lemma 3.4. *Let $(K(x), L_a^*) \in \Gamma$. Then $(K(x), L_a^*) \in E(\Gamma)$ if and only if $a * x = i_a x (= a\lambda_x)$.*

Proof. Since $(K(x), L_a^*)(K(x), L_a^*) = (K(i_x(a * x)), L^*_{(a*x)\lambda_a})$, it is easy to check that if $a * x = i_a \cdot x = a \cdot \lambda_x$, then

$$(K(i_x(a * x)), L^*_{(a*x)\lambda_a}) = (K(i_x \cdot i_a \cdot x), L^*_{a\lambda_x\lambda_a}) = (K(x), L_a^*).$$

Thus $(K(x), L_a^*) \in E(\Gamma)$. Conversely, if $(K(x), L_a^*) \in E(\Gamma)$, then $K(i_x(a * x)) = K(x)$ and so $i_x(a * x)\delta x$. Consequently, $i_x(a * x) = kxl$ for some $k \in E(x^+)$ and $l \in E(x^*)$. It follows that

$$x = x^+ \cdot i_x(a * x) \cdot x^* = i_x(a * xx^*) = i_x(a * x).$$

Hence $a * x = i_a x$. □

Lemma 3.5. *Suppose that $(K(x), L_a^*) \in \Gamma$, denote $u = (K(i_x), L^*_{x^{o+}})$ and $v = (K(x^{o*}), L^*_{\lambda_a})$, where $x = i_x x^o \lambda_x$, $a = i_a x^o \lambda_a$ and $i_x \mathcal{L} x^{o+}, \lambda_a \mathcal{R} x^{o*}$ for some $x^{o+}, x^{o*} \in E^o$. Then $u, v \in E(\Gamma)$ and $u\mathcal{R}^*(K(x), L_a^*)\mathcal{L}^* v$.*

Proof. By Lemma 3.4, $u, v \in E(\Gamma)$ is clear. Computing

$$\begin{aligned}(K(i_x), L^*_{x^{o+}})(K(x), L^*_a) &= (K(i_x(x^{o+} * x)), L^*_{(x^{o+}*x)\lambda_a}) \\ &= (K(i_x x^{o+} x), L^*_{x^{o+}x\lambda_a}) \quad (\text{since } x^{o+} \in E^o) \\ &= (K(x), L^*_{x^o\lambda_a}) \\ &= (K(x), L^*_a). \quad (\text{since } x^o\lambda_a \mathcal{L}^* i_a x^o \lambda_a = a).\end{aligned}$$

Suppose that $(K(y), L^*_b), (K(z), L^*_c) \in \Gamma^1$ are such that

$$(K(y), L^*_b)(K(x), L^*_a) = (K(z), L^*_c)(K(x), L^*_a).$$

This implies that

$$(K(i_y(b * x)), L^*_{(b*x)\lambda_a}) = (K(i_z(c * x)), L^*_{(c*x)\lambda_a}).$$

That is

$$i_y(b * x)\mathcal{R}^* i_z(c * x),\ i_y(b * x)\delta i_z(c * x) \quad \text{and} \quad (b * x)\lambda_a \mathcal{L}^*(c * x)\lambda_a.$$

From $(b*x)\lambda_a\mathcal{L}^*(c*x)\lambda_a$, we have $(b*x)\lambda_a\lambda_x\mathcal{L}^*(c*x)\lambda_a\lambda_x$ and thus $(b*x)\mathcal{L}^*(c*x)$. Consequently, $i_y(b * x)\mathcal{L}^* i_z(c * x)$ since $i_b i_y i_b = i_b$ and $i_c i_z i_c = i_c$. Hence

$$(i_y(b * x), i_z(c * x)) \in \mathcal{R}^* \cap \mathcal{L}^* \cap \delta = \mathcal{H}^* \cap \delta = l.$$

That is $i_y(b * x) = i_z(c * x)$. From $x\mathcal{R}^* i_x$ and (4) we deduce that $i_y(b * i_x) = i_z(c * i_x)$. Thus

$$b * i_x = i_b(b * i_x)\mathcal{L}^* i_y i_b(b * i_x) = i_z i_c(c * i_x)\mathcal{L}^* i_c(c * i_x) = c * i_x.$$

