

Minkowski-type inequalities for means generated by two functions and a measure

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Abstract. Given two continuous functions $f, g : I \rightarrow \mathbb{R}$ such that g is positive and f/g is strictly monotone, and a probability measure μ on the Borel subsets of $[0, 1]$, the two variable mean $M_{f,g;\mu} : I^2 \rightarrow I$ is defined by

$$M_{f,g;\mu}(x, y) := \left(\frac{f}{g}\right)^{-1} \left(\frac{\int_0^1 f(tx + (1-t)y) d\mu(t)}{\int_0^1 g(tx + (1-t)y) d\mu(t)} \right) \quad (x, y \in I).$$

The aim of this paper is to study Minkowski-type inequalities for these means, i.e., to find conditions for the generating functions $f_0, g_0 : I_0 \rightarrow \mathbb{R}$, $f_1, g_1 : I_1 \rightarrow \mathbb{R}$, ..., $f_n, g_n : I_n \rightarrow \mathbb{R}$, and for the measure μ such that

$$M_{f_0, g_0; \mu}(x_1 + \cdots + x_n, y_1 + \cdots + y_n) \underset{[\geq]}{\leq} M_{f_1, g_1; \mu}(x_1, y_1) + \cdots + M_{f_n, g_n; \mu}(x_n, y_n)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$ with $x_1 + \cdots + x_n, y_1 + \cdots + y_n \in I_0$. The particular case when the generating functions are power functions, i.e., when the means are generalized Gini means is also investigated.

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1. Introduction

Throughout this paper the classes of continuous strictly monotone and continuous positive real-valued functions defined on a nonempty open real interval I will be denoted by $\mathcal{CM}(I)$ and $\mathcal{CP}(I)$, respectively. Given two continuous functions $f, g : I \rightarrow \mathbb{R}$ with $g \in \mathcal{CP}(I)$, $f/g \in \mathcal{CM}(I)$ and a probability measure μ on the Borel subsets of $[0, 1]$, the two variable mean $M_{f,g;\mu} : I^2 \rightarrow I$ is defined by

$$M_{f,g;\mu}(x, y) := \left(\frac{f}{g}\right)^{-1} \left(\frac{\int_0^1 f(tx + (1-t)y) d\mu(t)}{\int_0^1 g(tx + (1-t)y) d\mu(t)} \right) \quad (x, y \in I).$$

If $\mu = \frac{\delta_0 + \delta_1}{2}$ (where δ_s denotes the Dirac measure concentrated at $s \in [0, 1]$), $\varphi \in \mathcal{CM}(I)$, and $p \in \mathcal{CP}(I)$, then

$$M_{p\varphi,p;\mu}(x, y) = \varphi^{-1} \left(\frac{p(x)\varphi(x) + p(y)\varphi(y)}{p(x) + p(y)} \right) \quad (x, y \in I),$$

which was introduced and studied by BAJRAKTAREVIĆ [Baj58], [Baj69]. In the particular case $p = 1$, we get the well-known quasi-arithmetic means (cf. [HLP34]).

If μ is the Lebesgue measure on $[0, 1]$ and $\varphi, \psi : I \rightarrow \mathbb{R}$ are continuously differentiable functions with $\psi' \in \mathcal{CP}(I)$ and $\varphi'/\psi' \in \mathcal{CM}(I)$, then, by the Fundamental Theorem of Calculus, one can easily see that

$$M_{\varphi',\psi';\mu}(x, y) = \begin{cases} \left(\frac{\varphi'}{\psi'}\right)^{-1} \left(\frac{\varphi(y) - \varphi(x)}{\psi(y) - \psi(x)} \right) & \text{if } x \neq y \\ x & \text{if } x = y \end{cases} \quad (x, y \in I),$$

which is called a Cauchy or difference mean in the literature (cf. [BM00], [Los00]). When $\psi(x) = x$, then this mean goes over into a Lagrangian mean (cf. [BM98], [Ber98]).

