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B-transform and its quasiasymptotics-applications to the convolution equation $(xu_{xx} + 2u_x + mu) * g = h,\ m \leq 0$

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Abstract. By using the B-transform we examine the equation $(xu_{xx} + 2u_x + mu) * g = h$, $g, h \in S'_+$, $m \leq 0$. We investigate the quasiasymptotic behaviour of the solution. We find the Laguerre series solution.

1. Introduction

The B-transform, or Bessel transform on the spaces of tempered distribution supported by $[0,\infty)$ was introduced by ZAVIALOV [10]; see also [9]. In [6] we have given different approaches to the B-tansform on the spaces of tempered distributions and ultradistributions supported by $[0,\infty)$ by using Laguerre expansions of their elements.

This paper is concerned with the convolution equation

$$(xu'' + 2u' + mu) * g = h,\ m \leq 0,$$

the quasiasymptotic behaviour of the solution and the Laguerre series solution.

In Section 2 we give the definition and the Laguerre representation of the B-transform on the spaces S'_+. In Section 3 we give relations between the quasiasymptotics at 0^+ (resp. ∞) and the B-transform as well as the applications of these notions to the qualitative analysis of an ordinary differential equation in S'_+, and consequently to the quoted convolution equation. In Section 4 we give an explicit method for solving this convolution equations based on the Laguerre expansions.

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2. Generalized B-transform

The basic test function space for the well-known space of tempered distributions supported by $[0,\infty)$, S'_+, is

$$S_+ = \{\phi \in C^\infty[0,\infty), \sup_{t\in[0,\infty)} t^k|\phi^{(n)}(t)| < \infty,\ k,n \in \mathbb{N}_\mathbf{0}\}.$$

We give below an equivalent definition of this space ([2], [4], [7], [11].)

Let $l_n = e^{-x/2}L_n(x)$, $x > 0$, $n \in \mathbb{N}_\mathbf{0}$ be the Laguerre orthonormal system in $L^2(\mathbb{R}_+)$, where

$$L_n = \sum_{m=0}^{n} \binom{n}{n-m} \frac{(-x)^m}{m!}, \quad x > 0,\ n \in \mathbb{N}_\mathbf{0},$$

are the Laguerre polynomials, and l_n are the eigenfunctions of the operator $\mathcal{R} = e^{x/2}Dxe^{-x}De^{x/2}$, for which $\mathcal{R}(l_n) = -nl_n$, $n \in \mathbb{N}_\mathbf{0}$.

S_+ is the space of smooth functions for which all the norms

$$\|\phi\|_k = \left(\int_0^\infty |\mathcal{R}^k\phi(x)|^2 dx\right)^{1/2}, \quad k \in \mathbb{N}_\mathbf{0}$$

are finite and the following holds:

$$(\mathcal{R}^k\phi, l_n) = (\phi, \mathcal{R}^k l_n), \quad k,n \in \mathbb{N}_\mathbf{0},\ (\mathcal{R}^{k+1} = \mathcal{R}(\mathcal{R}^k)).$$

In [6] we defined the B-transform on S'_+ dualizing the results for the b-transform on S_+:

$$\langle B[f], \phi\rangle = \langle f, b[\phi]\rangle, \quad \phi \in S_+,$$

where, if $\phi = \sum_{n=0}^\infty a_n l_n \in S_+$, then

$$b[\phi](t) = \phi(0) + 1/2\langle \phi(\tau), \sqrt{t/\tau}J_0'(\sqrt{t\tau})\rangle =$$
$$= -2\sum_{n=0}^\infty (-1)^n(2\sum_{i=n+1}^\infty a_i + a_n)l_n(t), \quad t > 0 \quad ([10],[6]).$$

So we obtain ([6])

$$B[f] = \sum_{n=0}^\infty [2\sum_{m=0}^{n-1}(-1)^m b_m + (-1)^n b_n]l_n,$$

being $f = \sum_{n=0}^\infty b_n l_n$.

