

Complete spacelike CMC hypersurfaces in a Lorentzian space form

By BIAOGUI YANG (Fuzhou) and XIMIN LIU (Guangzhou)

Abstract. Let $x : M^n \rightarrow \overline{M}_1^{n+1}(c)$ be a complete spacelike hypersurface immersed into a Lorentzian space form, where $\overline{M}_1^{n+1}(c)$ is a Lorentz–Minkowski space $\mathbb{L}^{n+1} = \mathbb{R}_1^{n+1}$, a de Sitter space $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$ or an anti-de Sitter space $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$, according to $c = 0$, $c = 1$ or $c = -1$, respectively. Let $\phi = \langle x, a \rangle$ and $\psi = \langle \vec{H}, a \rangle$, where $\vec{H}$ is the mean curvature vector field of M^n and a is a fixed nonzero vector in the corresponding pseudo-Euclidean space. We prove that if M^n has constant mean curvature (CMC), and $\phi = \lambda\psi$, for some real number λ , then M^n is a spacelike isoparametric hypersurface of $\overline{M}_1^{n+1}(c)$. Furthermore, it is either a totally umbilical hypersurface or a hyperbolic cylinder.

1. Introduction

Let $\mathbb{R}_t^{n+2}$ be an $(n+2)$ -dimensional pseudo-Euclidean space with index t endowed the indefinite inner product given by with flat semi-Euclidean metric

$$\langle x, y \rangle = - \sum_{i=1}^t x_i y_i + \sum_{j=t+1}^{n+2} x_j y_j,$$

where $(x_1, \dots, x_{n+2})$ is a rectangular coordinate system of $\mathbb{R}_t^{n+2}$. The de Sitter space and anti-de Sitter space [9] are defined by $\mathbb{S}_1^{n+1} = \{x \in \mathbb{R}_1^{n+2} | \langle x, x \rangle = 1\}$, $\mathbb{H}_1^{n+1} = \{x \in \mathbb{R}_2^{n+2} | \langle x, x \rangle = -1\}$, respectively, with constant sectional curvature

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$c = 1$ and $c = -1$, respectively. A hypersurface M^n is said to be spacelike if the induced metric on M^n from that of the ambient space is positive definite.

Constant mean curvature (CMC) hypersurfaces are often closely related to either an eigenvalue problem or a differential equation stemming from the Laplacian. At the same time, Maximal and CMC hypersurfaces play a chief role in relativity theory. There are many interesting results in the study of spacelike CMC hypersurfaces.

The study of this kind of hypersurface was inspired, in particular, by a conjecture posed by GODDARD [6], stating that every complete spacelike hypersurface with constant mean curvature in $\mathbb{S}_1^{n+1}$ must be totally umbilical. The first result in this direction was obtained by RAMANATHAN [10] in 1987. He showed that if the constant mean curvature H of a complete spacelike surface in $\mathbb{S}_1^3$ satisfies $H^2 \leq 1$, then the surface is totally umbilical. Independently, and still in 1987, AKUTAGAWA [2] proved Goddard's conjecture for the case $H^2 \leq 1$ if $n = 2$ and for the case $H^2 < 4(n-1)/n^2$ if $n > 2$. On the other hand, MONTIEL [8] proved the conjecture for the compact case.

In [3], ALÍAS, BRASIL and PERDOMO studied the quadric constant mean curvature hypersurfaces of spheres and gave a characterization of ones by a linear relation between two functions on the position vector and a Gauss map of ones. Inspired their works, we will investigate complete spacelike CMC hypersurfaces in a Lorentzian space form.

Let $x : M^n \rightarrow \overline{M}_1^{n+1}(c)$ be a complete spacelike hypersurface immersed into a Lorentzian space form, where $\overline{M}_1^{n+1}(c)$ is a Lorentz–Minkowski space $\mathbb{L}^{n+1} = \mathbb{R}_1^{n+1}$, a de Sitter space $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$ or an anti-de Sitter space $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ according to $c = 0$, $c = 1$ or $c = -1$, respectively. For some fixed nonzero vector $a \in \mathbb{R}_1^{n+1}$, $\mathbb{R}_1^{n+2}$ or $\mathbb{R}_2^{n+2}$, according to $c = 0$, $c = 1$ or $c = -1$, respectively, let $\phi = \langle x, a \rangle$ and $\psi = \langle \vec{H}, a \rangle$, where $\vec{H}$ is the mean curvature vector field of M^n . In this paper, we will prove that if M^n has constant mean curvature, and $\phi = \lambda\psi$, for some real number λ , then M^n is either a totally umbilical hypersurface or a hyperbolic cylinder. In fact, we prove the following main results.

