

Semi-parallel vector fields and conformally flat Randers metrics

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*This paper is dedicated to Professor Masao Hashiguchi on the occasion
of his 80th birthday*

Abstract. In this article, we investigate the conformal flatness of Randers metrics as an application of the geometry of Riemannian spaces admitting a semi-parallel vector field.

1. Introduction

A Riemannian manifold (M, g) is said to be conformally flat if there exists a local function $\sigma(x)$ defined on a neighborhood U of an arbitrary point in M such that the local metric $e^{\sigma(x)}g$ is a flat metric on U . As is well-known, the conformal-flatness is characterized by the vanishing of the Weyl's conformal curvature. On the other hand, YANO [Ya2] characterized the conformal-flatness by the existence of some special linear connection, that is, a Riemannian metric g is conformally flat if and only if there exists a semi-symmetric metrical connection ∇ whose curvature vanishes identically.

In Finsler geometry, the conformal curvature which characterize the conformal-flatness of Finsler metric has not been obtained yet. The conformal-flatness of a Finsler metric is characterized by [Ha-Ic2] as a generalization of

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Yano's theorem as follows: A Finsler manifold (M, L) is conformally flat if and only if there exists a semi-symmetric Finsler connection ∇ whose curvature vanishes identically.

A typical example of Finsler metrics are given by the *Funk metric* or *Hilbert metric* on a strictly convex domain. These metrics have some special properties, that is, these metrics are projectively flat, and they are of negative constant flag curvature. In particular, the Funk metric is a special class of the so-called *Randers metric* if the boundary of the domain is given by a quadratic equation. A positive function L defined on the total space TM of tangent bundle of a smooth manifold M is said to be a *Randers metric* if L is given by the form $L(X) = \sqrt{g(X, X)} + \beta(X)$ ($X \in TM$) for a Riemannian metric g on M and a one-form β on M . A characterization of Randers metric of negative constant flag curvature is given by [Ha-Sa-Sh] under some assumption.

Theorem 1.1 ([Ha-Sa-Sh]). *Let $L(X) = \sqrt{g(X, X)} + \beta(X)$ be a Randers metric on M with a one-form β satisfying $d\beta(E, X) = 0$ for all vector field X on M , where E is the dual of β with respect to g . Then (M, L) has negative constant flag curvature $K = -\rho^2/4$ if and only if*

(1) *the base Riemannian manifold (M, g) has negative constant sectional curvature $-\rho^2$,*

(2) *the one-form β is semi-parallel, that is, β satisfies*

$$\nabla^g \beta = \rho(g - \beta \otimes \beta), \quad (1.1)$$

where ∇^g is the Riemannian connection of (M, g) .

The form β satisfying (1.1) is a special case of the so-called *torse-forming one-form*, that is,

$$\nabla^g \beta = \rho(g + \varepsilon \beta \otimes \beta), \quad (1.2)$$

where ρ is a constant and $\varepsilon = \pm 1$ ([Ya1]). It is easily shown that one-form β satisfying (1.2) is closed. In the second section, we shall show that a connected complete Riemannian manifold which admits such a one-form β is given by the warped product space $M = N \times_{\psi(t)} \mathbb{R}$, where N is a totally umbilic hypersurface. In the third and fourth sections, we shall investigate the conformal flatness of Randers metric as an application of geometry of Riemannian manifolds admitting a semi-parallel vector field.

2. Semi-parallel vector fields

Let (M, g) be a connected complete Riemannian manifold of $\dim M = n$. In this section, we shall investigate the structure of a Riemannian manifold (M, g) admitting a semi-parallel vector field E .

Definition 2.1 ([Ha1]). A vector field E is said to be *semi-parallel* if it satisfies

$$\nabla_X^g E = \rho(X + \varepsilon\beta(X)E), \quad (2.1)$$

where β is the dual of E , that is, $\beta(X) = g(X, E)$ for any vector field X on M , and ρ is a constant and $\varepsilon = \pm 1$. If $\rho \equiv 0$, then E is a *parallel* vector field.

The condition (2.1) is equivalent to (1.2) which implies $d\beta = 0$, and thus $\beta = df$ for some local function f . Then E is the gradient vector field $\nabla^g f$ of f , that is, $g(X, \nabla^g f) = X(f)$ every vector field X , and E is given by

$$E = \nabla^g f = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \quad (2.2)$$

in local coordinate on M .