Recall ([8], [6]) that

$$B[f_\alpha] = 4^\alpha f_{-\alpha},\ \alpha \in \mathbb{R},$$

where

$$f_\alpha = \begin{cases} H(t)t^{\alpha-1}/\Gamma(\alpha) & \text{if} \quad \alpha > 0, \\ D^N f_{\alpha+N}(t) & \text{if} \quad \alpha \le 0,\ \alpha + N > 0,\ N \in \mathbb{N}, \end{cases}$$

and D is the distributional derivative.

3. The quasiasymptotics and the B-transform

Let us start with regularly varying functions at ∞ and 0^+ which were defined by J. Karamata in the early thirties as natural generalizations of power functions. The best reference concerning such functions in [1].

A function $\rho : (a, \infty) \to \mathbb{R}$, (resp. $\rho : (0, a) \to \mathbb{R}$), $a \in \mathbb{R}$ is *regularly varying at* ∞, (resp. at 0^+) if it is positive, measurable and there exists a real number α such that for each $x > 0$,

$$\lim_{k\to\infty} \frac{\rho(kx)}{\rho(k)} = x^\alpha \quad (\text{resp. } \lim_{\varepsilon\to 0} \frac{\rho(\varepsilon x)}{\rho(\varepsilon)} = x^\alpha).$$

Specially, when $\alpha = 0$, then ρ is *slowly varying at* ∞ (resp. at 0^+), and for such a function the letter "L" will be used. Recall some properties of regularly varying functions. A positive and measurable function $\rho : (a, \infty) \to \mathbb{R}$, (resp. $\rho : (0, a) \to \mathbb{R}$) is regularly varying at ∞, (resp. at 0^+) if and only if it can be written as

$$\rho(x) = x^\alpha L(x), \quad x > a \ \ (\text{resp. } x \in (0, a)),$$

for some real number α and some slowly varying function L at ∞ (resp. at 0^+). If $L(k),\ k \ge k_0$ is slowly varying at ∞ then $L(1/k), k \in (0, 1/k_0)$ is slowly varying at 0^+. The reverse assertion also holds.

The notion of quasiasymptotic behaviour at ∞ and 0^+ of distributions from S'_+ has been introduced by Vladimirov, Zavialov and Drožžinov [9]. Recall the definitions and the properties of this notion.

Let $f \in S'_+$ and $c(k) = k^\sigma L(k),\ k > 0$, (resp. $c(\varepsilon) = \varepsilon^\sigma L(\varepsilon),\ \varepsilon \in (0, \varepsilon_0)$), where $L(k), k \ge k_0$, (resp. $L(\varepsilon),\ \varepsilon \in (0, \varepsilon_0)$) is slowly varying at ∞ (resp. at 0^+). Then f has the quasiasymptotic behaviour at ∞ (resp. at 0^+) of order σ with respect to $c(k)$ (resp. $c(\varepsilon)$) if

$$\lim_{k\to\infty} \langle \frac{f(kx)}{k^\sigma L(k)}, \phi(x) \rangle = \langle C f_{\sigma+1}, \phi \rangle, \quad (\phi \in S_+),\ C \ne 0$$

(resp.

$$\lim_{\varepsilon\to\infty} \langle \frac{f(\varepsilon x)}{\varepsilon^\sigma L(\varepsilon)}, \phi(x) \rangle = \langle C f_{\sigma+1}, \phi \rangle, \quad (\phi \in S_+),\ C \ne 0).$$

We shall use the identity

$$B[f(\varepsilon x)](t) = k^2 B[f(x)](kt), \quad t > 0,\ \varepsilon = 1/k,\ k > 0. \tag{1}$$

Proposition 1. *Let $f \in S'_+$, and let $L(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0)$ be slowly warying at 0^+. Then the following conditions are equivalent:*

1. *f has quasiasymptotics at 0^+ (resp. at ∞) of order σ with respect to $\varepsilon^\sigma L(\varepsilon)$ (resp. $k^\sigma L(1/k)$).*
2. *Bf has quasiasymptotics at ∞ (resp. at 0^+) of order $-\sigma - 2$ with respect to $k^{-\sigma-2}L(1/k)$ (resp. $\varepsilon^{-\sigma-2}L(\varepsilon)$).*

Proof. Since $B : S'_+ \to S'_+$ is an isomorphism ([6]) we have to prove only the part of this assertion which corresponds to 0^+.

