

## On the Diophantine equation $ax^2 - by^2 = c$

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### 1. Introduction

In the paper [3] there has been given a matrix method for the study of some properties of the solutions in integers  $x, y$  of the Diophantine equation

$$(1.1) \quad ax^2 - by^2 = c.$$

The study of (1.1) was begun by Lagrange and continued by several authors, see C. U. JENSEN [5], P. KAPLAN [6], J. C. LAGARIAS [7], H. LIENEN [8], T. NAGELL [9], [10], [11] and many others.

From Theorems 2 and 3 of our paper [3] we get the following solvability criteria in integers  $x, y$  for (1.1) when  $c = 1$  or  $c = 2$ :

**Criterion 1.** *Let  $a > 1, b$  be positive integers such that  $(a, b) = 1$  and  $d = ab$  is not a square of a natural number. Moreover let  $\langle u_0, v_0 \rangle$  denote the least positive integer solution of Pell's equation*

$$(1.2) \quad u^2 - dv^2 = 1.$$

*Then equation (1.1) with  $c = 1$  has a solution in positive integers  $x, y$  iff*

$$(1.3) \quad 2a \mid u_0 + 1 \quad \text{and} \quad 2b \mid u_0 - 1.$$

We note that this result has been proved also by W. GÓRZNY [2], but in another way.

**Criterion 2.** *Let  $a, b$  be positive integers such that  $(a, b) = (a, 2) = (b, 2) = 1$  and  $d = ab$  is not a square of a natural number and let  $\langle u_0, v_0 \rangle$  denote the least positive integer solution of (1.2). Then the equation (1.1) with  $c = 2$  has a solution in positive integers  $x, y$  iff*

$$(1.4) \quad a \mid u_0 + 1 \quad \text{and} \quad b \mid v_0 - 1.$$

By using an idea contained in [3] we give in this paper a solvability criterion for (1.1) when  $c > 2$ . Namely, we reduce the problem of the solvability of (1.1) in integers  $x, y$  to the investigation of the integer solutions of the following Diophantine equation

$$u^2 - abv^2 = c^2.$$

We conclude the introduction by expressing our thanks to referee for the remarks incorporated in the present version of the paper.

## 2. Notations and Lemmas

Let  $d = ab$  and suppose that  $(a, b) = (b, c) = (c, a) = 1$ . In a similar way as in [3] we introduce the matrix

$$(2.1) \quad S = \begin{bmatrix} \sqrt{a}x & \frac{d}{\sqrt{a}}y \\ \frac{1}{\sqrt{a}}y & \sqrt{a}x \end{bmatrix}$$

associated with the Diophantine equation (1.1). The matrix  $S$  will be called a solvable matrix if  $x, y$  are integers such that  $(x, c) = 1$  and

$$(2.2) \quad \det S = ax^2 - by^2 = c.$$

In the case  $a = c = 1$  the solvable matrix  $S$  will be called Pell's solvable matrix. Hence

$$(2.3) \quad P = \begin{bmatrix} u & dv \\ v & u \end{bmatrix}$$

and

$$(2.4) \quad \det P = u^2 - dv^2 = 1.$$

Let  $\langle u_0, v_0 \rangle$  denote the least positive integer solution of (2.4), such a solution we will be called a primitive Pell's solution. Now we can define the primitive solution of (1.1).

The solution  $\langle x_0, y_0 \rangle$  of (1.1) will be called primitive solution, if  $ax_0^2 - by_0^2 = c$  and  $x_0 \leq x$  for any positive integer  $x$  satisfying (1.1). Let  $S_0, P_0$  be matrices associated with a primitive solution of (1.1) and a primitive Pell's solution, respectively.