Example 2.1. Let $\mathbb{B}$ be the unit ball in the n -dimensional Euclidean space $\mathbb{R}^n$:

$$\mathbb{B} = \left\{ (x^1, \dots, x^n) \in \mathbb{R}^n \mid 1 - \sum_{i=1}^n (x^i)^2 > 0 \right\}.$$

For every point $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, we set $\|x\|^2 = \sum_{i=1}^n (x^i)^2$ and $f(x) = \log(1 - \|x\|^2)$. The *Hilbert metric* g_H on $\mathbb{B}$ is defined by

$$g_H = \frac{(1 - \|x\|^2)(\sum_{i=1}^n dx^i)^2 + (\sum_{i=1}^n x^i dx^i)^2}{(1 - \|x\|^2)^2}, \quad (2.3)$$

which is a hyperbolic Riemannian metric induced on $\mathbb{B}$ from the Minkowski metric $g_M = \sum (dx^i)^2 - (dt)^2$ on the hyperboloid $\sum (x^i)^2 - t^2 = -1$ (cf. [KN]). As is well-known, the space $(\mathbb{B}, g_H)$ has negative constant curvature $K = -1$. The vector field $E = \nabla^g f$ satisfies (2.1), and the one-form β defined by

$$\beta = -\frac{1}{2}df = \frac{1}{1 - \|x\|^2} \sum_{i=1}^n x^i dx^i \quad (2.4)$$

satisfies (1.2) for the case of $\rho = 1$ and $\varepsilon = -1$ (cf. [Ok]). The norm $\|\beta\|$ of this one-form β is given by $\|\beta\| = \|x\| < 1$. $\square$

Since any torse-forming one-form β is closed, the integrability condition $\nabla_X^g \nabla_Y^g E - \nabla_Y^g \nabla_X^g E - \nabla_{[X,Y]}^g E = R^g(X, Y)E$ for (2.1) is given by

$$R^g(X, Y)E = -\varepsilon\rho^2[g(X, E)Y - g(Y, E)X]. \quad (2.5)$$

Then, the sectional curvature $K(X \wedge E)$ of the 2-plane $X \wedge E$ is given by

$$K(X \wedge E) = \frac{g(R^g(X, E)E, X)}{\|X \wedge E\|^2} = \frac{g(R^g(X, E)E, X)}{\|X\|^2\|E\|^2 - g(X, E)^2} = \varepsilon\rho^2 \quad (2.6)$$

for all $X \in TM$. In particular, if (M, g) is a Riemannian manifold of constant curvature $\varepsilon\rho^2$, that is, if its curvature R^g satisfies

$$R^g(X, Y)Z = \varepsilon\rho^2[g(Y, Z)X - g(X, Z)Y]$$

for all $X, Y, Z \in TM$, then the integrability condition (2.5) is satisfied, and thus, if (M, g) is of constant curvature, there exists a semi-parallel vector field around every point in M .

Every Riemannian manifold of constant curvature c can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$ (cf. [Ch]). In the sequel, we shall show that a Riemannian manifold (M, g) admitting a semi-parallel vector field E can be locally expressed as a warped product space $\mathbb{R} \times_\rho N$, where N is a totally umbilic submanifold.

We suppose that there exists a smooth function $f : M \rightarrow \mathbb{R}$ such that $E = \nabla^g f$ satisfies (2.1) and f has no critical points. Then the set $N = f^{-1}(t)$ is a hypersurface in M with normal vector field E for every $t \in \mathbb{R}$. Let $p \in N = f^{-1}(t)$ be an arbitrary point. The second fundamental form S of N at $p \in N$ with respect to the normal vector field $E = \nabla^g f$ is defined by

$$S(X, Y) = (\nabla_X^g Y)^\perp \quad (2.7)$$

for $X, Y \in T_p N$, where $\perp$ is the projection to the orthogonal line bundle spanned by E . Since ∇^g is metrical and $\beta(X) = g(X, E) = g(X, \nabla^g f) = 0$ for every $X \in T_p N$, we have

$$\begin{aligned} g(\nabla_X^g Y, E) &= Xg(Y, E) - g(Y, \nabla_X^g E) = X\beta(Y) - g(Y, \rho\{X + \varepsilon\beta(X)E\}) \\ &= -\rho g(X, Y), \end{aligned}$$

and thus we have

$$S(X, Y) = -\frac{\rho g(X, Y)}{\|E\|^2} E. \quad (2.8)$$

Hence we have

Lemma 2.1. *Let (M, g) be a connected complete Riemannian space. Suppose that there exists a smooth function $f : M \rightarrow \mathbb{R}$ such that its gradient $E = \nabla^g f$ is a semi-parallel vector field. Then the complete hypersurface $N = f^{-1}(t)$ is totally umbilical for every $t \in \mathbb{R}$. If E is parallel, that is, $\rho \equiv 0$, then N is a totally geodesic hypersurface.*

If a semi-parallel vector field E has constant length, then its length must be unit and $\varepsilon = -1$ (cf. [Ha1]). Thus, in the sequel, we shall restrict our discussions to the case of $\varepsilon = -1$. Then, since the gradient of f has constant norm $\|\nabla^g f\| = 1$, we have a splitting theorem by the same method as in [In], [Sa] and [Ya1] as follows.

Since the function f admits no critical points, $f^{-1}(t) = N$ is a complete hypersurface for every $t \in \mathbb{R}$, and M is diffeomorphic to the product space $N \times \mathbb{R}$. In the case of $\rho \equiv 0$, the gradient $\nabla^g f$ is parallel, and the Hessian $\nabla^g(df)$ vanishes everywhere. Such a function f is called an *affine function* (cf. [Sa]).