Theorem 1.1. *Let $x : M^n \rightarrow \mathbb{L}^{n+1}$ be a complete spacelike CMC hypersurface immersed into the Lorentz–Minkowski space $\mathbb{L}^{n+1}$. If for some nonzero constant vector $a \in \mathbb{R}_1^{n+1}$ and some real number λ , we have that $\phi = \lambda\psi$, then M^n is one of the following hypersurfaces, up to rigid motions:*

- (i) $\mathbb{R}^n = \{x \in \mathbb{R}_1^{n+1} : x_1 = 0\}$;
- (ii) $\mathbb{H}^n(\sinh t) = \{x \in \mathbb{R}_1^{n+1} : \|x\|^2 = -\sinh^2 t\}$, where $t \in \mathbb{R}$;

- (iii) $\mathbb{H}^k(\sinh t) \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}_1^{n+1} : -x_1^2 + x_2^2 + \cdots + x_{k+1}^2 = -\sinh^2 t\}$, where $t \in \mathbb{R}$.

Theorem 1.2. *Let $x : M^n \rightarrow \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$ be a complete spacelike CMC hypersurface immersed into the de Sitter space $\mathbb{S}_1^{n+1}$. If for some nonzero constant vector $a \in \mathbb{R}_1^{n+2}$ and some real number λ , we have that $\phi = \lambda\psi$, then M^n is either a totally umbilical hypersurface or a hyperbolic cylinder, i.e. M^n is one of the following hypersurfaces, up to rigid motions:*

- (i) $\mathbb{R}^n = \{(f(y) + \sinh t, f(y) + \cosh t, y) \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} : y \in \mathbb{R}^n\}$, where $t \in \mathbb{R}$ and $f(y) = -(e^t/2)\|y\|^2$;
- (ii) $\mathbb{S}^n(\cosh t) = \{(\sinh t, y \cosh t) \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} : y \in \mathbb{S}^n(1) \subset \mathbb{R}^{n+1}\}$, where $t \in \mathbb{R}$;
- (iii) $\mathbb{H}^n(\sinh t) = \{(y \sinh t, \cosh t) \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} : y \in \mathbb{H}^n(1) \subset \mathbb{R}_1^{n+1}\}$, where $t \in (0, +\infty)$;
- (iv) $\mathbb{H}^k(\sinh t) \times \mathbb{S}^{n-k}(\cosh t) = \{(y, z) \in \mathbb{R}_1^{k+1} \times \mathbb{R}^{n-k+1} : \|y\|^2 = -\sinh^2 t, \|z\|^2 = \cosh^2 t\}$, where $0 < k < n$, $t \in (0, +\infty)$.

Theorem 1.3. *Let $x : M^n \rightarrow \mathbb{H}_1^{n+1}(c) \subset \mathbb{R}_2^{n+2}$ be a complete spacelike CMC hypersurface immersed into the anti-de Sitter space $\mathbb{H}_1^{n+1}$. If for some nonzero constant vector $a \in \mathbb{R}_2^{n+2}$ and some real number λ , we have that $\phi = \lambda\psi$, then M^n is either a totally umbilical hypersurface or a hyperbolic cylinder, i.e. M^n is one of the following hypersurfaces, up to rigid motions:*

- (i) $\mathbb{H}^n(\sin t) = \{(\cos t, y \sin t) \in \mathbb{R}_2^{n+2} : y \in \mathbb{H}^n \subset \mathbb{R}_1^{n+1}\}$, where $t \in (0, \pi/2]$;
- (ii) $\mathbb{H}^k(\cos t) \times \mathbb{H}^{n-k}(\sin t) = \{(y, z) \in \mathbb{R}_1^{k+1} \times \mathbb{R}_1^{n-k+1} : \|y\|^2 = -\cos^2 t, \|z\|^2 = -\sin^2 t\}$, where $0 < k < n$, $t \in (0, \pi/2)$.