1.$\Longrightarrow$2. Let $\phi \in S_+$. Then, with $\varepsilon = 1/k,\ k \to \infty$,

$$\langle \frac{(Bf)(kt)}{k^{-\sigma-2}L(1/k)}, \phi(t)\rangle = \langle \frac{B[f(x/k)](t)}{k^{-\sigma-2+2}L(1/k)}, \phi(t)\rangle = \langle \frac{f(x/k)}{k^{-\sigma}L(1/k)}, b[\phi](x)\rangle$$
$$= \langle \frac{f(\varepsilon x)}{\varepsilon^\sigma L(\varepsilon)}, b[\phi](x)\rangle \to \langle C f_{\sigma+1}(x), b[\phi](x)\rangle = \langle CB[f_{\sigma+1}], \phi\rangle$$
$$= \langle \tilde{C} f_{-\sigma-1}, \phi\rangle, \text{ where } \tilde{C} = C4^{\sigma+1}.$$

2.$\Longrightarrow$1. For given $\phi \in S_+$ let $\phi = b[\psi], \psi \in S_+$. From $b[\phi] = b[b[\psi]] = \psi$ ([6]) and (1) we have

$$\langle \frac{f(x/k)}{k^{\sigma+2}L(k)}, \phi(t)\rangle = \langle \frac{(Bf)(kt)}{k^\sigma L(k)}, b[b[\psi]](x)\rangle \to \langle C f_{-\sigma-2+1}, \psi\rangle, \quad k \to \infty,$$

and this implies the assertion.

Now we shall use the B-transform and quasiasymptotics for the qualitative analysis of the following ordinary differential equation in S'_+:

$$x u_{xx} + 2u_x + mu = g,\ m \le 0. \tag{2}$$

Since

$$B[x u_{xx} + 2u_x](t) = (-t/4)\ B[u](t),\ t > 0, ([6]), \tag{3}$$

it is equivalent to the equation

$$(-t/4 + m)\ \tilde{u} = \tilde{g},\ \text{where } \tilde{g} = B[g] \in S'_+, \quad \text{and } \tilde{u} = B[u].$$

Let us remark that the equation

$$xp = q,\ q \in S'_+ \tag{4}$$

has solutions in S'_+ uniquely determined up to $C\delta,\ C \in \mathbb{C}$, and that the equation

$$(x + m)p = q,\ m > 0,\ q \in S'_+,$$

has the unique solution in S'_+.

We need the following

Proposition 2. (i) *Let $m = 0$ in* (2) *and g have quasiasymptotics at 0^+ with respect to $\varepsilon^\sigma L(\varepsilon)$, where $\sigma \neq -2, -1, 0, 1, \ldots$ then:*

1. *if $\sigma < -2$, then the solution u has quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma+1}L(\varepsilon)$;*
2. *if $\sigma > -2$ then there exists numbers $b_j, j = 0, 1, \ldots, p$ such that $u + \sum_{j=0}^p b_j f_j$ has quasiasymptotics with respect to $\varepsilon^{\sigma+1}L(\varepsilon)$.*

(ii) *Let $m < 0$ in* (2) *and g have quasiasymptotics at 0^+ with respect to $\varepsilon^\sigma L(\varepsilon)$, $\sigma \in \mathbb{R}$. Then the solution u has quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma+1}L(\varepsilon)$.*

Proof. (i) 1. Since $\tilde{g}$ has quasiasymptotics at ∞ with respect to $k^{-\sigma-2}L(1/k)$ and $\sigma < -2$, by [9] (p. 144, the first part of Lemma 2), it follows that $\tilde{u}$ has quasiasymptotics at ∞ with respect to $k^{-\sigma-3}L(1/k)$, because $-\sigma - 3 > -1$. Proposition 1 implies that u has quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma+1}L(\varepsilon)$.