Consider now the setting when $I = \mathbb{R}_+$ and the functions f, g are power functions, more precisely, for $p, q \in \mathbb{R}$, define

$$\begin{aligned} f(x) &= x^p, & g(x) &= x^q & \text{if } p &\neq q, \\ f(x) &= x^p \ln x, & g(x) &= x^p & \text{if } p &= q. \end{aligned} \quad (1)$$

Then the mean $M_{f,g;\mu}$ reduces to the following generalization of the so-called Gini means:

$$G_{p,q;\mu}(x, y) := \begin{cases} \left(\frac{\int_0^1 (tx + (1-t)y)^p d\mu(t)}{\int_0^1 (tx + (1-t)y)^q d\mu(t)} \right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \exp \left(\frac{\int_0^1 (tx + (1-t)y)^p \ln(tx + (1-t)y) d\mu(t)}{\int_0^1 (tx + (1-t)y)^p d\mu(t)} \right) & \text{if } p = q. \end{cases}$$

In the particular case when $\mu = \frac{1}{2}(\delta_0 + \delta_1)$, the mean $G_{p,q;\mu}$ goes over into the standard Gini mean (cf. [Gin38]) defined as

$$G_{p,q;\mu}(x, y) = G_{p,q}(x, y) := \begin{cases} \left(\frac{x^p + y^p}{x^q + y^q} \right)^{\frac{1}{p-q}} & \text{if } p \neq q \\ \exp \left(\frac{x^p \ln x + y^p \ln y}{x^p + y^p} \right) & \text{if } p = q \end{cases} \quad (x, y \in \mathbb{R}_+).$$

The other particular case of great importance is when μ is equal to the Lebesgue measure λ . Then

$$G_{p,q;\lambda}(x, y) = S_{p+1,q+1}(x, y) \quad (x, y \in \mathbb{R}_+),$$

where $S_{p,q}$ is the so-called Stolarsky mean (cf. [Sto75]) given by

$$S_{p,q}(x, y) := \begin{cases} \left(\frac{q(x^p - y^p)}{p(x^q - y^q)} \right)^{\frac{1}{p-q}} & \text{if } (p-q)pq \neq 0 \\ \exp \left(-\frac{1}{p} + \frac{x^p \ln x - y^p \ln y}{x^p - y^p} \right) & \text{if } p = q \neq 0 \\ \left(\frac{x^p - y^p}{p(\ln x - \ln y)} \right)^{\frac{1}{p}} & \text{if } p \neq 0, q = 0 \\ \left(\frac{x^q - y^q}{q(\ln x - \ln y)} \right)^{\frac{1}{q}} & \text{if } p = 0, q \neq 0 \\ \sqrt{xy} & \text{if } p = q = 0 \end{cases} \quad (x, y \in \mathbb{R}_+).$$

In [Los71], the first author obtained Minkowski-type inequalities for Bajraktarević means. Investigating Minkowski-type inequalities for the standard Gini means, CZINDER and the second author obtained the following result (cf. [CP00, Theorem 5]).

Theorem A. *Let $n \geq 2$ and $p_0, p_1, \dots, p_n, q_0, q_1, \dots, q_n \in \mathbb{R}$. Then*

$$G_{p_0, q_0}(x_1 + \dots + x_n, y_1 + \dots + y_n) \leq G_{p_1, q_1}(x_1, y_1) + \dots + G_{p_n, q_n}(x_n, y_n)$$

holds for all $x_1, \dots, x_n, y_1, \dots, y_n > 0$ if and only if

- (a) $0 \leq \min\{p_1, q_1, \dots, p_n, q_n\}$,
- (b) $\min\{p_0, q_0\} \leq \min\{1, p_1, q_1, \dots, p_n, q_n\}$,
- (c) $\max\{1, p_0 + q_0\} \leq \min\{p_1 + q_1, \dots, p_n + q_n\}$.

The particular case $p_0 = p_1 = \dots = p_n$, $q_0 = q_1 = \dots = q_n$, i.e., when all the Gini means are the same, was investigated by the authors in [LP96]. It is interesting to note that the characterization of the reversed Minkowski-type inequality even in this particular setting is *still unknown*.

In the context of Stolarsky means, in [LP98], we obtained the following result (formulated in the case $n = 2$ only).