Let γ_E be the integral curve of E through $p \in N = f^{-1}(0)$. By the assumption of $\|E\|^2 = \beta(E) = 1$, we get $\nabla_E^g E = 0$, and thus the integral curve γ_E of E is a geodesic orthogonal to the hypersurface N . Since (M, g) is complete, $\gamma_E(t)$ is defined for all $t \in \mathbb{R}$. Then, since $E(f) = g(E, \nabla^g f) = \|E\|^2 = 1$ and

$$E(f) = \frac{d}{dt} f(\gamma_E(t)),$$

we have $f(\gamma_E(t)) = f(p) + t = t$ for all $t \in \mathbb{R}$.

Moreover, the integral curve γ_E is a minimal geodesic between the hypersurface $f^{-1}(0)$ and $f^{-1}(t)$ (cf. [Sa]). Indeed, let $c : [0, l] \rightarrow M$ be any smooth curve parametrized by its arc-length s joining a point $p = c(0) \in f^{-1}(0)$ and a point $q = c(l) \in f^{-1}(t)$. Then the length $L(c)$ of c satisfies

$$\begin{aligned} L(c) &= \int_0^l \|\dot{c}(s)\| ds \geq \left| \int_0^l g(\dot{c}(s), \nabla^g f(c(s))) ds \right| \\ &= \left| \int_0^l \frac{d}{ds} f(c(s)) ds \right| = |f(q) - f(p)| = t \\ &= \int_0^t \|E\| dt = L(\gamma_E), \end{aligned}$$

namely $\text{dist}(N, f^{-1}(t)) = L(\gamma_E)$.

Lemma 2.2 ([Sa]). *Let (M, g) be a connected complete Riemannian manifold. Suppose that there exists a smooth function $f : M \rightarrow \mathbb{R}$ such that its gradient $\nabla^g f$ is a semi-parallel vector field of unit length. Then f is the signed distance function to the connected complete hypersurface $N = f^{-1}(0)$.*

With respect to a suitable local coordinate system $(x^1, \dots, x^n)$ on $M \cong N \times \mathbb{R}$, the given metric g has the form

$$g = \sum_{i,j=1}^{n-1} g_{ij}(x^1, \dots, x^n) dx^i \otimes dx^j + g_{nn}(x^1, \dots, x^n) dx^n \otimes dx^n.$$

Since the orthogonal trajectories to N defined by $x^i = \text{constant}$ ($i = 1, \dots, n-1$) are geodesics, the coefficients $\Gamma_{\beta\gamma}^\lambda$ of the Riemannian connection ∇^g of (M, g) satisfy

$$\Gamma_{nn}^i = \frac{1}{2} \sum_{\lambda=1}^n g^{\lambda i} \left(\frac{\partial g_{\lambda n}}{\partial t} + \frac{\partial g_{n\lambda}}{\partial t} - \frac{\partial g_{nn}}{\partial x^\lambda} \right) = 0,$$

and this implies $g_{nn} = g_{nn}(x^n)$. Replacing $\sqrt{g_{nn}(x^n)} dx^n$ by dt , the metric has the following form

$$g = \sum_{i,j=1}^{n-1} g_{ij}(x, t) dx^i \otimes dx^j + dt \otimes dt. \quad (2.9)$$

Denoting by $g_0 = dt \otimes dt$ the canonical metric on $\mathbb{R}$, the map $f : (M, g) \longrightarrow (\mathbb{R}, g_0)$ is a Riemannian submersion.

With respect to such a coordinate system $(x, t) = (x^1, \dots, x^{n-1}, t)$, we have $E = \partial/\partial t$ and

$$\nabla_{\frac{\partial}{\partial x^j}}^g \frac{\partial}{\partial x^i} = \sum_{h=1}^{n-1} \Gamma_{ij}^h(x, t) \frac{\partial}{\partial x^h} + \Gamma_{ij}^n(x, t) E.$$

Consequently the second fundamental form S is given by

$$S \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \Gamma_{ij}^n(x, t) E,$$

and (2.9) implies that $\Gamma_{ij}^n = \rho(x, t) g_{ij}$, that is,

$$\frac{1}{2} \sum_{\lambda=1}^n g^{\lambda n} \left(\frac{\partial g_{i\lambda}}{\partial x^j} + \frac{\partial g_{\lambda j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\lambda} \right) = \rho g_{ij}.$$

Since $g^{in} = g_{jn} = 0$ for $i, j = 1, \dots, n-1$ and $g^{nn} = 1$, we have

$$\frac{\partial g_{ij}}{\partial t} = -2\rho g_{ij}(x, t).$$

This implies that $g_{ij}(x, t) = \psi(t)^2 g_{ij}^*(x)$ for a Riemannian metric $g_N = \sum g_{ij}^*(x) dx^i \otimes dx^j$ on N , where ψ is given by $\psi(t) = e^{-\rho t}$. Thus the metric g is of the form

$$g = (e^{-\rho t})^2 \sum_{i,j=1}^{n-1} g_{ij}^*(x) dx^i \otimes dx^j + dt \otimes dt = (e^{-\rho t})^2 g_N + dt \otimes dt. \quad (2.10)$$

In particular, if $\rho = 0$, then g is the product metric $g = g_N + dt \otimes dt$ on the product space $M = N \times \mathbb{R}$. Consequently Lemma 2.1 and 2.2 imply the following.