Therefore

$$\begin{aligned}(K(y), L^*_b)(K(i_x), L^*_{x^{o+}}) &= (K(i_y(b * i_x)), L^*_{(b*i_x)x^{o+}}) = (K(i_z(c * i_x)), L^*_{(c*i_x)x^{o+}})\\ &= (K(z), L^*_c)(K(i_x), L^*_{x^{o+}}).\end{aligned}$$

By Corollary 1.2, $u\mathcal{R}^*(K(x), L^*_a)$. □

Dually, we may show that $v\mathcal{L}^*(K(x), L^*_a)$.

Lemma 3.6. *Γ is an abundant semigroup.*

Proof. It follows from Lemma 3.5 immediately. □

Lemma 3.7. *Let $W = \{(K(s), L^*_s) : s \in S^o\}$. Then W is isomorphic to S^o and W is a quasi-adequate $*$-subsemigroup of Γ with $E(W)=\{(K(s), L^*_s) : s \in E^o\}$.*

Proof. Clearly $W \subseteq \Gamma$. Let $(K(s), L^*_s), (K(t), L^*_t) \in W$. It is easy to see that

$$(K(s), L^*_s)(K(t), L^*_t) = (K(i_s st), L^*_{st\lambda_t}) = (K(st), L^*_{st}) \in W.$$

Therefore W is a subsemigroup. For any $s \in S^o$, define $s\varphi = (K(s), L^*_s)$, it is evident that φ is an isomorphism. Thus $S^o \cong W$.

To show that W is a $*$-subsemigroup, let $(K(s), L^*_s) \in W$. By Lemma 3.4 and Lemma 3.5, $u = (K(s^+), L^*_{s^+}) \in E(W)$ and $u\mathcal{R}^*(K(s), L^*_s)$. Similarly, $v = (K(s^*), L^*_{s^*}) \in E(W)$ and $v\mathcal{L}^*(K(s), L^*_s)$. That $E(W) = \{(K(s), L^*_s) : s \in E^o\}$ is obvious. □

Lemma 3.8. *Let* $(K(x_1), L^*_{a_1}), (K(x_2), L^*_{a_2}) \in \Gamma$. *Then*

(1) $(K(x_1), L^*_{a_1})\mathcal{R}^*(K(x_2), L^*_{a_2})$ *if and only if* $x_1\mathcal{R}^*x_2$.

(2) $(K(x_1), L^*_{a_1})\mathcal{L}^*(K(x_2), L^*_{a_2})$ *if and only if* $a_1\mathcal{L}^*a_2$.

Proof. To prove (1), by Lemma 3.5, it is equivalent to show that

$$(K(i_{x_1}), L^*_{x_1{}^{o+}})\mathcal{R}^*(K(i_{x_2}), L^*_{x_2{}^{o+}}) \quad \text{if and only if } x_1\mathcal{R}^*x_2.$$

Now $u_1 = (K(i_{x_1}), L^*_{x_1{}^{o+}})\mathcal{R}^*(K(i_{x_2}), L^*_{x_2{}^{o+}}) = u_2$

$\Longleftrightarrow$ $u_1u_2 = u_2$ and $u_2u_1 = u_1$, that is $(K(i_{x_1}x_1{}^{o+}i_{x_2}), L^*_{x_1{}^{o+}i_{x_2}x_2{}^{o+}}) = (K(i_{x_2}), L^*_{x_2{}^{o+}})$ and $(K(i_{x_2}x_2{}^{o+}i_{x_1}), L^*_{x_2{}^{o+}i_{x_1}x_1{}^{o+}}) = (K(i_{x_1}), L^*_{x_1{}^{o+}})$

$\Longleftrightarrow$ $(K(i_{x_1}i_{x_2}), L^*_{x_1{}^{o+}i_{x_2}}) = (K(i_{x_2}), L^*_{x_2{}^{o+}})$ and $(K(i_{x_2}i_{x_1}), L^*_{x_2{}^{o+}i_{x_1}}) = (K(i_{x_1}), L^*_{x_1{}^{o+}})$ since $i_{x_1}\mathcal{L}x_1{}^{o+}, i_{x_2}\mathcal{L}x_2{}^{o+}$ and $x_1{}^{o+}i_{x_2}, x_2{}^{o+}i_{x_1} \in E^o$.