Theorem B. *Let $n \geq 2$ and $p, q \in \mathbb{R}$. Then the inequality*

$$S_{p, q}(x_1 + \dots + x_n, y_1 + \dots + y_n) \begin{matrix} \leq \\ \geq \end{matrix} S_{p, q}(x_1, y_1) + \dots + S_{p, q}(x_n, y_n)$$

holds for all $x_1, \dots, x_n, y_1, \dots, y_n > 0$ if and only if

$$3 \begin{matrix} \leq \\ \geq \end{matrix} p + q \quad \text{and} \quad 1 \begin{matrix} \leq \\ \geq \end{matrix} \min\{p, q\}.$$

In order that the more general inequality

$$S_{p_0, q_0}(x_1 + \dots + x_n, y_1 + \dots + y_n) \begin{matrix} \leq \\ \geq \end{matrix} S_{p_1, q_1}(x_1, y_1) + \dots + S_{p_n, q_n}(x_n, y_n)$$

be valid for all $x_1, \dots, x_n, y_1, \dots, y_n > 0$, CZINDER and the second author obtained necessary conditions and also sufficient conditions for the parameters $p_0, p_1, \dots, p_n, q_0, q_1, \dots, q_n \in \mathbb{R}$ in [CP03].

Motivated by the above preliminaries, the aim of this paper is to study Minkowski-type inequalities for the means $M_{f, g; \mu}$, i.e., our purpose is to find

conditions for the generating functions $f_0, g_0 : I_0 \rightarrow \mathbb{R}$, $f_1, g_1 : I_1 \rightarrow \mathbb{R}$, $\dots$, $f_n, g_n : I_n \rightarrow \mathbb{R}$, and for the measure μ such that

$$M_{f_0, g_0; \mu}(x_1 + \dots + x_n, y_1 + \dots + y_n) \begin{matrix} \leq \\ [\geq] \end{matrix} M_{f_1, g_1; \mu}(x_1, y_1) + \dots + M_{f_n, g_n; \mu}(x_n, y_n) \quad (2)$$

be valid for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$ with $x_1 + \dots + x_n, y_1 + \dots + y_n \in I_0$. In the main results of the paper we give sufficient conditions (which, in a certain sense, are also necessary) for (2) to hold. As an important particular case, we also consider Minkowski-type inequalities involving the generalized Gini means.

2. Main results

In order to describe the regularity conditions related the two generating functions f, g of the mean $M_{f, g; \mu}$ in a convenient way, we say that the pair (f, g) of functions is in the class $\mathcal{C}_1(I)$ if f, g are continuously differentiable functions such that $g \in \mathcal{CP}(I)$ and the *Wronski determinant*

$$\begin{vmatrix} f'(x) & f(x) \\ g'(x) & g(x) \end{vmatrix} = g^2(x) \left(\frac{f(x)}{g(x)} \right)' \quad (x \in I) \quad (3)$$

does not vanish on I . Obviously, the latter condition implies that f/g is strictly monotone, i.e., $f/g \in \mathcal{CM}(I)$. For $(f, g) \in \mathcal{C}_1(I)$, we define the *deviation function* $\mathcal{D}_{f, g}^* : I^2 \rightarrow \mathbb{R}$ by

$$\mathcal{D}_{f, g}^*(x, y) := \frac{\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix}}{\begin{vmatrix} f'(y) & f(y) \\ g'(y) & g(y) \end{vmatrix}} = \frac{g(x) \left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \right)}{g(y) \left(\frac{f(y)}{g(y)} \right)'} \quad (x, y \in I). \quad (4)$$

Clearly, we have that $\mathcal{D}_{f, g}^*(x, y) \begin{matrix} < \\ \equiv \\ > \end{matrix} 0$ if and only if $x \begin{matrix} < \\ \equiv \\ > \end{matrix} y$.

The next result characterizes the mean $M_{f, g; \mu}$ via an implicit equation and signifies the role of the function $\mathcal{D}_{f, g}^*$ (cf. [LP08]).