Theorem 2.1. *Let (M, g) be a connected complete Riemannian manifold. Then, there exists a smooth function $f : M \rightarrow \mathbb{R}$ such that its gradient $\nabla^g f$ is a semi-parallel vector field of unit length if and only if there exists a connected complete hypersurface N which is totally umbilic with constant mean curvature ρ and M is isometric to the warped product space $N \times_{\psi(t)} \mathbb{R}$ with the warping function $\psi(t) = e^{-\rho t}$. In this case, the function f is the signed distance function to N .*

By the exponential function, the manifold $N \times \mathbb{R}$ is diffeomorphic to $N \times \mathbb{R}_+$ by sending $N \times \mathbb{R} \ni (x, t) \rightarrow (x, e^t) \in N \times \mathbb{R}_+$.

Example 2.2. Let $\mathbb{H} = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n > 0\}$ be the upper half plane with the Riemannian connection ∇^g of the Poincaré metric

$$g_P = \left(\frac{1}{x^n} \right)^2 \sum_{\alpha=1}^n dx^\alpha \otimes dx^\alpha. \quad (2.11)$$

Setting $t = \log x^n$, the metric g_P is written as

$$g_P = (e^{-t})^2 \sum_{i=1}^{n-1} dx^i \otimes dx^i + dt \otimes dt. \quad (2.12)$$

Comparing this metric with (2.11), we define a function $f : \mathbb{H} \rightarrow \mathbb{R}$ by $f(x) = \log x^n = t$. Since ∇^g is given by the Christoffel symbols

$$\Gamma_{jk}^i = -\frac{1}{x^n} (\delta_{jn} \delta_k^i + \delta_{kn} \delta_j^i - \delta_{jk} \delta_n^i),$$

we can easily show that the gradient

$$\nabla^g f = \frac{\partial}{\partial t} = x^n \frac{\partial}{\partial x^n}$$

is a semi-parallel vector field on $\mathbb{H}$ of constant norm $\|\nabla^g f\| = 1$, namely $E = \nabla^g f$ satisfies $\nabla_X^g E = \beta(X)E - X$. The hypersurface $f^{-1}(t)$ is a totally umbilical hypersurface with constant mean curvature $H = 1$ for every $t \in \mathbb{R}$. $\square$

3. Finsler metrics and connections

3.1. Finsler metrics. First we shall review the variational problems from [He]. Let $\pi : TM \rightarrow M$ be the tangent bundle of a connected smooth manifold M . We denote by $v = (x, y)$ the points in TM if $y \in \pi^{-1}(x) = T_x M$. We denote by $z(M)$ the zero section of TM , and by $TM^\times$ the slit tangent bundle $TM \setminus z(M)$. We introduce a coordinate system on TM as follows. Let $U \subset M$ be an open set with local coordinate $(x^1, \dots, x^n)$. By setting $v = \sum_{i=1}^n y^i (\partial/\partial x^i)_x$ for every $v \in \pi^{-1}(U)$, we introduce a local coordinate $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^n)$ on $\pi^{-1}(U)$.

Let $L : TM \times \mathbb{R} \rightarrow \mathbb{R}$ be a Lagrangian. The *Cartan form* θ_L of L is defined by

$$\theta_L = \sum_{i=1}^n \frac{\partial L}{\partial y^i} (dx^i - y^i dt) + L dt. \quad (3.1)$$

For every smooth curve $c : [0, 1] \rightarrow M$, we define a natural lift $\phi : [0, 1] \rightarrow TM \times \mathbb{R}$ by $\phi(t) = (c'(t), t)$. Since $\phi^*(dx^i - y^i dt) = 0$, we obtain

$$\int_0^1 \phi^*(\theta_L) = \int_0^1 L(c'(t), t) dt.$$

For an arbitrary proper variational field X , we denote by $\phi_s : (-\varepsilon, \varepsilon) \times I \rightarrow TM \times \mathbb{R}$ the variation of ϕ . The critical point of this variation is calculated as follows:

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \int_0^1 \phi_s^* \theta_L &= \int_0^1 \frac{d}{ds} \Big|_{s=0} \phi_s^* \theta_L = \int_0^1 [\phi^*(\iota_X d\theta_L) + d\phi^*(\iota_X \theta_L)] \\ &= \int_0^1 \phi^*(\iota_X d\theta_L), \end{aligned}$$

since X is proper. Then we define a 2-form Θ_L on $TM \times \mathbb{R}$ by $\Theta_L = d\theta_L$:

$$\Theta_L = \sum \left[d \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} dt \right] \wedge (dx^i - y^i dt). \quad (3.2)$$

Then $\phi = \phi(t)$ is called a *characteristic curve* of L if it satisfies $\phi^* \Theta_L = 0$. A smooth curve $c = c(t)$ on M is called a *extremal curve* of L if c is the projection of a characteristic curve $\phi = \phi(t)$ into M . A smooth curve $c = c(t)$ is a extremal if and only if it satisfies the *Euler-Lagrange equation*:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^i} \right) = \frac{\partial L}{\partial x^i}.$$