By (2.1) and (2.3) we have

$$(2.5) \quad S_0 = \begin{bmatrix} \sqrt{a}x_0 & \frac{d}{\sqrt{a}}y_0 \\ \frac{1}{\sqrt{a}}y_0 & \sqrt{a}x_0 \end{bmatrix}$$

$$(2.6) \quad P_0 = \begin{bmatrix} u_0 & dv_0 \\ v_0 & u_0 \end{bmatrix}$$

From (2.5) and (2.6) we obtain

$$(2.7) \quad S_1 = S_0P_0 = P_0S_0 = \begin{bmatrix} \sqrt{a}x_1 & \frac{d}{\sqrt{a}}y_1 \\ \frac{1}{\sqrt{a}}y_1 & \sqrt{a}x_1 \end{bmatrix}$$

where

$$(2.8) \quad x_1 = x_0u_0 + by_0v_0, \quad y_1 = y_0u_0 + ax_0v_0.$$

From (2.7) and Cauchy's Theorem on the product of determinants we get

$$(2.9) \quad \det S_1 = \det S_0 \cdot \det P_0 = \det P_0 \cdot \det S_0 = ax_1^2 - by_1^2 = c,$$

because  $\det S_0 = c$  and  $\det P_0 = 1$ . From (2.9) it follows that the numbers  $x_1, y_1$  given in (2.8) are solutions of (1.1).

Now we define the singular solution of (1.1).

*Definition 1.* The solution  $\langle u, v \rangle$  of (1.1) will be called a singular solution of (1.1) if

$$(2.10) \quad x_0 < u < x_1$$

where  $x_1$  is given by (2.8) and  $\langle x_0, y_0 \rangle$  is the primitive solution of (1.1).

We can prove the following

**Lemma 1.** *Let  $c > 2$  not be a square of a natural number and suppose that equation (1.1) has a primitive solution in positive integers  $x_0, y_0$  such that  $(x_0, c) = 1$ . Then there exists a singular solution  $\langle u, v \rangle$  of (1.1).*

PROOF. Let  $d = ab$  and

$$(2.11) \quad u = x_0u_0 - by_0v_0, \quad v = y_0u_0 - ax_0v_0.$$

It is easy to see that by (2.6) we have

$$(2.12) \quad P_0^{-1} = \begin{bmatrix} u_0 & -dv_0 \\ -v_0 & u_0 \end{bmatrix}$$

and  $\det P^{-1} = 1$ , thus by (2.7) and (2.8) it follows that the  $\langle u, |v| \rangle$  given by (2.11) is a solution of (1.1).

Since  $u_0^2 - abv_0^2 = 1$  then  $u_0 > \sqrt{ab}v_0$  and

$$u = x_0u_0 - by_0v_0 > \sqrt{ab}v_0x_0 - by_0v_0 = v_0\sqrt{b}(\sqrt{a}x_0 - \sqrt{b}y_0).$$

On the other hand from the  $ax_0^2 - by_0^2 = c$ ,  $c > 2$  follows that  $\sqrt{a}x_0 - \sqrt{b}y_0 > 0$  and we obtain  $u > 0$ . Then from (2.11) and (2.8) we have

$$(2.13) \quad 0 < u < x_1.$$

We remark that  $v \neq 0$ . Indeed, suppose that  $v = 0$  then by (1.1) we have  $au^2 = c$ . Since  $(a, c) = 1$  thus  $a = 1$  and  $u^2 = c$  contradicting our assumption that  $c$  is not a square of a positive integer. Since  $\langle x_0, y_0 \rangle$  is a primitive solution of (1.1), by (2.13) and the definition of a primitive solution we obtain

$$(2.14) \quad x_0 \leq u < x_1.$$

Suppose that in (2.14) we have  $u = x_0$ . Then by (2.11) it follows that

$$(2.15) \quad x_0(u_0 - 1) = by_0v_0.$$

On the other hand, since  $\langle u, |v| \rangle$  is a solution of  $ax^2 - by^2 = c$  by (2.11) we have

$$ax^2 - b(ax_0v_0 - y_0u_0)^2 = c.$$

From the last equality we obtain

$$(2.16) \quad ax_0^2 - ax_0^2(abv_0^2) + 2au_0x_0(by_0v_0) - u_0^2(by_0^2) = c.$$

From the assumptions we have  $ax_0^2 - by_0^2 = c$  and  $u_0^2 - abv_0^2 = 1$  and therefore  $by_0^2 = ax_0^2 - c$  and  $abv_0^2 = u_0^2 - 1$ .

