

Some applications of Bochner formula to submanifolds of a unit sphere

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Abstract. The aims of this paper are to give estimates for the eigenvalue of the Laplacian on submanifolds in a unit sphere $S^{n+p}(1)$ and to give some new characterizations of spheres in $S^{n+p}(1)$.

1. Introduction

Let M be an n -dimensional closed oriented manifold immersed in a unit sphere $S^{n+p}(1)$, where p is the codimension. One knows that minimal submanifolds and submanifolds with constant scalar curvature $n(n-1)$ in $S^{n+p}(1)$ were studied by many others, they are very interesting topics since they come from the variational problems in differential geometry.

A well-known result about minimal submanifolds in $S^{n+p}(1)$ is due to SIMONS [11], if M is a minimal hypersurface in $S^{n+1}(1)$ and

$$S < n,$$

then $S \equiv 0$ and M is a unit sphere $S^n(1)$. Many years later, LEUNG [8] has proved

Theorem 1.1 ([8]). *Let M be a closed minimal submanifold in $S^{n+p}(1)$. Let f be an eigenfunction of the Laplacian on M corresponding to a non-zero eigenvalue λ , then*

$$\int_M (\lambda + S - n) |\nabla f|^2 dv \geq 0, \quad (1.1)$$

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equality holds if and only if either M is totally geodesic and λ is a non-zero eigenvalue or $n = 2$ and $n + p = 2m$ and M is isometric to $S^2\left(\sqrt{\frac{m(m+1)}{2}}\right)$ and λ is a non-zero eigenvalue, where S denotes the squared norm of the second fundamental form of M .

On the other hand, by the study of Cheng–Yau’s self-adjoint operator and some new estimates, LI [6] has obtained that if M is an n -dimensional closed hypersurface with constant scalar curvature $n(n-1)$ in $S^{n+1}(1)$ and

$$S < \frac{n^2}{2(n-2)},$$

then $S \equiv 0$ and M is a unit sphere $S^n(1)$.

By comparing the results of Simons, Li and Leung, it is nature to ask the following:

Question 1. Does there exist the similar integral inequality with (1.1) for the eigenvalue of the Laplacian on submanifolds with constant scalar curvature $n(n-1)$ in $S^{n+p}(1)$?

In this paper, we answer the above question and prove the following integral inequality:

Theorem 1.2. *Let M be a closed submanifold in $S^{n+p}(1)$ with constant scalar curvature $n(n-1)$. Let f be an eigenfunction of the Laplacian on M corresponding to a non-zero eigenvalue λ , then*

$$\int_M \left(\lambda + \frac{2(n-2)}{n} S - n \right) |\nabla f|^2 dv \geq 0, \quad (1.2)$$

equality holds if and only if M is a geodesic sphere $S^n(1)$.

Finally, we do not assume that M has constant scalar curvature $n(n-1)$ and obtain some other integral inequality which also involves the non-zero eigenvalue of the Laplacian. In fact, we prove

Theorem 1.3. *Let M be a closed submanifold in $S^{n+p}(1)$. Let f be an eigenfunction of the Laplacian on M corresponding to a non-zero eigenvalue λ , then*

$$\int_M \left(\lambda + \frac{n^2}{4(n-1)} \rho^2 - n \right) |\nabla f|^2 dv \geq 0, \quad (1.3)$$

equality holds if and only if M is a sphere, where $\rho^2 := S - nH^2$ is a non-negative function which vanishes exactly at the umbilical points of M , H is the mean curvature of M .

Remark 1. We do not assume that ρ^2 is constant in the above Theorem.