2. If $\sigma > -2$ then there exist $p \in \mathbb{N}$ and numbers $a_j, j = 0, 1, \ldots, p$, such that $\tilde{u} + \sum_{j=0}^p a_j \delta^{(j)}$ has quasiasymptotics at ∞ with respect to $k^{-\sigma-3}L(1/k)$. This follows from [9] (p. 144, the second part of Lemma 2).

Thus $u + \sum_{j=0}^p b_j x_+^{j-1}/\Gamma(j)$, where $b_j = 4^{-j}a_j$, has quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma+1}L(\varepsilon)$ because $B[\delta^{(j)}] = 4^{-j}f_j$, $j \in \mathbb{N}$.

(ii) Since for every $\phi \in S_+$

$$\frac{1}{k^{-\sigma-3}L(1/k)}\langle\left(\frac{\tilde{g}(t)}{-t/4+m}\right)(kx), \phi(x)\rangle = \langle\frac{\tilde{g}(kx)}{k^{-\sigma-2}L(1/k)}, \frac{1}{-x/4+m/k}\phi(x)\rangle$$

and

$$\frac{1}{-x/4+m/k}\phi(x) \to \phi(x) \text{ in } S_+ \text{ as } k \to \infty,$$

the proof of the assertion easily follows from Proposition 1.

The main part of the paper is a qualitative analysis of the equation

$$((xu)'' + mu) * h = g, \ m \leq 0, \tag{5}$$

where g and h are from S'_+. For example, the equation

$$\sum_{k=0}^m a_k((xu)'' + mu)^{(k)} = g \quad (\text{with } h = \sum_{k=0}^m a_k\delta^{(k)}) \tag{6}$$

is of this form.

Proposition 3. *Assume that $g, h \in S'_+$ and the Laplace transform of h, $(\mathcal{L}h)\,(x+iy),\ x \in \mathbb{R}, y > 0$, has a bounded argument. Let h have quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma_1} L_1(\varepsilon)$ and g have quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma_2} L_2(\varepsilon)$.*

1. *Let $m = 0$ in* (5). *If $\sigma_1 - \sigma_2 > 1$ then* (5) *has the solution u with quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma_2-\sigma_1} L_2(\varepsilon)/L_1(\varepsilon)$.*
   If $\sigma_1 - \sigma_2 < 1$ and $\sigma_1 - \sigma_2 \neq 1, 0, -1, -2, \dots$ then there are numbers $b_j \in \mathbb{C}, j = 0, 1, \dots, p$, such that (5) *has the solution u such that $u + \sum_{j=0}^{p} b_j f_j$ has quasiassymptotics at 0^+ with respect to $\varepsilon^{\sigma_2-\sigma_1} L_2(\varepsilon)/L_1(\varepsilon)$.*
2. *Let $m < 0$ in* (5). *Then* (5) *has the solution u with quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma_2-\sigma_1} L_2(\varepsilon)/L_1(\varepsilon)$.*

Proof. First, we will prove if the Laplace transform of h has a bounded argument, then the same holds for $B[h]$. Namely by

$$b[e^{iz\tau}](t) = e^{-\frac{ti}{4z}},\ t > 0,\ \mathrm{Im} z > 0 \quad ([9],\text{p.}41)$$

we have

$$\mathcal{L}(B[f])(z) = \langle (B[f])(\tau), e^{i\tau z}\rangle = \langle f(t), b[e^{iz\tau}]\rangle = \mathcal{L}f\left(-\frac{1}{4z}\right),\ \mathrm{Im} z > 0,$$

which implies the assertion.