Secondly we shall investigate the case where the Lagrangian L is independent of the parameter t and, moreover, L satisfies the homogeneity condition $L(\lambda X) = \lambda L(X)$ for all $\lambda > 0$. In this case, the extremal of L is independent of the parametrization, and, since the homogeneity of L implies $\sum y^i \partial L / \partial y^i - L = 0$, the Cartan form θ_L is given by

$$\theta_L = \sum_{i=1}^n \frac{\partial L}{\partial y^i} dx^i. \quad (3.3)$$

Definition 3.1. A function $L : TM \rightarrow \mathbb{R}$ is called a *Finsler metric* on M if

1. $L(x, y) \geq 0$, and $L(x, y) = 0$ if and only if $y = 0$,
2. $L(x, \lambda y) = \lambda L(x, y)$ for $\forall \lambda \in \mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$,
3. $L(x, y)$ is smooth on $TM^\times$

are satisfied. The pair (M, L) is called a *Finsler space*.

For each $X \in T_x M$, its norm $\|X\|$ is defined by $\|X\| = L(x, X)$. The length $l(c)$ of a smooth curve $c = c(t)$ is defined by $l(c) = \int_0^1 L(c'(t)) dt$. An extremal curve in a Finsler manifold (M, L) is called a *geodesic* in (M, L) .

Example 3.1. Let g be a Riemannian metric on M . Since there exists a 1-form β satisfying $\beta(X) \leq \sqrt{g(X, X)}$, the function $L = \sqrt{g(X, X)} + \beta(X)$ defines a Finsler metric on M so-called *Randers metric*. A Randers metric L is strictly convex if and only the norm $\|\beta\|$ of β with respect to the metric g satisfies $\|\beta\| < 1$. $\square$

Let L and $\tilde{L}$ be two Finsler metrics on M . Then (M, L) is *projectively equivalent* to $(M, \tilde{L})$ if (M, L) and $(M, \tilde{L})$ have the same geodesics as point sets. An appropriate "sufficient condition" for (M, L) to be projectively equivalent to $(M, \tilde{L})$ is

$$\Theta_L = \Theta_{\tilde{L}}. \quad (3.4)$$

Definition 3.2. A Finsler space (M, L) is said to be *strictly projective-equivalent* to a Finsler space $(M, \tilde{L})$ if (3.4) is satisfied.

The condition (3.4) is written in the form:

$$\sum_{i=1}^n d \left(\frac{\partial L}{\partial y^i} \right) \wedge dx^i = \sum_{i=1}^n d \left(\frac{\partial \tilde{L}}{\partial y^i} \right) \wedge dx^i.$$

Therefore we have

$$\frac{\partial^2 (\tilde{L} - L)}{\partial y^i \partial y^j} = 0, \quad \frac{\partial^2 (\tilde{L} - L)}{\partial y^i \partial x^j} = \frac{\partial^2 (\tilde{L} - L)}{\partial x^i \partial y^j}.$$

These equations imply $\theta_{\tilde{L}} = \theta_L + \pi^* \beta$ for a closed one-form β on M and

$$\tilde{L}(X) = L(X) + \beta(X) \quad (3.5)$$

for all $X \in TM$. Consequently we obtain the following theorem:

Theorem 3.1. *Let L and $\tilde{L}$ be two Finsler metrics on a smooth manifold M . Then (M, L) is strictly projective-equivalent to $(M, \tilde{L})$ if and only if $\tilde{L}$ is given by (3.5) for a closed one-form β on M . In particular, a Finsler manifold (M, L) is strictly projective-equivalent to a Riemannian manifold (M, g) if and only if there exists a closed one-form β on M satisfying*

$$L(X) = \sqrt{g(X, X)} + \beta(X) \quad (3.6)$$

for all $X \in TM$.

3.2. Chern-Finsler connection. Let $V = \ker \pi_*$ be the *vertical sub-bundle* of the tangent bundle over TM , where π_* is the differential of the projection $\pi : TM \rightarrow M$. Since we have the natural identification $V \cong \pi^* TM = \{(y, v) \in TM \times TM \mid v \in T_{\pi(y)} M\}$ and V is locally spanned by $\{e_j = \partial/\partial y^j\}$ ($j = 1, \dots, n$) on each $\pi^{-1}(U)$, we may consider the differential π_* as a V -valued one-form $\pi_* = \sum e_i \otimes dx^i$ on TM . We denote by $A^k(V)$ the space of smooth V -valued k -forms.

A Finsler metric L is said to be *convex* if $G = L^2/2$ is *strictly convex* on each tangent space $T_x M$, that is, the Hessian (G_{ij}) defined by

$$G_{ij}(x, y) = \frac{\partial^2 G}{\partial y^i \partial y^j} \quad (3.7)$$

is positive-definite.