Substituting the last equality and (2.15) in to (2.16) we obtain

$$(2.17) \quad ax_0^2 - ax_0^2(u_0^2 - 1) + 2au_0x_0^2(u_0 - 1) - u_0^2(ax_0^2 - c) = c.$$

From (2.17) we get

$$2ax_0^2 - 2ax_0^2u_0 = c(1 - u_0^2)$$

and consequently

$$2ax_0^2(1 - u_0) = c(1 - u_0)(1 + u_0).$$

Since  $u_0 \neq 1$ , the last equality implies

$$(2.18) \quad 2ax_0^2 = c(u_0 + 1).$$

Since  $(a, c) = 1$  and  $(x_0, c) = 1$ , by (2.18) we get  $c \mid 2$ , thus  $c \leq 2$ , and this is impossible, because  $c > 2$ . Therefore  $u \neq x_0$  and by (2.14) and the Definition 1 our Lemma follows.

**Lemma 2.** *Let  $S_1, S_2$  be the matrices associated with the solutions  $\langle x_1, y_1 \rangle$  and  $\langle x_2, y_2 \rangle$  of (1.1). Then the matrix  $R = S_1 S_2 = S_2 S_1$  has the form*

$$R = \begin{bmatrix} x_3 & dy_3 \\ y_3 & x_3 \end{bmatrix}$$

where

$$x_3 = ax_1x_2 + by_1y_2, \quad y_3 = x_1y_2 + y_1x_2$$

and  $R$  is associated with the solution  $\langle x_3, y_3 \rangle$  of the Diophantine equation

$$u^2 - dv^2 = c^2$$

where  $d = ab$ .

PROOF. We have

$$(2.19) \quad R = S_1 S_2 = S_2 S_1 = \begin{bmatrix} \sqrt{a}x_1 & \frac{d}{\sqrt{a}}y_1 \\ \frac{1}{\sqrt{a}}y_1 & \sqrt{a}x_1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{a}x_2 & \frac{d}{\sqrt{a}}y_2 \\ \frac{1}{\sqrt{a}}y_2 & \sqrt{a}x_2 \end{bmatrix}.$$

From (2.19) we get

$$(2.20) \quad R = \begin{bmatrix} ax_1x_2 + by_1y_2 & d(x_1y_2 + y_1x_2) \\ x_1y_2 + y_1x_2 & ax_1x_2 + by_1y_2 \end{bmatrix}.$$

Putting in (2.20)

$$(2.21) \quad x_3 = ax_1x_2 + by_1y_2, \quad y_3 = x_1y_2 + y_1x_2$$

we get

$$(2.22) \quad R = \begin{bmatrix} x_3 & dy_3 \\ y_3 & x_3 \end{bmatrix}.$$

From (2.19) and the assumptions of our Lemma we get  $\det S_1 = \det S_2 = c$  and therefore by Cauchy's theorem on the product of determinants we obtain

$$(2.23) \quad \det R = \det S_1 \cdot \det S_2 = c^2.$$

On the other hand by (2.22) it follows that  $\det R = x_3^2 - dy_3^2$  and therefore by (2.23) we get

$$x_3^2 - dy_3^2 = c^2, \quad \text{where } d = ab$$

and the proof is complete.

**Lemma 3.** *All positive integral solutions of the equation*

$$x^2 - dy^2 = z^2$$

are given by the formulas

$$x = (am^2 + bn^2)\varrho, \quad y = 2mn\varrho, \quad z = (am^2 - bn^2)\varrho$$

if  $d = ab$  is even, or

$$x = \frac{1}{2}(am^2 + bn^2)\varrho, \quad y = mn\varrho, \quad z = \frac{1}{2}(am^2 - bn^2)\varrho$$

if  $d = ab$  is odd and  $\varrho$  is any integer when  $m$  and  $n$  are odd, but  $\varrho$  is even when one of  $m$  and  $n$  is even and the other is odd. In all cases  $m, n$  are positive integers and relatively prime.

For the proof see [1], Th. 40, p. 41.