$\Longleftrightarrow i_{x_1}i_{x_2}\mathcal{R}^*i_{x_2}, i_{x_2}i_{x_1}\mathcal{R}^*i_{x_1}$

$\Longleftrightarrow x_1\mathcal{R}^*x_2$ since $x_1\mathcal{R}^*i_{x_1}, x_2\mathcal{R}^*i_{x_2}$.

(2) can be proved similarly. □

Lemma 3.9. *Let* $g = (K(x), L^*_a) \in \Gamma$. *Then*

$$C_W(g) = \{(K(y), L^*_y) \in W : y \in C_{S^o}(x) \cap C_{S^o}(a)\}.$$

Proof. Let $V = \{(K(y), L^*_y) \in W : y \in C_{S^o}(x) \cap C_{S^o}(a)\}$ and $(K(y), L^*_y) \in V$. Since $y \in C_{S^o}(x) \cap C_{S^o}(a)$, there exist $e, f \in E(R)$ and $i_a, \lambda_a \in E(R)$ such that $x = eyf$ and $a = i_a y \lambda_a$, where $e\mathcal{L}y^+, f\mathcal{R}y^*$ for some $y^+, y^* \in E^o$. It follows that

$$(K(x), L^*_a) = (K(e), L^*_{y^+})(K(y), L^*_y)(K(y^*), L^*_{\lambda_a}).$$

Furthermore, by Lemma 3.8 we have

$$(K(e), L^*_{y^+})\mathcal{L}(K(y^+), L^*_{y^+})\mathcal{R}^*(K(y), L^*_y)$$

and

$$(K(y^*), L^*_{\lambda_a})\mathcal{R}(K(y^*), L^*_{y^*})\mathcal{L}^*(K(y), L^*_y).$$

Hence $(K(y), L^*_y) \in C_W(g)$ and so $V \subseteq C_W(g)$.

Conversely, let $(K(y), L^*_y) \in C_W(g)$. Then there exist $(K(y_1), L^*_{b_1}), (K(y_2), L^*_{b_2}) \in E(\Gamma)$ such that

$$(K(x), L^*_a) = (K(y_1), L^*_{b_1})(K(y), L^*_y)(K(y_2), L^*_{b_2})$$

and

$$(K(y_1), L^*_{b_1})\mathcal{L}(K(y), L^*_y)^+ \quad \text{for some } (K(y), L^*_y)^+ \in E(W),$$

$$(K(y_2), L^*_{b_2})\mathcal{R}(K(y), L^*_y)^* \quad \text{for some } (K(y), L^*_y)^* \in E(W).$$

By Lemma 1.3, $(K(y_1), L^*_{b_1})\mathcal{R}^* g\mathcal{L}^*(K(y_2), L^*_{b_2})$. Hence by Lemma 3.7, $y_1\mathcal{R}^* x$ and $a\mathcal{L}^* b_2$.

On the other hand, by Lemma 3.7 there exist $x', x'' \in E^o$ such that

$$(K(y), L^*_y)^+ = (K(x'), L^*_{x'}) \quad \text{with } x'\mathcal{R}^* y$$

and

$$(K(y), L^*_y)^* = (K(x''), L^*_{x''}) \quad \text{with } x''\mathcal{L}^* y.$$

It follows that

$$(K(x'), L^*_{x'})(K(x), L^*_a)(K(x''), L^*_{x''}) = (K(y), L^*_y),$$

and consequently, $x'x\lambda_a x''(\mathcal{R}^* \cap \delta)y$ and $x'x\lambda_a x''\mathcal{L}^* y$. Thus $y = x' \cdot x \cdot \lambda_a x''$ since $\mathcal{R}^* \cap \mathcal{L}^* \cap \delta = l$.