Lemma 1. *Let $(f, g) \in \mathcal{C}_1(I)$ and μ be a Borel probability measure on $[0, 1]$. Then for all $x, y \in I$ and $u \in [x, y]$,*

$$M_{f,g;\mu}(x, y) \begin{matrix} \leq \\ \equiv \\ \geq \end{matrix} u \quad \text{if and only if} \quad \int_0^1 \mathcal{D}_{f,g}^*(tx + (1-t)y, u) d\mu(t) \begin{matrix} \leq \\ \equiv \\ \geq \end{matrix} 0. \quad (5)$$

As a consequence of (5), we have the identity

$$\int_0^1 \mathcal{D}_{f,g}^*(tx + (1-t)y, M_{f,g;\mu}(x, y)) d\mu(t) = 0 \quad (x, y \in I). \quad (6)$$

By Lemma 2 below, the function $\mathcal{D}_{f,g}^*$ is also connected to the sequence of means $M_{f,g;m_k}$, where (m_k) is the sequence of measures defined by

$$m_k := \left(1 - \frac{1}{k}\right) \delta_0 + \frac{1}{k} \delta_1 \quad (k \in \mathbb{N}). \quad (7)$$

For its proof, the reader should consult [LP08].

Lemma 2. *Let $(f, g) \in \mathcal{C}_1(I)$. Then*

$$\lim_{k \rightarrow \infty} k [M_{f,g;m_k}(x, y) - y] = \mathcal{D}_{f,g}^*(x, y) \quad (x, y \in I). \quad (8)$$

Now we can formulate our main result which gives a sufficient condition for the general Minkowski-type inequality (2) which does not involve the measure μ .

Theorem 3. *Let $I_0, I_1, \dots, I_n$ be open real intervals, and let $(f_i, g_i) \in \mathcal{C}_1(I_i)$ for $i = 0, 1, \dots, n$. Then the following three assertions are equivalent:*

(i) *For all Borel probability measures μ on $[0, 1]$,*

$$M_{f_0,g_0;\mu}(x_1 + \dots + x_n, y_1 + \dots + y_n) \begin{matrix} \leq \\ [\geq] \end{matrix} M_{f_1,g_1;\mu}(x_1, y_1) + \dots + M_{f_n,g_n;\mu}(x_n, y_n) \quad (9)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$ with $x_1 + \dots + x_n, y_1 + \dots + y_n \in I_0$.

(ii) *For all $k \in \mathbb{N}$,*

$$M_{f_0,g_0;m_k}(x_1 + \dots + x_n, y_1 + \dots + y_n) \begin{matrix} \leq \\ [\geq] \end{matrix} M_{f_1,g_1;m_k}(x_1, y_1) + \dots + M_{f_n,g_n;m_k}(x_n, y_n) \quad (10)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$ with $x_1 + \dots + x_n, y_1 + \dots + y_n \in I_0$ (where (m_k) is the sequence of measures defined by (7)).

(iii)

$$\mathcal{D}_{f_0, g_0}^*(x_1 + \cdots + x_n, y_1 + \cdots + y_n) \underset{[\geq]}{\leq} \mathcal{D}_{f_1, g_1}^*(x_1, y_1) + \cdots + \mathcal{D}_{f_n, g_n}^*(x_n, y_n) \quad (11)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$ with $x_1 + \cdots + x_n, y_1 + \cdots + y_n \in I_0$.

PROOF. The implication (i) $\Rightarrow$ (ii) is obvious. To prove (ii) $\Rightarrow$ (iii), for $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$ with $x_1 + \cdots + x_n, y_1 + \cdots + y_n \in I_0$, use (10) and Lemma 2 to get

$$\begin{aligned} & \mathcal{D}_{f_0, g_0}^*(x_1 + \cdots + x_n, y_1 + \cdots + y_n) \\ &= \lim_{k \rightarrow \infty} k [M_{f_0, g_0; m_k}(x_1 + \cdots + x_n, y_1 + \cdots + y_n) - (y_1 + \cdots + y_n)] \\ &\underset{[\geq]}{\leq} \lim_{k \rightarrow \infty} k [(M_{f_1, g_1; m_k}(x_1, y_1) - y_1) + \cdots + (M_{f_n, g_n; m_k}(x_n, y_n) - y_n)] \\ &= \mathcal{D}_{f_1, g_1}^*(x_1, y_1) + \cdots + \mathcal{D}_{f_n, g_n}^*(x_n, y_n), \end{aligned}$$

which proves (11).