### 3. Result

In this part of our paper we prove the following

**Theorem.** *Let  $a, b$  and  $c > 2$  be positive integers such that  $(a, b) = (b, c) = (c, a) = 1$  and  $d = ab$  is not a square of an integer.*

*Then the equation*

$$(3.1) \quad ax^2 - by^2 = c$$

has a solution in positive integers  $x, y$  with  $(x, y) = 1$  iff there exists an integer solution  $\langle u, v \rangle$  of the equation

$$(3.2) \quad u^2 - dv^2 = c^2$$

PROOF. Suppose that the assumptions of our Theorem are fulfilled and let the equation (3.2) have an integer solution  $\langle u, v \rangle$ . By Lemma 3 it follows that all positive integer solutions of (3.2) are given by the formulae

$$(3.3) \quad u = (am^2 + bn^2)\varrho, \quad v = 2mn\varrho, \quad c = (am^2 - bn^2)\varrho$$

if  $d = ab$  is even, or

$$(3.4) \quad u = \frac{1}{2}(am^2 + bn^2)\varrho, \quad v = mn\varrho, \quad c = \frac{1}{2}(am^2 - bn^2)\varrho$$

if  $d = ab$  is odd, where  $\varrho$  is any integer when  $m$  and  $n$  are odd, but  $\varrho$  is even when one of  $m$  and  $n$  is even and the other is odd. In all cases  $(m, n) = 1$ .

Let  $d = ab$  be even. Then by (3.3) in the case  $\varrho = 1$  we obtain

$$u = am^2 + bn^2, \quad c = am^2 - bn^2$$

and consequently

$$\frac{u+c}{2} - \frac{u-c}{2} = am^2 - bn^2 = c,$$

so denote that the equation  $ax^2 - by^2 = c$ , has a solution in positive integers  $m, n$  such that  $(m, n) = 1$ .

Let  $d = ab$  be odd. Then by (3.4) in the case  $\varrho = 2\varrho_1$  we have

$$u = (am^2 + bn^2)\varrho_1, \quad c = (am^2 - bn^2)\varrho_1$$

where  $(m, n) = 1$  and  $m, n$  are different parity. Thus for  $\varrho_1 = 1$  we obtain

$$\frac{u+c}{2} - \frac{u-c}{2} = am^2 - bn^2 = c,$$

and we get a solution in positive integers  $m, n$  of the equation  $ax^2 - by^2 = c$ . Now we can assume that the equation (3.1) has a primitive solution  $\langle x_0, y_0 \rangle$  such that  $(x_0, y_0) = 1$  and  $(x_0, c) = 1$ . Then there exists a solution  $\langle x_1, y_1 \rangle$  given by (2.8). Since  $(x_0, c) = 1$  then by Lemma 1 we obtain that there exists a singular solution  $\langle u, v \rangle$  of (3.1).

By Lemma 2 it follows that there exists a solution in positive integers of the equation (3.2). The proof is complete.

#### 4. Application

Let  $K = Q(\sqrt{d})$ ,  $d > 0$  be a given quadratic number field and let  $h$  denote the class-number of this field. Then from well-known results of C. S. HERZ [4], (Cf. [12], p. 483) it follows that if  $h = 1$  then

$$(4.1) \quad d = p, \quad 2q, \quad qr$$

where  $p$  is a prime and  $q \equiv r \equiv 3 \pmod{4}$  are primes

From this results follows that for the investigation of the famous Gauss problem concerning the existence of infinitely many real quadratic number fields with class-number  $h = 1$  it suffices to consider one of the cases given in (4.1). Consider the case  $d = p \equiv 3 \pmod{4}$ . Then if  $R_K$  is the ring of all integers of  $K = Q(\sqrt{p})$  and if  $\alpha \in R$  then for some rational integers  $x, y$  we have

$$(4.2) \quad \alpha = x + y\sqrt{p} \quad \text{and} \quad N(\alpha) = x^2 - py^2.$$