2. Preliminaries

In order to prove our results, we introduce some preliminaries and notations in this section. Let $S^{n+p}(1)$ be a sphere of constant sectional curvature one. Let M be an n -dimensional closed hypersurface in $S^{n+p}(1)$. For any $p \in M$, we choose a local orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in $S^{n+p}(1)$ around p , such that $e_1, \dots, e_n$ are tangent to M and $e_{n+1}, \dots, e_{n+p}$ are normal to M . Take the corresponding dual coframe $\omega_1, \dots, \omega_{n+p}$. In this paper, we make the following convention on the range of indices:

$$1 \leq A, B, C \leq n+p; \quad 1 \leq i, j, k \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

The structure equations of $S^{n+p}(1)$ are (see [7])

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} = \omega_{BA}, \quad (2.1)$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B. \quad (2.2)$$

Restricted to M , we have $\omega_\alpha = 0$, thus

$$0 = d\omega_\alpha = \sum_i \omega_{\alpha i} \wedge \omega_i, \quad (2.3)$$

from Cartan's lemma, we obtain

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \quad (2.4)$$

We then get the structure equations of M as follows:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = \omega_{ji}, \quad (2.5)$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l, \quad (2.6)$$

where R_{ijkl} is the component of the curvature tensor of induced metric on M . If h_{ij}^α denotes the component of the second fundamental form of M , S denotes the squared norm of the second fundamental form, $\vec{H}$ denotes the mean curvature vector and H denotes the mean curvature of M , then we have

$$S = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2, \quad \vec{H} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{n} \sum_k h_{kk}^\alpha, \quad H = |\vec{H}|. \quad (2.7)$$

M is called minimal if $\vec{H} \equiv 0$, i.e., $\sum_k h_{kk}^\alpha = 0$ for all α .
The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha} (h_{ik}^{\alpha}h_{jl}^{\alpha} - h_{il}^{\alpha}h_{jk}^{\alpha}), \quad (2.8)$$

$$R_{ik} = (n-1)\delta_{ik} + n \sum_{\alpha} H^{\alpha} h_{ik}^{\alpha} - \sum_{j,\alpha} h_{ij}^{\alpha} h_{jk}^{\alpha}, \quad (2.9)$$

$$R = n(n-1) + n^2 H^2 - S, \quad (2.10)$$

where R is the scalar curvature.

The Codazzi equations are

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha}, \quad (2.11)$$

where the covariant derivative of h_{ij}^{α} is defined by

$$\sum_k h_{ijk}^{\alpha} \omega_k = dh_{ij}^{\alpha} + \sum_k h_{kj}^{\alpha} \omega_{ki} + \sum_k h_{ik}^{\alpha} \omega_{kj} + \sum_{\beta} h_{ij}^{\beta} \omega_{\beta\alpha}. \quad (2.12)$$

3. Proofs of theorems

Firstly, we prove the following Lemma.

Lemma 1 (Cf. Theorem B of [8]). *Let M be a closed submanifold in $S^{n+p}(1)$. Let f be an eigenfunction on M corresponding to a non-zero eigenvalue λ , then*

$$\lambda \int_M |\nabla f|^2 dv \geq \frac{n}{n-1} \int_M \text{Ric}(\nabla f, \nabla f). \quad (3.1)$$

PROOF OF LEMMA 1. For any smooth function $u : M \rightarrow R$, one has from Bochner formula that

$$\frac{1}{2} \Delta(|\nabla u|^2) = \text{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla(\Delta u) \rangle + |\text{Hess } u|^2. \quad (3.2)$$

Since M is closed, one deduces from (3.2) and $\Delta f + \lambda f = 0$ that

$$\int_M \text{Ric}(\nabla f, \nabla f) dv - \lambda \int_M |\nabla f|^2 dv + \int_M |\text{Hess } f|^2 dv = 0. \quad (3.3)$$

Let I denotes the identity operator on the tangent bundle TM of M . By a direct calculation, we obtain for any $t \in R$,

$$\int_M |\text{Hess } f - t f I|^2 dv = \int_M (|\text{Hess } f|^2 - 2t f \Delta f + n f^2 t^2) dv$$

$$\begin{aligned}
&= \int_M |\text{Hess } f|^2 dv + \left(2t + \frac{n}{\lambda} t^2\right) \int_M |\nabla f|^2 dv \\
&\geq \int_M |\text{Hess } f|^2 dv + \left(2 \times \left(-\frac{\lambda}{n}\right) + \frac{n}{\lambda} \left(-\frac{\lambda}{n}\right)^2\right) \int_M |\nabla f|^2 dv \\
&= \int_M |\text{Hess } f|^2 dv - \frac{\lambda}{n} \int_M |\nabla f|^2 dv,
\end{aligned} \tag{3.4}$$