By using (3), and $B[f * g] = B[f] * B[g]$, (see [6]), (5) becomes

$$(-t/4 + m)\tilde{u} * \tilde{h} = \tilde{g}. \tag{7}$$

We shall prove only part 1 since part 2 simply follows. Let $m = 0$. We shall use a theorem from [9], page 198. In the one-dimensional case this theorem reads as follows:

"Let $\mathcal{K} \in S'_+$ has quasiasymptotics at ∞ with respect to $k^\alpha L_1(k)$ with the limit $C_1 f_{\alpha+1},\ C_1 \neq 0$, and $f \in S'_+$ has quasiasymptotics at ∞ with respect to $k^\beta L_2(k)$ with the limit $C_2 f_{\beta+1},\ C_2 \neq 0$. Let the Laplace transform of $\mathcal{K}$, $\mathcal{L}\mathcal{K}(x+iy)$, has a bounded argument in $\mathbb{R} + i\mathbb{R}_+$. Then the convolution equation $\mathcal{K} * u = f$ has the solution $u \in S'_+$ which has quasiasymptotics at ∞ with respect to $k^{(\beta-\alpha-1} L_2(k)/L_1(k)$ with the limit $(C_2/C_1) f_{\beta-\alpha}$."

Since the Laplace transform of $\tilde{h}$ has a bounded argument, it follows that there exists $\tilde{s} \in S'_+$ such that $\tilde{s} * \tilde{h} = \tilde{g}$, and

$$\frac{\tilde{s}(kx)}{k^{\sigma_1-\sigma_2-1} L_2(1/k)/L_1(1/k)} \to \mathrm{const} \cdot f_{\beta-\alpha},\ k \to \infty \text{ in } S'_+,$$

because $\tilde{g}$ has quasiasymptotics with respect to $k^{\sigma_2-2} L_2(1/k)$ and $\tilde{h}$ has quasiasymptotics with respect to $k^{\sigma_1-2} L_1(1/k)$.

Let u be the solution of

$$xu'' + 2u' = s \iff (-t/4)\tilde{u} = \tilde{s}.$$

As in the proof of Proposition 2. we have the following situations:

1. Since $\tilde{s}$ has quasiasymptotics at ∞ with respect to $k^{-(\sigma_2-\sigma_1-1)-2} L_2(1/k)/L_1(1/k)$, if $\sigma_2-\sigma_1 < -1$ it follows that $\tilde{u}$ has quasiasymptotics at ∞ with respect to $k^{-(\sigma_2-\sigma_1-1)-3}L_2(1/k)/L_1(1/k)$ and by Proposition 1, h has the quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma_2-\sigma_1}L_2(\varepsilon)/L_1(\varepsilon)$.

2. If $\sigma_2 - \sigma_1 > -1$ and $\sigma_1 - \sigma_2 \neq 1, 0-1, -2\dots$ then as in the proof of Proposition 2 we conclude that there are numbers $a_j, j = 0,\dots,p$, such that $\tilde{u}+\sum_{j=0}^{p} a_j\delta^j$ has quasiasymptotics at ∞ with respect to $k^{(\sigma_2-\sigma_1-1)-3}L_2(1/k)/L_1(1/k)$ which implies that

$$u + \sum_{j=0}^{p} b_j x_+^{j-1}/\Gamma(j), \quad (b_j = 4^j a_j,\ j = 0,\dots,p),$$

has quasiasymptotics at 0^+ with respect to $\varepsilon^{\sigma_2-\sigma_1}L_2(\varepsilon)/L_1(\varepsilon)$.