On the other hand it is well-known that if  $D_K$  is the discriminant of  $K$  then for every rational prime  $q$  we have

$$(4.3) \quad (q) = P^2, \quad N(P) = q \quad \text{if} \quad q \mid D_K$$

and if  $q \nmid D_K$  then

$$(4.4) \quad (q) = P_1 P_2, \quad P_1 \neq P_2, \quad N(P_1) = N(P_2) = q \text{ if } \left(\frac{D_K}{q}\right) = +1$$

$$(4.5) \quad (q) = P, \quad N(P) = q^2 \text{ if } \left(\frac{D_K}{q}\right) = -1$$

where  $P, P_1, P_2$  are prime ideals in  $R_K$  and  $\left(\frac{a}{b}\right)$  denotes the Legendre symbol. In the case  $d = p \equiv 3 \pmod{4}$  we have  $D = 4d = 4p$ . From (4.3) we have  $q = 2$  or  $p$  and if  $P = (\alpha)$  then  $N(P) = N((\alpha)) = |N(\alpha)|$  and conversely. By (4.2) we obtain that this condition is equivalent to the condition that the equation  $|x^2 - py^2| = 2$  or  $p$  has a solution in integers  $x, y$ . But it is easy to see that the equation  $|x^2 - py^2| = p$  has always the solution  $x = 0, y = \pm 1$  and it remains to investigate the equations

$$(4.6) \quad x^2 - py^2 = 2, \quad x^2 - py^2 = -2.$$

Let  $\langle u_0, v_0 \rangle$  be the primitive solutions of Pell's equation  $u^2 - pv^2 = 1$ , then we have  $(u_0 - 1)(u_0 + 1) = pv_0^2$  and we obtain

$$(4.7) \quad p \mid u_0 - 1 \quad \text{or} \quad p \mid u_0 + 1.$$

From (4.7) and Criterion 2 we get that one of the equations (4.6) has a solution in integers  $x, y$ . Therefore we can investigate the cases (4.4) and (4.5). Similarly as in the above case we obtain that if one of the equations

$$(4.8) \quad x^2 - py^2 = q, \quad x^2 - py^2 = -q.$$

has a solution in integers  $x, y$  for every odd prime  $q \neq p$  such that  $\left(\frac{D_K}{q}\right) = \left(\frac{p}{q}\right) = +1$  then every prime ideal  $P$  of  $R_K$  is principal and consequently any integer ideal is also principal and we get that in this case  $h = 1$ .

Applying our Theorem to (4.8) we get the following

**Corollary.** *Let  $K = Q(\sqrt{p})$ , where  $p \equiv 3 \pmod{4}$  is a prime. If the equation*

$$u^2 - pv^2 = q^2$$

*has an integer solution  $\langle u, v \rangle$  for every odd prime  $q \neq p$ , such that  $\left(\frac{q}{p}\right) = +1$ , then  $h = 1$ .*

## References

- [1] L. E. DICKSON, Introduction to the Theory of Numbers, *New York*, 1957.
- [2] W. GÓRZNY, On the equation  $D_1 x^2 - D_2 y^2 = 1$ , *Discuss. Math.* **4** (1981), 109–111.
- [3] A. GRELAK and A. GRZYTCZUK, Some remarks on matrices and Diophantine equation  $A^2 - By^2 = C$ , *Discuss. Math.* **10** (1990), 13–27.



- [4] C. S. HERZ, Construction of class fields, Seminar on Complex Multiplication, Lectures Notes in Math. 21, *Springer-Verlag*, 1966.
- [5] C. U. JENSEN, On the solvability of a certain class of non-Pellian equations, *Math. Scand.* **10** (1962), 71–84.
- [6] P. KAPLAN, A propos des équations antipelliennes, *Enseign. Math.* (1983), 323–328.
- [7] J. C. LAGARIAS, On the computational complexity of determining the solvability or unsolvability of the equation  $X^2 - DY^2 = -1$ , *Trans. Amer. Math. Soc.* **260** (1980), 485–508.
- [8] H. LIENEN, The quadratic form  $x^2 - 2py^2$ , *J. Number Theory* (1978), 10–15.
- [9] T. NAGELL, On a special class of Diophantine equations of the second degree, *Arkiv. Math.* (1954), 51–65.
- [10] T. NAGELL, Contributions to the theory of a category of Diophantine equations of the second degree with two unknowns, *Nova Acta Soc. Sci. Uppsala* (1955), 1–38.
- [11] T. NAGELL, Sur la solubilité en nombres entiers des équations du second degré à deux indéterminées, *Acta Arith.* (1971), 105–114.
- [12] W. NARKIEWICZ, Elementary and Analytic Theory of Algebraic Numbers, *PWN, Warszawa*, 1990.

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