2. Preliminaries and auxiliary results

In this section, we give some formulas and notions of submanifolds in the space forms by using the method of moving frames. Let $x : M \rightarrow \overline{M}_1^{n+1}(c) \subset \mathbb{R}_t^{n+2}$ be a isometric immersion from Riemannian manifold M^n to Lorentz space forms $\overline{M}_1^{n+1}(c)$ with constant sectional curvature c . Let $\nabla, \overline{\nabla}, \tilde{\nabla}$ be the Levi-Civita connection on $M^n, \overline{M}_1^{n+1}(c)$ and $\mathbb{R}_t^{n+2}$.

For any $p \in M$, we can choose a local orthonormal frame fields $e_1, \dots, e_{n+2}$ ($e_{n+2} = x$ when $c = \pm 1$) in a neighborhood U of M such that $e_1, \dots, e_n$ are tangential to M , e_{n+1} is a unit timelike normal vector field of M^n . In the following

we shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C \leq n+2, \quad 1 \leq i, j, k, l \leq n, \quad n+1 \leq \alpha, \beta, \gamma \leq n+2.$$

Let ω_A be the corresponding dual frame. The smooth connection 1-forms are denoted by ω_{AB} . Then we have the structure equations of $\mathbb{R}_t^{n+2}$

$$\begin{cases} dx = \sum_A \varepsilon_A \omega_A e_A, \\ de_A = \sum_B \varepsilon_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \\ d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB}, \end{cases} \quad (2.1)$$

where $\varepsilon_A = \langle e_A, e_A \rangle$, $\varepsilon_i = 1$, $\varepsilon_{n+1} = -1$, $\varepsilon_{n+2} = c$. A well-known argument shows that the forms ω_{in+1} may be expressed as $\omega_{in+1} = \sum_j h_{ij} \omega_j$, $h_{ij} = h_{ji}$. From (2.1) we obtain structure equations of M in $M_1^{n+1}(c)$

$$\begin{cases} dx = \sum_i \omega_i e_i, \\ de_i = \sum_j \omega_{ij} e_j - \sum_j h_{ij} \omega_j e_{n+1} - c \omega_i x, \\ de_{n+1} = - \sum_{i,j} h_{ij} \omega_j e_i. \end{cases} \quad (2.2)$$

The second fundamental form is defined

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j, \quad (2.3)$$

and the square of the length of h is given by $S = |h|^2 = \sum_{i,j} h_{ij}^2$.

The Gauss and Codazzi equations are

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \quad (2.4)$$

$$h_{ijk} = h_{ikj}, \quad (2.5)$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{kj} \omega_{ki} + \sum_k h_{ik} \omega_{kj}. \quad (2.6)$$

Associated to the shape operator A of M one has n invariants S_r , $1 \leq r \leq n$, given by the equality

$$\det(tI - A) = \sum_{k=0}^n (-1)^k S_k t^{n-k}.$$

If $p \in M$ and e_k is basis of $T_p M$ formed by eigenvectors of the shape operator A_p , with corresponding eigenvalues λ_k , one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[x_1, \dots, x_n]$ is the r -th elementary symmetric polynomial on the indeterminates $x_1, \dots, x_n$. The r -th mean curvature of M is given by

$$H_r = \frac{1}{\binom{n}{r}} S_r.$$

In particular, when $r = 1$

$$H_1 = \frac{1}{n} \sum_i \lambda_i = \frac{1}{n} S_1 = H$$

is nothing but the mean curvature of M .

The classical Newton transformations $P_r : \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ are defined inductively from the shape operator A by

$$P_r = \begin{cases} I, & r = 0, \\ S_r I - A \circ P_{r-1}, & r = 1, \dots, n, \end{cases} \quad (2.7)$$

where I denotes the identity transformation in $\mathcal{X}(M)$. Equivalently,

$$P_r = \sum_{j=0}^r (-1)^j S_{r-j} A^j. \quad (2.8)$$

Thus $P_n = 0$ by Cayley–Hamilton Theorem. Moreover, since P_r is a polynomial in A for every r , it is also self-adjoint and commutes with A . Therefore, all bases of $T_p M$, diagonalizing A at $p \in M$, also diagonalize all of the P_r at p . Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\text{span}\{e_i\}^\perp \subset T_p M$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k}, \quad (2.9)$$

where

$$S_k(A_i) = \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_1, \dots, j_k \neq i}} \lambda_{j_1} \dots \lambda_{j_k}.$$

With the above notions, it is also immediate to check that $P_r(e_i) = \sum_j T_{ij}^r e_j = S_r(A_i)e_i$, where $T_{ij}^r = \langle P_r(e_i), e_j \rangle$. Thus, according to [4], [5], [11], [12], we have the following lemma.