4. Laguerre series solution of convolution equations

We can find the Laguerre series solution of (5) by using (7) and the approximation formulas for the convolution given in [7]. Recall, if $f = \sum_{n=0}^{\infty} b_n l_n \in S'_+$, $g = \sum_{n=0}^{\infty} c_n l_n \in S'_+$ then

$$f * g = \sum_{n=0}^{\infty}\Big(\sum_{p+q=n} b_p c_p - \sum_{p+q=n-1} b_p c_q\Big) l_n, \tag{8}$$

where as usual $\sum_{p+q=-1} = 0$, and thus ([6])

$$B[f*g] = \sum_{n=0}^{\infty}\Big[\sum_{p+q=n}\Big(2\sum_{k=p}^{n-1}(-1)^k + (-1)^n\Big)b_p c_q - \sum_{p+q=n-1}\Big(2\sum_{k=p}^{n-1}(-1)^k + (-1)^n\Big)b_p c_q)\Big] l_n.$$

Let in (7)

$$(-t/4+m)\tilde{u} = \sum_{n=0}^{\infty} b_n l_n, \quad \tilde{h} = \sum_{n=0}^{\infty} c_n l_n, \quad \tilde{g} = \sum_{n=0}^{\infty} d_n l_n.$$

Then (8) gives the system of equations

$b_0c_0 = d_0,\ b_1c_0 + b_0c_1 - b_0c_0 = d_1,\ b_2c_0 + b_1c_1 + b_0c_2 - b_1c_0 - b_0c_1 = d_2,\ \dots$

which is discussed in [7]. With $\tilde{u} = \sum_{n=0}^{\infty} x_n l_n$, and since

$$-tl_n = (n+1)l_{n+1} - (2n+1)l_n + nl_{n-1},$$

(see [3] p. 188, formula (8)), it follows

$$\sum_{n=0}^{\infty} x_n[-1/4t + m]l_n = \sum_{n=0}^{\infty} b_n l_n$$

or

$$\sum_{n=0}^{\infty}[nx_{n-1} - (2n+1-4m)x_n + (n+1)x_{n+1}]l_n = \sum_{n=0}^{\infty} 4b_n l_n.$$

This gives the system of equations

$$(-1+4m)x_0 + x_1 = 4b_0$$

$$x_0 + (-3+4m)x_1 + 2x_2 = 4b_1$$

$$\dots$$

$$ix_{i-1} + (-2i-1+4m)x_i + (i+1)x_{i+1} = 4b_i,\ i \in \mathbb{N}. \tag{9}$$

whose solution gives the coefficients of the Laguerre series of the inverse of the solution $\tilde{u}(x)$.

Summing (9) till $i = n$ we obtain the recurrence relation

$$x_1 = 4b_0 - (4m-1)x_0,$$

$$(n+1)x_{n+1} - (n+1)x_n + 4m\sum_{i=0}^{n} x_i = 4\sum_{i=0}^{n} b_i \quad n \in \mathbb{N},\ m \le 0,$$

which gives the solution $\tilde{u}$.

If $m = 0$ then one can easily obtain, in view of $x_1 = x_0 + 4b_0$,

$$x_{n+1} = x_0 + \sum_{i=0}^{n} b_i \sum_{j=i}^{n} 4/(j+1), \quad n \in \mathbb{N}_{\mathbf{0}},$$

and the inverse for $\tilde{u}$ is

$$u(x) = \sum_{n=0}^{\infty}(2\sum_{i=0}^{n-1}(-1)^i x_i + (-1)^n x_n)l_n(x), (see[6]).$$

Finally, since $\delta = \sum_{n=0}^{\infty} l_n$ the solution is

$$u(x) = x_0\delta + \sum_{n=0}^{\infty}(2\sum_{j=0}^{n-1}(-1)^j \sum_{k=0}^{j-1} b_i \sum_{k=i}^{j-1} 4/(k+1)$$

$$+ (-1)^n \sum_{i=0}^{n-1} b_i \sum_{k=i}^{n-1} 4/(k+1)])l_n.$$

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