(iii) $\Rightarrow$ (i) Let $u_1, v_1 \in I_1, \dots, u_n, v_n \in I_n$ with $u_1 + \cdots + u_n, v_1 + \cdots + v_n \in I_0$. Substituting

$$x_i := tu_i + (1-t)v_i, \quad y_i := M_{f_i, g_i; \mu}(u_i, v_i) \quad (i = 1, \dots, n)$$

into (11) and integrating on $[0, 1]$ with respect to t by the measure μ , we get

$$\begin{aligned} & \int_0^1 \mathcal{D}_{f_0, g_0}^*(t(u_1 + \cdots + u_n) + (1-t)(v_1 + \cdots + v_n), y_1 + \cdots + y_n) d\mu(t) \\ &\underset{[\geq]}{\leq} \int_0^1 \mathcal{D}_{f_1, g_1}^*(tu_1 + (1-t)v_1, y_1) d\mu(t) + \cdots \\ &\quad + \int_0^1 \mathcal{D}_{f_n, g_n}^*(tu_n + (1-t)v_n, y_n) d\mu(t). \end{aligned} \quad (12)$$

By Lemma 1 and the choice of $y_1, \dots, y_n$, the right hand side of this inequality is zero. Thus, we obtain from (12) that

$$\int_0^1 \mathcal{D}_{f_0, g_0}^*(t(u_1 + \cdots + u_n) + (1-t)(v_1 + \cdots + v_n), y_1 + \cdots + y_n) d\mu(t) \underset{[\geq]}{\leq} 0.$$

This inequality, by Lemma 1 again, yields that

$$\begin{aligned} & M_{f_0, g_0; \mu}(u_1 + \cdots + u_n, v_1 + \cdots + v_n) \\ &\underset{[\geq]}{\leq} y_1 + \cdots + y_n = M_{f_1, g_1; \mu}(u_1, v_1) + \cdots + M_{f_n, g_n; \mu}(u_n, v_n), \end{aligned}$$

which proves (9) on the domain indicated. $\square$

The following result concerns generalized quasi-arithmetic means when a stronger statement can be obtained.

Theorem 4. *Let $I_0, I_1, \dots, I_n$ be open intervals with $I_1 + \dots + I_n \subseteq I_0$. Assume that $f_i : I_i \rightarrow I_0$ are continuously differentiable functions such that $f'_i(x) \neq 0$ if $x \in I_i$ $i = 0, 1, \dots, n$ (the latter conditions ensure that $(f_i, 1) \in \mathcal{C}_1(I_i)$ for $i = 0, 1, \dots, n$). Then the following three assertions are equivalent:*

(i) *For all Borel probability measures μ on $[0, 1]$,*

$$M_{f_0, 1; \mu}(x_1 + \dots + x_n, y_1 + \dots + y_n) \underset{[\geq]}{\leq} M_{f_1, 1; \mu}(x_1, y_1) + \dots + M_{f_n, 1; \mu}(x_n, y_n)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$.

(ii) *For all $k \in \mathbb{N}$,*

$$M_{f_0, 1; m_k}(x_1 + \dots + x_n, y_1 + \dots + y_n) \underset{[\geq]}{\leq} M_{f_1, 1; m_k}(x_1, y_1) + \dots + M_{f_n, 1; m_k}(x_n, y_n)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$ (where m_k is the sequence of measures defined by (7)).

(iii)

$$\mathcal{D}_{f_0, 1}^*(x_1 + \dots + x_n, y_1 + \dots + y_n) \underset{[\geq]}{\leq} \mathcal{D}_{f_1, 1}^*(x_1, y_1) + \dots + \mathcal{D}_{f_n, 1}^*(x_n, y_n) \quad (13)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$.