Lemma 2.1. *For each $1 \leq r \leq n-2$*

- (a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$;
- (b) $\text{tr}(P_r) = (n-r)S_r$;
- (c) $\text{tr}(A \circ P_r) = (r+1)S_{r+1}$;
- (d) $\text{tr}(A^2 \circ P_r) = S_1 S_{r+1} - (r+2)S_{r+2}$;
- (e) $\text{tr}(P_r \circ \nabla_X A) = \langle \nabla S_{r+1}, X \rangle$ for $X \in \mathcal{X}(M)$.

Associated to each Newton transformation P_r , we consider the second-order linear differential operator $L_r : C^\infty(M) \rightarrow C^\infty(M)$ given by

$$L_r(f) = \text{tr}(P_r \circ \nabla^2 f),$$

where $\nabla^2 f : \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and given by

$$\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathcal{X}(M).$$

Using that P_r is a symmetrical operator, we have

$$L_r(fg) = fL_r g + gL_r f + 2\langle P_r(\nabla f), \nabla g \rangle \quad (2.10)$$

for every $f, g \in C^\infty(M)$.

Since $\vec{H}$ is parallel to e_{n+1} , then $\vec{H} = H e_{n+1}$. Furthermore,

$$\phi = \langle x, a \rangle, \quad \psi = H \langle e_{n+1}, a \rangle,$$

where a is a fixed vector in $\mathbb{R}_1^{n+2}$ if $c = 1$; or a is a fixed vector $\mathbb{R}_2^{n+2}$ if $c = -1$. Then we have that

Proposition 2.2. *If M^n is a spacelike hypersurface of $\overline{M}_1^{n+1}(c)$ with non-zero mean curvature vector field. Then the gradients of functions ϕ and ψ are give by*

$$\nabla \phi = a^\top, \quad \nabla \psi = \nabla(\ln |H|)\psi - HA(a^\top), \quad (2.11)$$

where $a^\top$ denotes the tangential component of a .

PROOF. Since

$$\langle \nabla \phi, e_i \rangle = e_i(\phi) = \langle \tilde{\nabla}_{e_i} x, a \rangle = \langle dx(e_i), a \rangle = \langle e_i, a \rangle,$$

and

$$\begin{aligned} \langle \nabla \langle e_{n+1}, a \rangle, e_i \rangle &= \langle \tilde{\nabla}_{e_i} e_{n+1}, a \rangle = \langle de_{n+1}(e_i), a \rangle \\ &= - \left\langle \sum_j h_{ij} e_j, a \right\rangle = - \langle A e_i, a \rangle = - \langle e_i, A(a^\top) \rangle, \end{aligned}$$

thus

$$\begin{aligned} \nabla \phi &= \sum_i \langle e_i, a \rangle e_i = a^\top, \quad \nabla \langle e_{n+1}, a \rangle = -A(a^\top) \\ \nabla \psi &= \nabla \langle H \langle e_{n+1}, a \rangle \rangle = \nabla H \langle e_{n+1}, a \rangle + H \nabla \langle e_{n+1}, a \rangle \\ &= \nabla H \langle e_{n+1}, a \rangle - H A(a^\top) = \nabla (\ln |H|) \psi - H A(a^\top). \quad \square \end{aligned}$$

Now, we have the following result.

Proposition 2.3. *If M^n is a spacelike hypersurface of $\overline{M}_1^{n+1}(c)$ with nonzero mean curvature vector field. For each $1 \leq r \leq n-2$, we have*

$$L_r \phi = -c(n-r)S_r \phi - H^{-1}(r+1)S_{r+1}\psi, \quad (2.12)$$

$$\begin{aligned} L_r \psi &= c(r+1)S_{r+1}H\phi + (H^{-1}L_r H + S_1 S_{r+1} - (r+2)S_{r+2})\psi \\ &\quad - H \langle \nabla S_{r+1}, a \rangle - 2 \langle A \circ P_r(\nabla H), a \rangle. \end{aligned} \quad (2.13)$$