First since $(K(y_1), L^*_{b_1})\mathcal{L}(K(y), L^*_y)^+ = (K(x'), L^*_{x'})$, we have $b_1\mathcal{L}^* x'$. Hence $(K(y_1), L^*_{x'}) = (K(y_1), L^*_{b_1}) \in E(\Gamma)$ and so there exists $z \in C_{S^o}(y_1) \cap C_{S^o}(x')$. And from $(K(y_1), L^*_{x'}) \in E(\Gamma)$ by Lemma 3.4, $x' * y_1 = x'y_1 = x'\lambda_{y_1} \in E^o$, and so $y_1 x' y_1 = y_1$ since $y_1\mathcal{L}^* x'y_1 = x'\lambda_{y_1}\mathcal{R}^* x'$. Thus y_1 is regular. It is evident that $y_1 = i_{y_1}x' \cdot x'\lambda_{y_1} \in IE^oE^o \subseteq I = E(R)$ and $x'\mathcal{L} i_{y_1}x'\mathcal{R} y_1\mathcal{R}^* x$.

Next since $(K(y_2), L^*_{b_2})\mathcal{R}(K(y), L^*_y)^* = (K(x''), L^*_{x''})$, we have $y_2\mathcal{R}^* x''$ and

$$(K(x''), L^*_{x''})(K(y_2), L^*_{b_2}) = (K(y_2), L^*_{b_2}).$$

That is

$$(K(x''y_2), L^*_{x''y_2\lambda_{b_2}}) = (K(y_2), L^*_{b_2}).$$

From $y_2\mathcal{R}^* x''$ we have $y_2 = x''y_2 \in S^o$ and so $b_2\mathcal{L}^* x''y_2\lambda_{b_2} = y_2\lambda_{b_2}$. Consequently,

$$y_2\lambda_{b_2}y_2 = (y_2\lambda_{b_2}) * y_2 = y_2\lambda_{b_2}\lambda_{y_2} = y_2.$$

Thus y_2 is regular and $y_2 = y_2\lambda_{b_2} \cdot y_2^{o*}\lambda_{y_2} \in \Lambda E^o \subseteq E^o$, where $y_2^o \in C_{S^o}(y_2) \cap C_{S^o}(b_2)$, $y_2 = i_{y_2}y_2^o\lambda_{y_2}$ and $\lambda_{y_2}\mathcal{R} y_2^{o*}$ for some $y_2^{o*} \in E^o$. Therefore $\lambda_a\mathcal{R}\lambda_a x''\mathcal{L} x''$ since λ_a and x'' are in the same rectangular band and $\lambda_a x'' \in \Lambda E^o \subseteq E^o$.

Finally, since $(K(x), L^*_a) \in \Gamma$, there exists $x^o \in C_{S^o}(x) \cap C_{S^o}(a)$ such that $x = i_x x^o \lambda_x$, $a = i_a x^o \lambda_a$ and $i_a\mathcal{L} x^{o+}, \lambda_a\mathcal{R} x^{o*}$ for some $x^{o+}, x^{o*} \in E^o$. Thus

$$x\mathcal{L}^*\lambda_x\mathcal{L} x^{o*}\lambda_x\mathcal{R} x^{o*}\mathcal{R}\lambda_a\mathcal{R}\lambda_a x''\mathcal{L} x''\mathcal{L}^* y.$$

Consequently,

$$i_{y_1}x' \cdot y \cdot x^{o*}\lambda_x = i_{y_1}x' \cdot x' \cdot x \cdot \lambda_a x'' \cdot x^{o*}\lambda_x = i_{y_1}x' \cdot x \cdot x^{o*}\lambda_x = x,$$

moreover, $i_{y_1}x'\mathcal{L} x'\mathcal{R}^* y$ and $x^{o*}\lambda_x\mathcal{R}\lambda_a x''\mathcal{L}^* y$. Therefore $y \in C_{S^o}(x)$. Similarly, we have $y \in C_{S^o}(a)$ and hence $C_W(g) \subseteq V$. □