In the sequel, we assume the convexity of L . Then each fiber $T_x M$ is a Riemannian space with a metric $G_x = \sum G_{ij}(x, y) dy^i \otimes dy^j$, and the family $\{G_x\}_{x \in M}$ defines a metric G on V by $G(Y, Z) = \sum G_{ij} Y^i Z^j$ for every section $Y = \sum Y^i e_i$ and $Z = \sum Z^j e_j$. We define a symmetric tensor $C : \otimes^3 V \rightarrow \mathbb{R}$ by

$$C(e_i, e_j, e_k) = \frac{1}{2} \frac{\partial G_{ij}}{\partial y^k} := C_{ijk}. \quad (3.8)$$

It is trivial C vanishes identically if and only if G is a Riemannian metric on M .

The multiplier group $\mathbb{R}^+ \cong \{cI \in GL(TM); c \in \mathbb{R}^+\} \subset GL(TM)$ acts on the total space by multiplication $m_\lambda : TM \ni v = (x, y) \rightarrow \lambda v = (x, \lambda y) \in TM$ for $\forall \lambda \in \mathbb{R}^+$. This action induces a canonical vector field $\mathcal{E}$ defined by

$$\mathcal{E}(x, y) = \sum_{i=1}^n y^i \frac{\partial}{\partial y^i},$$

which is called the *tautological section* of V . By the homogeneity of L , we have

$$\mathcal{E}(L) = \frac{d}{dt} \Big|_{t=0} L(x, y + t\mathcal{E}) = L.$$

Moreover it is easily shown that $L = \sqrt{G(\mathcal{E}, \mathcal{E})}$ and $C(\mathcal{E}, \bullet, \bullet) \equiv 0$.

Definition 3.3 ([Ba-Ch-Sh]). The *Chern connection* on (M, L) is a connection $D : \Gamma(V) \rightarrow A^1(V)$ uniquely determined by the following conditions.

(1) D is *almost G -compatible*:

$$DG = 2C. \quad (3.9)$$

(2) D is *symmetric*:

$$D\pi_* = 0, \quad (3.10)$$

where we consider π_* as a V -valued one-form on TM .

We define $\theta \in A^1(V)$ by $\theta = D\mathcal{E}$. Then, $H = \ker \theta$ defines a *horizontal sub-bundle* H which is complementary to V . Denoting by $\omega_j^i = \sum_{k=1}^n \Gamma_{jk}^i(x, y) dx^k$ the connection form of the Chern connection D with respect to the frame $\{e_1, \dots, e_n\}$, the differentiation d_D in the horizontal direction is given by

$$d_DF := \sum_{k=1}^n \left(\frac{\partial F}{\partial x^k} - \sum_{l=1}^n y^l \Gamma_{lk}^i(x, y) \frac{\partial F}{\partial y^i} \right) dx^k$$

for any smooth function F on $TM^\times$. Then the Chern–Finsler connection D of (M, L) satisfies

$$d_DL \equiv 0. \quad (3.11)$$

Remark 3.1. In the case of $C = 0$, the metric L is the norm function of a Riemannian metric g , and the Chern connection D is given by $D = \pi^* \nabla^g$ for the Riemannian connection ∇^g of (M, g) . Then we have

$$d_{\nabla^g} L_g \equiv 0, \quad (3.12)$$

where L_g is defined by $L_g(X) = \sqrt{g(X, X)}$ for every $X \in TM$. $\square$

Let D be the Chern–Finsler connection of a Finsler space (M, L) . The 2-plane $\mathcal{F}(X)$ spanned by $X \in V$ and $\mathcal{E}$ is called the *flag* with the *flagpole* $\mathcal{E}$. For the curvature tensor R of D , the sectional curvature

$$K(X \wedge \mathcal{E}) = \frac{\langle R(X, \mathcal{E})\mathcal{E}, X \rangle}{\|X\|^2 \|\mathcal{E}\|^2 - \langle X, \mathcal{E} \rangle^2}$$

is called the *flag curvature* of the flag $\mathcal{F}(X)$. A Finsler manifold (M, L) is said to be of *constant flag curvature* if $K(X \wedge \mathcal{E})$ is constant for every $X \in V$.

Example 3.2 ([Ok]). The Hilbert metric g_H defined by (2.3) has negative constant curvature. The *Funk metric* $L_{\mathbb{B}}$ on $\mathbb{B}$ is defined by

$$L_{\mathbb{B}}(X) = \sqrt{g_H(X, X)} + \beta(X) \quad (3.13)$$

for the one-form β defined by (2.4). The norm $\|\beta\|_H$ with respect to g_H is given by $\|\beta\|_H = \|x\| < 1$. Since β is closed, the Funk metric $L_{\mathbb{B}}$ is strictly projectively-equivalent to Hilbert metric g_H . Furthermore, β satisfies the condition (1.1). Therefore, by Theorem 1.1, $(\mathbb{B}, L_{\mathbb{B}})$ has negative constant flag curvature $K = -1/4$. $\square$

3.3. Berwald spaces and Wagner spaces. A Finsler metric L is said to be *flat* or *locally Minkowski* if its Chern-Finsler connection D is flat, that is, its curvature R vanishes identically. The flatness of L is equivalent to the fact that there exists an open covering of M such that the metric L is independent of the base point $x \in M$ (cf. [Ma]).