PROOF. Since

$$\phi_i = e_i \langle x, a \rangle = \langle e_i, a \rangle, \quad (2.14)$$

then

$$\begin{aligned} \sum_j \phi_{ij} \omega_j &= d\phi_i + \sum_j \phi_j \omega_{ji} = \langle de_i, a \rangle - \sum_j \langle e_j, a \rangle \omega_{ij} \\ &= \left\langle \sum_j \omega_{ij} e_j - \sum_j h_{ij} \omega_j e_{n+1} - c \omega_i x, a \right\rangle - \sum_j \langle e_j, a \rangle \omega_{ij} \\ &= -c\phi \sum_j \delta_{ij} \omega_j - H^{-1} \psi \sum_j h_{ij} \omega_j. \end{aligned}$$

Thus

$$\phi_{ij} = -c\phi \delta_{ij} - H^{-1} \psi h_{ij}. \quad (2.15)$$

With that in mind we calculate

$$\begin{aligned} L_r \phi &= \sum_{i,j} T_{ij}^r \phi_{ij} = -c\phi \sum_{i,j} T_{ij}^r \delta_{ij} - H^{-1} \psi \sum_{i,j} T_{ij}^r h_{ij} \\ &= -c\phi \operatorname{tr}(P_r) - H^{-1} \psi \operatorname{tr}(A \circ P_r) = -c(n-r)S_r \phi - (r+1)S_{r+1}H^{-1}\psi. \end{aligned}$$

Next we let $\eta = \langle e_{n+1}, a \rangle$

$$\eta_i = e_i \langle e_{n+1}, a \rangle = \langle de_{n+1}(e_i), a \rangle = - \sum_j h_{ij} \langle e_j, a \rangle, \quad (2.16)$$

thus we can make the suggestive calculation

$$\begin{aligned} \sum_j \eta_{ij} \omega_j &= d\eta_i + \sum_j \eta_j \omega_{ji} = - \sum_j dh_{ij} \langle e_j, a \rangle - \sum_j h_{ij} \langle de_j, a \rangle - \sum_j \eta_j \omega_{ij} \\ &= - \sum_{j,k} (h_{ijk} \omega_k - h_{kj} \omega_{ki} - h_{ik} \omega_{kj}) \langle e_j, a \rangle \\ &\quad - \left\langle \sum_{j,k} h_{ij} \omega_{jk} e_k - \sum_{j,k} h_{ij} h_{jk} \omega_k e_{n+1} - c \sum_j h_{ij} \omega_j x, a \right\rangle + \sum_{j,k} h_{jk} \langle e_k, a \rangle \omega_{ij} \\ &= - \sum_{j,k} h_{ijk} \omega_j \langle e_k, a \rangle + c\phi \sum_j h_{ij} \omega_j + \eta \sum_{j,k} h_{ik} h_{jk} \omega_j. \end{aligned}$$

This shows that

$$\eta_{ij} = -h_{ijk} \langle e_k, a \rangle + c\phi h_{ij} + \eta \sum_k h_{ik} h_{jk}. \quad (2.17)$$

With this, then we have

$$\begin{aligned} L_r \eta &= \sum_{i,j} T_{ij}^r \eta_{ij} = - \sum_{i,j,k} T_{ij}^r h_{ijk} \langle e_k, a \rangle + c\phi \sum_{i,j} T_{ij}^r h_{ij} + \eta \sum_{i,j,k} T_{ij}^r h_{ik} h_{jk} \\ &= - \sum_{i,j,k} T_{ij}^r h_{ijk} \langle e_k, a \rangle + c\phi \operatorname{tr}(A \circ P_r) + \eta \operatorname{tr}(A^2 \circ P_r) \\ &= - \langle \nabla S_{r+1}, a \rangle + c\phi(r+1)S_{r+1} + \eta(S_1 S_{r+1} - (r+2)S_{r+2}). \end{aligned}$$

Thus

$$\begin{aligned} L_r \psi &= L_r(H\eta) = HL_r\eta + 2\langle P_r(\nabla H), \nabla \eta \rangle + \eta L_r H \\ &= H(-\langle \nabla S_{r+1}, a \rangle + c\phi(r+1)S_{r+1} + \eta(S_1 S_{r+1} - (r+2)S_{r+2})) \\ &\quad + 2\langle P_r(\nabla H), -A(a^\top) \rangle + H^{-1} L_r H \psi \\ &= c(r+1)S_{r+1} H \phi + (H^{-1} L_r H + S_1 S_{r+1} - (r+2)S_{r+2}) \psi \\ &\quad - 2\langle A \circ P_r(\nabla H), a \rangle - H \langle \nabla S_{r+1}, a \rangle. \end{aligned} \quad \square$$

3. Proof of the main theorems

In this section, using Proposition 2.3, we will prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 in a union form.