Corollary 3.10. *W is an abundant transversal of Γ.*

Proof. It follows from Lemma 3.7 and Lemma 3.9 immediately. $\square$

Lemma 3.11. *For any $g \in E(\Gamma)$ and $h \in E(W)$,*

$$C_W(h)C_W(g) \subseteq C_W(gh) \quad \text{and} \quad C_W(g)C_W(h) \subseteq C_W(hg).$$

Proof. Let $g = (K(x), L_a^*) \in E(\Gamma)$ and $h = (K(p), L_p^*) \in E(W)$ with $p \in E^o$. Then

$$gh = (K(i_x ap), L^*_{ap\lambda_p}) = (K(i_x ap), L^*_{ap}).$$

By Lemma 3.9, for any $g^o \in C_W(g), h^o \in C_W(h)$, there exist $y \in C_{S^o}(x) \cap C_{S^o}(a), q \in C_{S^o}(p)$ such that $g^o = (K(y), L_y^*)$ and $h^o = (K(q), L_q^*)$, furthermore, it is obvious that $q \in E^o$. Thus

$$h^o g^o = (K(q), L_q^*)(K(y), L_y^*) = (K(qy), L^*_{qy}).$$

Since $y \in C_{S^o}(x) \cap C_{S^o}(a)$, there exist $i_x, \lambda_x \in E(R)$ and $i_a, \lambda_a \in E(R)$ such that $x = i_x y \lambda_x$, $a = i_a y \lambda_a$, where $i_x \mathcal{L} y^+, \lambda_x \mathcal{R} y^*$ and $i_a \mathcal{L} y^{+\prime}, \lambda_a \mathcal{R} y^{*\prime}$ for some $y^+, y^*, y^{+\prime}, y^{*\prime} \in E^o$.

Also, $g = (K(x), L_a^*) \in E(\Gamma)$ gives

$$\begin{aligned}
& a * x = i_a \cdot x \\
\Longrightarrow\ & a * u_x = i_a i_x && \text{(since } x\mathcal{R}^* i_x \text{ and (4))} \\
\Longrightarrow\ & i_x(a * i_x) = i_x i_a i_x \\
\Longrightarrow\ & i_x i_a y(\lambda_a * e_x) = i_x \\
\Longrightarrow\ & i_x y(\lambda_a * i_x) = i_x && \text{(since } i_x i_a y = i_x i_a y^+ y \text{ and } i_x \mathcal{L} i_a y^+ \in I) \\
\Longrightarrow\ & y^+ y(\lambda_a * i_x) = y^+ && \text{(since } i_x \mathcal{L} y^+) \\
\Longrightarrow\ & y(\lambda_a * i_x) y = y^+ y = y.
\end{aligned}$$

From $y(\lambda_a * i_x)y = y$ we deduce that $y^{*\prime}(\lambda_a * i_x)y = y^{*\prime}$. Hence $(\lambda_a * i_x)y = y^{*\prime}$ since $y^{*\prime}\mathcal{R}\lambda_a$ and consequently

$$(\lambda_a * i_x)y(\lambda_a * i_x) = y^{*\prime}(\lambda_a * i_x) = \lambda_a * i_x.$$

It follows that y is an inverse in S^o of $(\lambda_a * i_x)$ and thus $y \in E^o$ since $\lambda_a * i_x \in E^o$ and S^o is quasi-adequate. Therefore $a = i_a y \lambda_a \in E^o E^o \Lambda \subseteq \Lambda$. From condition (QA2) we have $qy \in C_{S^o}(p)C_{S^o}(a) \subseteq C_{S^o}(ap)$. Since $i_x i_a y \in IE^o E^o \subseteq$

I,, $i_x i_a y \mathcal{L} y^{*\prime}$ and $\lambda_a * i_x \in E^o, \lambda_a * i_x \mathcal{R} y^{*\prime}$, from $i_x = i_x i_a x^o \cdot y^{*\prime} \cdot (\lambda_a * i_x)$ we deduce that $y^{*\prime} \in C_{S^o}(i_x)$. Thus

$$qy = qyy^{*\prime} \in C_{S^o}(ap)C_{S^o}(i_x) \subseteq C_{S^o}(i_x ap)$$

and consequently $qy \in C_{S^o}(ap) \cap C_{S^o}(i_x ap)$. Hence from Lemma 3.9 we have $h^o g^o \in C_W(gh)$. Dually we may show that $g^o h^o \in C_W(hg)$. □