Definition 3.4. A Finsler metric L is said to be *Berwald* if the Chern-Finsler connection D is given by $D = \pi^*\nabla$ for a symmetric linear connection ∇ in TM .

The following theorem plays an important role in the study of Berwald spaces.

Theorem 3.2 ([Sz]). *Let (M, L) be a Berwald space. Then there exists a Riemannian metric g satisfying $D = \pi^*\nabla^g$, where ∇^g is the Riemannian connection of (M, g) .*

It is obvious that if (M, L) is a Berwald space, then (M, L) is projective equivalent to the associated Riemannian space (M, g) . In Theorem 3.2, without loss of generality, we may assume $L(X) > L_g(X)$ for every $X \in TM^\times$. Then we have

Theorem 3.3. *Suppose that a Berwald space (M, L) is strictly projective-equivalent to the associated Riemannian space (M, g) . Then L has the form of*

$$L(X) = \sqrt{g(X, X)} + \beta(X) \quad (3.14)$$

for a parallel one-form β on (M, g) .

PROOF. We assume that (M, L) is strictly projective-equivalent to (M, L_g) , that is, $\Theta_L = \Theta_{L_g}$. Then, by Theorem 3.1, there exists a closed one-form $\beta = \sum \beta_k(x) dx^k$ on M satisfying (3.5). Furthermore, by Theorem 3.2, the Chern-Finsler connection D is given by $D = \pi^*\nabla^g$ for the Riemannian connection ∇^g

of the associated Riemannian space (M, g) . Therefore we have $d_{\nabla^g} L = 0$. Consequently, because of (3.12), the one-form β satisfies

$$\begin{aligned} 0 &= d_{\nabla^g} [\beta(\mathcal{E})] = \sum_{k=1}^n \left(\frac{\partial \beta(\mathcal{E})}{\partial x^k} - \sum_{l=1}^n y^l \Gamma_{lk}^i(x, y) \frac{\partial \beta(\mathcal{E})}{\partial y^i} \right) dx^k \\ &= \sum_{k,l=1}^n y^l \left(\frac{\partial \beta_l}{\partial x^k} - \sum_{h=1}^n \Gamma_{kl}^h(x) \beta_h \right) dx^k = (\nabla^g \beta)(\mathcal{E}). \end{aligned}$$

Consequently β must be parallel with respect to ∇^g . $\square$

A Finsler metric (M, L) is said to be *generalized Berwald* if there exists a linear connection ∇ with torsion T such that $D = \pi^* \nabla$ (cf. [Ha2]). By the same argument as that in [Sz], we can prove the following:

Theorem 3.4 ([Ai]). *Let (M, L) be a generalized Berwald space whose Chern connection D is given by $D = \pi^* \nabla$ for a linear connection ∇ with torsion T . Then there exists a Riemannian metric g such that ∇ is compatible with g .*

In particular, a generalized Berwald space is said to be a *Wagner space* if the linear connection ∇ is *semi-symmetric*, that is, there exists a one-form γ such that its torsion T is given by

$$T(X, Y) = \gamma(X)Y - \gamma(Y)X. \quad (3.15)$$

Since a semi-symmetric linear connection that is compatible with a Riemannian metric g is uniquely determined, we obtain the following as an application of Theorem 3.4.

Theorem 3.5. *Let (M, L) be a Wagner space, and let ∇ be the semi-symmetric linear connection whose torsion form T is given by (3.15). Then there exists a Riemannian metric g such that ∇ is given by*

$$\nabla_X Y = \nabla_X^g Y - \gamma(Y)X + g(X, Y)\gamma^\#, \quad (3.16)$$

where $\gamma^\#$ is the dual of γ with respect to g .

The notion of Wagner spaces has a deep relation with the conformal flatness of Finsler spaces.

4. Conformally flat Randers metrics

A *conformal change* of Finsler metric L is defined by the change $L \mapsto \tilde{L} = e^{\sigma(x)} L$ for a smooth function $\sigma(x)$ on M .

Definition 4.1. A Finsler space (M, L) is said to be *locally conformal to a Berwald space* if there exists a local function $\sigma_U(x)$ on an open subset $U \subset M$ such that $\tilde{L}_U = e^{\sigma_U(x)}L$ is Berwald on U . A Finsler space (M, L) is said to be *conformally flat* if (M, L) is locally conformal to a flat Finsler space.

The following theorem is fundamental in the rest of this paper.

Theorem 4.1 ([Ha-Ic2]). *A Finsler space (M, L) is locally conformal to a Berwald space if and only if (M, L) is a Wagner space with respect to a closed one-form β . In particular, (M, L) is conformally flat if and only if (M, L) is a Wagner space whose semi-symmetric connection is flat.*

Let E be a semi-parallel vector field on a Rimeannian space (M, g) with unit length, that is,

$$\nabla_X^g E = \rho[X - \beta(X)E] \quad (4.1)$$

for a constant ρ . The dual β of E is a closed one-form satisfying (1.1). For this one-form β , we define a linear connection ∇ by

$$\nabla_X Y = \nabla_X^g Y + \rho[g(X, Y)E - \beta(Y)X] \quad (4.2)$$

for the Riemannian connection ∇^g of (M, g) .