Since $\phi = \lambda\psi$, if $H = 0$ or $\lambda = 0$, then $\phi = 0$ and M^n is a totally geodesic submanifold in $\overline{M}_1^{n+1}(c)$. In the following, we suppose $\lambda H \neq 0$. From Proposition 2.3, we obtain

$$\begin{aligned} L_r\phi &= -c(n-r)S_r\phi - (r+1)S_{r+1}H^{-1}\psi \\ &= -[c\lambda(n-r)S_r + (r+1)S_{r+1}H^{-1}]\psi \\ &= \lambda\{c(r+1)S_{r+1}H\phi + (H^{-1}L_rH + S_1S_{r+1} - (r+2)S_{r+2})\psi \\ &\quad - H\langle\nabla S_{r+1}, a\rangle - 2\langle A \circ P_r(\nabla H), a\rangle\} \\ &= -H\langle\nabla S_{r+1}, a\rangle\lambda + [c(r+1)S_{r+1}H\lambda^2 + (S_1S_{r+1} - (r+2)S_{r+2})\lambda]\psi. \end{aligned}$$

This imply

$$\begin{aligned} \{c(r+1)S_{r+1}H\lambda^2 + [S_1S_{r+1} - (r+2)S_{r+2} + c(n-r)S_r]\lambda + (r+1)S_{r+1}H^{-1}\}\psi \\ = \lambda\langle\nabla S_{r+1}, a\rangle H. \end{aligned}$$

If $\psi = 0$, then $\phi = 0$ and M^n is a totally geodesic submanifold in $\overline{M}_1^{n+1}(c)$. There is nothing to prove. Therefore we can assume $\psi \neq 0$, then

$$\begin{aligned} c(r+1)S_{r+1}H\lambda^2 + [S_1S_{r+1} - (r+2)S_{r+2} + c(n-r)S_r]\lambda + (r+1)S_{r+1}H^{-1} \\ = \lambda\psi^{-1}\langle\nabla S_{r+1}, a\rangle H. \end{aligned} \quad (3.1)$$

Taking $r = 0$, and since H is constant, then

$$cnH^2\lambda^2 + (S_1^2 - 2S_2 + cn)\lambda + n = 0.$$

Thus

$$S_2 = \frac{n}{2\lambda}[cH^2\lambda^2 + (nH^2 + c)\lambda + 1]$$

is constant.

Using (3.1), by inductive method, we show that S_r is constant for every $1 \leq r \leq n$. This means that M^n is a complete spacelike isoparametric hypersurface of $\overline{M}_1^{n+1}(c)$. According to Theorem 1 and Theorem 2 in [7] or by the congruence theorem of ABE, KOIKE and YAMAGUCHI [1], we conclude that M^n is either a totally umbilical hypersurface or a hyperbolic cylinder, i.e. M^n is one of the following hypersurfaces, up to rigid motions:

- (i) $\mathbb{R}^n$, $\mathbb{H}^n(\sinh t)$, $\mathbb{H}^k(\sinh t) \times \mathbb{R}^{n-k}$ in $\mathbb{L}^{n+1}$;
- (ii) $\mathbb{R}^n$, $\mathbb{S}^n(\cosh t)$, $\mathbb{H}^n(\sinh t)$, $\mathbb{H}^k(\sinh t) \times \mathbb{S}^{n-k}(\cosh t)$ in $\mathbb{S}_1^{n+1}$;
- (iii) $\mathbb{H}^n(\sin t)$, $\mathbb{H}^k(\cos t) \times \mathbb{H}^{n-k}(\sin t)$ in $\mathbb{H}_1^{n+1}$.

4. Some examples

In this section, we give some examples for hypersurfaces appearing in the main theorems and verify further our results.

Example 4.1. let $f : \mathbb{L}^{n+1} \rightarrow \mathbb{R}$ be a real function defined by

$$f(x_0, \dots, x_n) = \delta_1(-x_0^2 + x_2^2 + \dots + x_k^2) + x_{k+1}^2 + \delta_2(x_{k+2}^2 + \dots + x_n^2), \quad (4.1)$$

where $\delta_1, \delta_2 \in \{0, 1\}$ and $\delta_1^2 + \delta_2^2 \neq 0$. Taking $r > 0$ and $\epsilon = \pm 1$, the set $M^n = f^{-1}(\epsilon r^2)$ is a hypersurface of $\mathbb{L}^{n+1}$ provided $(\delta_1, \delta_2, \epsilon) \neq (0, 1, -1)$.