Lemma 3.12. *W is a multiplicative quasi-adequate transversal of Γ.*

Proof. Let $g = (K(x), L^*_a), h = (K(y), L^*_b) \in \Gamma$. For any $(K(x_1), L^*_{a_1}) \in I_g$ and $(K(y_1), L^*_{b_1}) \in \Lambda_h$, by the proof of Lemma 3.9 we have $x_1 \in E(R) = I$ and $y_1 \in E^o$. Consequently from the proof of Lemma 3.11 we have $a_1, b_1 \in E(L) = \Lambda$. Furthermore, there exists $g^o \in C_W(g)$ such that $(K(x_1), L^*_{a_1})\mathcal{L} g^{o+} = (K(e^o), L^*_{e^o})$ for some $e^o \in E^o$ with $a_1 \mathcal{L} e^o$. Thus $\lambda_{a_1} \mathcal{L} a_1 \mathcal{L} e^o$ and $\lambda_{a_1} \in E^o$. It follows that

$$(K(y_1), L^*_{b_1})(K(x_1), L^*_{a_1}) = (K(i_{y_1}(b_1 * x_1), L^*_{(b_1 * x_1)\lambda_{a_1}}).$$

Since $b_1 \in E(L)$ and $x_1 \in E(R)$, by Theorem 3.1(1) $b_1 * x_1 \in E^o$ and thus

$$i_{y_1}(b_1 * x_1) \in E^o \quad \text{and} \quad (b_1 * x_1)\lambda_{a_1} \in E^o.$$

For any $x^o, y^o \in E^o$, if $(K(x^o), L^*_{y^o}) \in \Gamma$, it is readily to see that

$$(K(x^o), L^*_{y^o}) = (K(x^o y^o), L^*_{x^o y^o}) \in E(W).$$

Therefore $\Lambda_h I_g \in E(W)$. Combining with Corollary 3.10 and Lemma 3.11 implies that W is a multiplicative quasi-adequate transversal of Γ. □

Now we turn to prove the converse part of Theorem 3.1. Let S be an abundant semigroup with a multiplicative quasi-adequate transversal S^o. Let

$$R = \{x \in S : (\exists \lambda_x \in \Lambda_x)\lambda_x \in E^o\} \quad \text{and} \quad L = \{a \in S : (\exists i_a \in I_a)\ i_a \in E^o\}.$$

Then by Proposition 2.7, R and L are quasi-adequate semigroups with a common quasi-adequate transversal S^o which is a right ideal of R and a left ideal of L.

For every $(a, x) \in L \times R$, put $a * x = ax$. Then $a * x = ax = i_a a x \lambda_x \in S^o$ for some $i_a, \lambda_x \in E^o$, and if $x \in E(R) = I$, $a \in E(L) = \Lambda$, then $a*x = ax \in \Lambda I \subseteq E^o$, since S^o is a multiplicative quasi-adequate transversal of S. Thus the mapping $*$ satisfies (1) and clearly it also satisfies (2), (3) and (4). Therefore we get an abundant semigroup Γ in the way of the direct part of Theorem 3.1. Finally we shall prove that Γ is isomorphic to S.

Let $(K(x), L_a^*) \in \Gamma$. Define $\theta : \Gamma \longrightarrow S$ by $(K(x), L_a^*)\theta = i_x a$, where $i_x \in I_x$ and $i_x \mathcal{L} x^{o+}$ for some $x^o \in C_{S^o}(x) \cap C_{S^o}(a)$ and some $x^{o+} \in E^o$. It is easy to see that the definition of θ is not dependent on the choice of i_x.