Then we have

Proposition 4.1. *The curvature tensor R of the connection ∇ defined by (4.2) is given by*

$$R(X, Y)Z = R^g(X, Y)Z + \rho^2[g(Y, Z)X - g(X, Z)Y] \quad (4.3)$$

in term of the curvature R^g of ∇^g .

PROOF. Because of $\nabla_X E = 0$, we have

$$\begin{aligned} \nabla_X \nabla_Y Z &= \nabla_X^g \nabla_Y^g Z + \rho[g(X, \nabla_Y^g Z)E - \beta(\nabla_Y^g Z)X - X(\beta(Z))Y \\ &\quad + X(g(Y, Z))E - \beta(Z)\nabla_X^g Y] - \rho^2\beta(Z)[g(X, Y)E - \beta(Y)X], \\ \nabla_Y \nabla_X Z &= \nabla_Y^g \nabla_X^g Z + \rho[g(Y, \nabla_X^g Z)E - \beta(\nabla_X^g Z)Y - Y(\beta(Z))X \\ &\quad + Y(g(X, Z))E - \beta(Z)\nabla_Y^g X] - \rho^2\beta(Z)[g(Y, X)E - \beta(X)Y] \end{aligned}$$

and

$$\nabla_{[X, Y]} Z = \nabla_{[X, Y]}^g Z + \rho[g([X, Y], Z)E - \beta(Z)[X, Y]].$$

Furthermore, because of

$$\begin{aligned} g(X, \nabla_Y^g Z) + X(g(Y, Z)) - Y(g(X, Z)) - g(Y, \nabla_X^g Z) - g([X, Y], Z) \\ = g(\nabla_X^g Y, Z) - g(\nabla_Y^g X, Z) - g([X, Y], Z) = 0, \end{aligned}$$

$$-\beta(\nabla_Y^g Z) + Y(\beta(Z)) = g(Z, \nabla_Y^g E)$$

and

$$-X(\beta(Z)) + \beta(\nabla_X^g Z) = -g(Z, \nabla_X^g E),$$

we obtain

$$\begin{aligned} R(X, Y)Z &= R^g(X, Y)Z + \rho[g(Z, \nabla_Y^g E)X - g(Z, \nabla_X^g E)Y] \\ &\quad + \rho^2\beta(Z)[\beta(Y)X - \beta(X)Y] \\ &= R^g(X, Y)Z + \rho^2[g(Z, Y)X - g(Z, X)Y]. \end{aligned}$$

By this proposition, we see that the connection ∇ defined by (4.2) is flat if and only if (M, g) is of negative constant curvature $K = -\rho^2$. Then we have

Theorem 4.2. *Suppose that a Riemannian space (M, g) admits a semi-parallel vector field E of unit length. Then the Finsler metric L defined by*

$$L(X) = \sqrt{g(X, X)} + c \cdot \beta(X) \quad (0 < c < 1) \quad (4.4)$$

is locally conformal to a Berwald metric. Furthermore, if (M, g) is a space of negative constant curvature, then (M, L) is conformally flat.

PROOF. Suppose that (M, g) admits a vector field E satisfying (4.1). Then it is easily shown that ∇ defined by (4.2) is compatible with g , and that ∇ has the torsion $T(X, Y) = \rho[\beta(X)Y - \beta(Y)X]$. Furthermore, by direct computation, we can show that the dual β of E is parallel with respect to ∇ . Therefore, similarly to the proof of Theorem 3.3, we have

$$d_{\nabla}L = 0.$$

Consequently, by Theorem 4.1, L is locally conformal to a Berwald metric.

In particular, if (M, g) is a space of negative constant curvature, then (2.6) implies the constant curvature K must be $K = -\rho^2$. Therefore (4.3) implies that the connection ∇ defined by (4.2) is flat. Consequently (M, L) is conformally flat. $\square$

Example 4.1. Let $\mathbb{H}$ the upper half plane with the Poincaré metric g_P . For the function $f(x^1, \dots, x^n) = \log x^n$, the one-form β defined by $\beta = c \cdot df$ for a constant c such that $0 < c < 1$ has constant norm $\|\beta\|_P = c < 1$ with respect to g_P . We shall define a Randers metric $L_{\mathbb{H}}$ on $\mathbb{H}$ by

$$L_{\mathbb{H}}(X) = \sqrt{g_P(X, X)} + c \cdot df(X) \quad (0 < c < 1). \quad (4.5)$$

Then, as shown in Example 2.2, $\nabla^g f$ is semi-parallel vector field on $\mathbb{H}$ with unit length. Since $(\mathbb{H}, g_P)$ is negative constant curvature -1 and the form β satisfies the condition (4.1), $(\mathbb{H}, L_{\mathbb{H}})$ is conformally flat. In deed, $L_{\mathbb{H}}$ is given by

$$L_{\mathbb{H}}(X) = \frac{1}{x^n} \sqrt{\sum_{i=1}^n (X^i)^2} + c \frac{X^n}{x^n} \quad (0 < c < 1)$$

for every $X \in T\mathbb{H}$. □

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