A straightforward computation shows the unit normal vector field is written as

$$e_{n+1} = \frac{1}{r}(\delta_1 x_0, \dots, \delta_1 x_k, x_{k+1}, \delta_2 x_{k+2}, \dots, \delta_2 x_n). \quad (4.2)$$

Moreover, the principal curvatures of M^n are given by

$$\begin{aligned} \lambda_1 &= \dots = \lambda_k = -\frac{\delta_1}{r}, \\ \lambda_{k+1} &= \dots = \lambda_n = -\frac{\delta_2}{r}, \end{aligned}$$

and we also have that

$$H = -\frac{k\delta_1 + (n-k)\delta_2}{nr}.$$

Thus

- (1) when $(\delta_1, \delta_2, \epsilon) = (0, 1, 1)$, $M^n = \mathbb{L}^k \times \mathbb{S}^{n-k}(r)$, and if we take $a = (0, \dots, 0, a_{k+1}, \dots, a_{n+1}) \in \mathbb{R}_1^{n+1}$, then we have $\phi = -\frac{nr^2}{n-k}\psi$;
- (2) when $(\delta_1, \delta_2, \epsilon) = (1, 0, -1)$, $M^n = \mathbb{H}^k(r) \times \mathbb{R}^{n-k}$, and if we take $a = (a_1, \dots, a_{k+1}, 0, \dots, 0) \in \mathbb{R}_1^{n+1}$, then we have $\phi = -\frac{nr^2}{k}\psi$;
- (3) when $(\delta_1, \delta_2, \epsilon) = (1, 1, -1)$, $M^n = \mathbb{H}^n(r)$, and if we take

$$a = (a_1, \dots, a_{n+1}) \in \mathbb{R}_1^{n+1},$$

then we have $\phi = -r^2\psi$.

Example 4.2. Given any integer $k \in \{1, \dots, n-1\}$ and any real number $r \in (0, 1)$, let

$$\begin{aligned} M^n &= \{(x, y) \in \mathbb{L}^{k+1} \times \mathbb{R}^{n-k+1} : \|x\|^2 = -r^2 \text{ and } \|y\|^2 = 1 + r^2\} \\ &= \mathbb{H}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1+r^2}). \end{aligned}$$

It is not difficult to see that for any $(x, y) \in M^n$ one gets

$$T_{(x,y)}M^n = \{(v, w) \in \mathbb{L}^{k+1} \times \mathbb{R}^{n-k+1} : \langle x, v \rangle = 0 \text{ and } \langle y, w \rangle = 0\}.$$

Therefore, the unit timelike normal vector field e_{n+1} is given by

$$e_{n+1}(x, y) = \left(\frac{\sqrt{1+r^2}}{r}x, -\frac{r}{\sqrt{1+r^2}}y \right).$$

Moreover, the principal curvatures of M^n are given by

$$\begin{aligned} \lambda_1 = \dots = \lambda_k &= -\frac{\sqrt{1+r^2}}{r}, \\ \lambda_{k+1} = \dots = \lambda_n &= \frac{r}{\sqrt{1+r^2}}, \end{aligned}$$

and we also have that

$$H = \frac{(n-2k)r^2 - k}{nr\sqrt{1+r^2}}.$$

Thus, when $H \neq 0$, i.e. $n \leq 2k$, or $n > 2k$ and $r \neq \sqrt{\frac{k}{n-2k}}$,

(i) if we take $a = (a_1, \dots, a_{k+1}, 0, \dots, 0) \in \mathbb{R}_1^{n+2}$, then we have that

$$\phi = \frac{nr^2}{(n-2k)r^2 - k}\psi; \quad (4.3)$$

(ii) if we take $a = (0, \dots, 0, a_{k+2}, \dots, a_{n+2}) \in \mathbb{R}_1^{n+2}$, then we have that

$$\phi = -\frac{n(1+r^2)}{(n-2k)r^2 - k}\psi. \quad (4.4)$$

Similarly, we have the following example.