it follows that

$$\int_M |\text{Hess } f|^2 dv = \int_M |\text{Hess } f - t f \mathbf{I}|^2 dv + \frac{\lambda}{n} \int_M |\nabla f|^2 dv \geq \frac{\lambda}{n} \int_M |\nabla f|^2 dv. \tag{3.5}$$

Equality holds if and only if

$$\text{Hess } f = -\frac{\lambda}{n} f \mathbf{I}. \tag{3.6}$$

Thus, we conclude from (3.3) and (3.5) that

$$\lambda \int_M |\nabla f|^2 dv \geq \frac{n}{n-1} \int_M \text{Ric}(\nabla f, \nabla f). \quad \square$$

In order to prove Theorem 1.2 and Theorem 1.3, we need the following Lemma which can be found in [9].

Lemma 2. *Let M^n be a submanifold of $S^{n+p}(1)$. Let Ric denotes the function that assign to each point of M the minimum Ricci curvature. Then*

$$\text{Ric} \geq \frac{n-1}{n} \left\{ n + 2nH^2 - S - \frac{n-2}{\sqrt{n-1}} \sqrt{nH^2} \sqrt{S - nH^2} \right\}. \tag{3.7}$$

PROOF OF THEOREM 1.2. Since M has constant scalar curvature $n(n-1)$, one has from the Gauss equation (2.10) that

$$n^2 H^2 = S.$$

According to Lemma 2 and the above equation, we obtain

$$\begin{aligned}
\text{Ric} &\geq \frac{n-1}{n} \left\{ n + 2nH^2 - S - \frac{(n-2)\sqrt{n-1}}{n-1} \sqrt{nH^2} \sqrt{S - nH^2} \right\} \\
&= (n-1) \{1 - (2n-4)H^2\} = (n-1) \left\{ 1 - \frac{2n-4}{n^2} S \right\}.
\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.1), we infer

$$\int_M \left(\lambda + \frac{2(n-2)}{n} S - n \right) |\nabla f|^2 dv \geq 0.$$

On the other hand, we obtain that equality holds if and only if M is a sphere by using of OBATA's result [10]. Combining with $S = n^2 H^2$, one concludes that $M = S^n(1)$. This completes the proof of Theorem 1.2. $\square$

PROOF OF THEOREM 1.3. Using a well-known inequality, we have, for an arbitrary real number $a > 0$,

$$2|H||\rho| \leq aH^2 + \frac{1}{a}\rho^2. \quad (3.9)$$

By using of Lemma 2, we infer

$$\begin{aligned} \text{Ric} &\geq \frac{n-1}{n} \left\{ n + nH^2 - \rho^2 - \frac{n(n-2)}{\sqrt{n(n-1)}|H|\rho} \right\} \\ &\geq \frac{n-1}{n} \left\{ n + \left(n - \frac{n(n-2)}{2\sqrt{n(n-1)}a} \right) H^2 + \left(-1 - \frac{n(n-2)}{2\sqrt{n(n-1)}a} \right) \rho^2 \right\}. \end{aligned} \quad (3.10)$$

Choosing $a = \frac{2\sqrt{n(n-1)}}{n-2}$, one obtains from (3.10)

$$\text{Ric} \geq \frac{n-1}{n} \left\{ n - \frac{n^2}{4(n-1)} \rho^2 \right\}. \quad (3.11)$$

Hence, we have from the above arguments and Lemma 1

$$\int_M \left(\lambda + \frac{n^2}{4(n-1)} \rho^2 - n \right) |\nabla f|^2 dv \geq 0.$$

Moreover, we then have that equality in the above formula implies that M is a sphere according to OBATA's result [10]. This finishes the proof of Theorem 1.3. $\square$

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