We first have to show that θ is well-defined. From the proof of Lemma 3.2, we have if $(K(x), L_a^*) \in \Gamma$, then $i_x a = x\lambda_a$. If $(K(x), L_a^*) = (K(y), L_b^*)$, then $R_x^* = R_y^*$, $\delta(x) = \delta(y)$ and $L_a^* = L_b^*$. From $x\mathcal{R}^* y$ and $x\delta y$ we deduce that there exists $h \in E(y^*)$ such that $x = yh$, moreover, $h\mathcal{L}^* x$. Thus $x\lambda_a = yh\lambda_a = i_y y^o \lambda_y h \lambda_a$ and $y\lambda_b = i_y y^o \lambda_y \lambda_b$. Since $y \in R$ we have $\lambda_y \in E^o$ and consequently

$$\lambda_y \cdot h \cdot \lambda_a \in E^o I \cdot \Lambda \subseteq E^o \Lambda \subseteq \Lambda \quad \text{and} \quad \lambda_y \lambda_b \in E^o \Lambda \subseteq \Lambda.$$

It is easy to check that $\lambda_y h \lambda_a$ and $\lambda_y \lambda_b$ in the same $\mathcal{H}^*$-class and so $\lambda_y h \lambda_a = \lambda_y \lambda_b$. Hence $x\lambda_a = y\lambda_b$ and therefore θ is well-defined.

For any $(K(x), L_a^*), (K(y), L_b^*) \in \Gamma$. Then

$$\begin{aligned}
[(K(x), L_a^*)(K(y), L_b^*)]\theta &= (K(i_x a y), L_{ay\lambda_b}^*)\theta = i_{i_x ay} \cdot ay\lambda_b = i_x \cdot i_{ay} \cdot ay\lambda_b \\
&= i_x ay\lambda_b = i_x a i_y b \qquad\qquad (\text{since } y\lambda_b = i_y b) \\
&= (K(x), L_a^*)\theta \cdot (K(y), L_b^*)\theta,
\end{aligned}$$

and so θ is a homomorphism.

For every $x \in S$, it is easy to check that $xx^{o*} \in R$ and $x^{o+}x \in L$, where $x = i_x x^o \lambda_x, i_x \mathcal{L} x^{o+}$, $\lambda_x \mathcal{R} x^{o*}$ for some $x^{o+}, x^{o*} \in E^o$. Moreover, from

$$xx^{o*} = i_x x^o \lambda_x x^{o*} = i_x x^o x^{o*} \quad \text{and} \quad x^{o+}x = x^{o+} i_x x^o \lambda_x = x^{o+} x^o \lambda_x$$

we deduce that $x^o \in C_{S^o}(xx^{o*}) \cap C_{S^o}(x^{o+}x)$. Thus $(K(xx^{o*}), L_{x^{o+}x}^*) \in \Gamma$ and

$$(K(xx^{o*}), L_{x^{o+}x}^*)\theta = i_{xx^{o*}} \cdot x^{o+}x = i_x x^{o+} x = i_x x = x.$$

Hence θ is surjective.

Now let $(K(x), L_a^*), (K(y), L_b^*) \in \Gamma$ be such that $(K(x), L_a^*)\theta = (K(y), L_b^*)\theta$, that is $i_x a = i_y b$. Then $x\mathcal{R}^* i_x \mathcal{R}^* i_x a = i_y b \mathcal{R}^* i_y \mathcal{R}^* y$ and $a\mathcal{L}^* i_x a = i_y b \mathcal{L}^* b$. Thus $R_x^* = R_y^*$ and $L_a^* = L_b^*$. From $i_x a = i_y b$ we deduce that $x\lambda_a = y\lambda_b$, and consequently

$$y = y\lambda_b \lambda_y = x\lambda_a \lambda_y = x^+ \cdot x \cdot x^* \lambda_a \lambda_y.$$

Since $x^* \lambda_a \lambda_y$ is idempotent in R and $x^* \lambda_a \lambda_y \cdot \lambda_a \lambda_x = x^*$, this implies that $x^* \lambda_a \lambda_y \in E(x^*)$ and so $x\delta y$. Hence $K(x) = K(y)$ and $L_a^* = L_b^*$. Therefore θ is injective and θ is an isomorphism.

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