Example 4.3. Given any integer $k \in \{1, \dots, n-1\}$ and any real number $r \in (0, 1)$, let

$$\begin{aligned} M^n &= \{(x, y) \in \mathbb{L}^{k+1} \times \mathbb{L}^{n-k+1} : \|x\|^2 = -r^2 \text{ and } \|y\|^2 = -1+r^2\} \\ &= \mathbb{H}^k(r) \times \mathbb{H}^{n-k}(\sqrt{1-r^2}). \end{aligned}$$

It is not difficult to see that for any $(x, y) \in M^n$ one gets

$$T_{(x,y)}M^n = \{(v, w) \in \mathbb{L}^{k+1} \times \mathbb{L}^{n-k+1} : \langle x, v \rangle = 0 \text{ and } \langle y, w \rangle = 0\}.$$

Therefore, the unit timelike normal vector field e_{n+1} is given by

$$e_{n+1}(x, y) = \left(\frac{\sqrt{1-r^2}}{r}x, -\frac{r}{\sqrt{1-r^2}}y \right).$$

Moreover, the principal curvatures of M^n are given by

$$\begin{aligned}\lambda_1 &= \cdots = \lambda_k = -\frac{\sqrt{1-r^2}}{r}, \\ \lambda_{k+1} &= \cdots = \lambda_n = \frac{r}{\sqrt{1-r^2}},\end{aligned}$$

and we also have that

$$H = \frac{nr^2 - k}{nr\sqrt{1-r^2}}.$$

Thus, when $H \neq 0$, i.e. $r \neq \sqrt{\frac{k}{n}}$,

(i) if we take $a = (a_1, \dots, a_{k+1}, 0, \dots, 0) \in \mathbb{R}_1^{n+2}$, then we have that

$$\phi = \frac{nr^2}{nr^2 - k}\psi; \quad (4.5)$$

(ii) if we take $a = (0, \dots, 0, a_{k+2}, \dots, a_{n+2}) \in \mathbb{R}_1^{n+2}$, then we have that

$$\phi = -\frac{n(1-r^2)}{nr^2 - k}\psi. \quad (4.6)$$

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References

- [1] N. ABE, N. KOIKE and S. YAMAGUCHI, Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form, *Yokohama Math. J.* **35** (1987), 123–136.
- [2] K. AKUTAGAWA, On spacelike hypersurfaces with constant mean curvature in the de Sitter space, *Math. Z.* **196** (1987), 13–19.
- [3] L. J. ALÍAS, A. BRASIL and O. PERDOMO, A characterization of quadric constant mean curvature hypersurfaces of spheres, *J. Geom. Anal.* **18** (2008), 687–703.
- [4] J. L. BARBOSA and A. G. COLARES, Stability of hypersurfaces with constant r -mean curvature, *Ann. Global Anal. Geom.* **15** (1997), 277–297.

- [5] A. CAMINHA, On spacelike hypersurfaces of constant sectional curvature Lorentz manifolds, *J. Geom. Phys.* **56** (2006), 1144–1174.
- [6] A. J. GODDARD, Some remarks on the existence of spacelike hypersurfaces of constant mean curvature, *Math. Proc. Cambridge Phil. Soc.* **82** (1977), 489–495.
- [7] Z. Q. LI and X. H. XIE, Spacelike isoparametric hypersurfaces in Lorentzian space forms, *Front. Math. China* **1** (2006), 130–137.
- [8] S. MONTIEL, An integral inequality for compact spacelike hypersurfaces in the de Sitter space and applications to the case of constant mean curvature, *Indiana Univ. Math. J.* **37** (1988), 909–917.
- [9] B. O'NEILL, Semi-Riemannian Geometry with Applications to Relativity, *Academic Press, New York*, 1983.
- [10] J. RAMANATHAN, Complete spacelike hypersurfaces of constant mean curvature in the de Sitter space, *Indiana Univ. Math. J.* **36** (1987), 349–359.
- [11] R. REILLY, Variational properties of functions of the mean curvatures for hypersurfaces in space forms, *J. Diff. Geom.* **8** (1973), 465–477.
- [12] H. ROSENBERG, Hypersurfaces of constant in space forms, *Bull. Sci. Math.* **117** (1993), 217–239.

BIAOGUI YANG
 SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE
 FUJIAN NORMAL UNIVERSITY
 FUZHOU, 350108
 P.R. CHINA

E-mail: bgyang@163.com

XIMIN LIU
 DEPARTMENT OF MATHEMATICS
 SOUTH CHINA UNIVERSITY OF TECHNOLOGY
 GUANGZHOU, 510641
 P.R. CHINA

E-mail: ximinliu@scut.edu.cn

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