(iv) *The function $F : f_1(I_1) \times \dots \times f_n(I_n) \rightarrow \mathbb{R}$ defined by*

$$F(u_1, \dots, u_n) := f_0(f_1^{-1}(u_1) + \dots + f_n^{-1}(u_n)) \quad (u_i \in f_i(I_i), i = 1, \dots, n)$$

is $\begin{smallmatrix} \text{concave} \\ \text{[convex]} \end{smallmatrix}$ on its domain provided that f_0 is increasing and $\begin{smallmatrix} \text{convex} \\ \text{[concave]} \end{smallmatrix}$ on its domain provided that f_0 is decreasing.

PROOF. The equivalence (i) $\iff$ (ii) $\iff$ (iii) follows from the previous theorem. To complete the proof we show that (iii) and (iv) are equivalent too. Assume, for the sake of definiteness that f_0 is increasing and the upper inequality sign holds in (13). By known characterizations of differentiable concave functions (see [RV73, p. 98, Theorem A], or [NP06, p. 141, Theorem 3.9.1], F is concave if and only if

$$F(u) - F(v) \leq \sum_{i=1}^n \partial_i F(v)(u_i - v_i) \quad (14)$$

holds for all $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n)$ in the domain of F , where $\partial_i F$ denotes the partial derivative of F with respect to its i th variable. A simple calculation shows that

$$\partial_i F(v) = f'_0(f_1^{-1}(v_1) + \dots + f_n^{-1}(v_n)) \frac{1}{f'_i(f_i^{-1}(v_i))}.$$

Dividing (14) by $f'_0(f_1^{-1}(v_1) + \dots + f_n^{-1}(v_n)) > 0$ and then substituting $f_i^{-1}(u_i) =: x_i, f_i^{-1}(v_i) =: y_i$, we obtain exactly (13), which proves the equivalence we claimed. $\square$

3. Minkowski-type inequalities for generalized Gini means

Theorem 5. *Let $n \geq 2$ and $p_0, p_1, \dots, p_n, q_0, q_1, \dots, q_n \in \mathbb{R}$. Then the following three assertions are equivalent:*

(i) *For all Borel probability measures μ on $[0, 1]$,*

$$G_{p_0, q_0; \mu}(x_1 + \dots + x_n, y_1 + \dots + y_n) \leq G_{p_1, q_1; \mu}(x_1, y_1) + \dots + G_{p_n, q_n; \mu}(x_n, y_n) \quad (15)$$

holds for all $x_1, y_1, \dots, x_n, y_n > 0$.

(ii) *For all $k \in \mathbb{N}$,*

$$\begin{aligned} G_{p_0, q_0; m_k}(x_1 + \dots + x_n, y_1 + \dots + y_n) \\ \leq G_{p_1, q_1; m_k}(x_1, y_1) + \dots + G_{p_n, q_n; m_k}(x_n, y_n) \end{aligned} \quad (16)$$

holds for all $x_1, y_1, \dots, x_n, y_n > 0$.

$$(iii) \quad (a) \quad 0 \leq \min\{p_1, q_1, \dots, p_n, q_n\},$$

$$(b) \quad \min\{p_0, q_0\} \leq \min\{1, p_1, q_1, \dots, p_n, q_n\},$$

$$(c) \quad \max\{1, p_0, q_0\} \leq \max\{p_i, q_i\}, \quad (i = 1, \dots, n). \quad (17)$$

Concerning the reversed Minkowski inequality, we have the following result.

Theorem 6. *Let $n \geq 2$ and $p_0, p_1, \dots, p_n, q_0, q_1, \dots, q_n \in \mathbb{R}$. Then the following three assertions are equivalent:*

(i) *For all Borel probability measures μ on $[0, 1]$,*

$$G_{p_0, q_0; \mu}(x_1 + \dots + x_n, y_1 + \dots + y_n) \geq G_{p_1, q_1; \mu}(x_1, y_1) + \dots + G_{p_n, q_n; \mu}(x_n, y_n) \quad (18)$$

holds for all $x_1, y_1, \dots, x_n, y_n > 0$.

(ii) For all $k \in \mathbb{N}$,

$$\begin{aligned} G_{p_0, q_0; m_k}(x_1 + \cdots + x_n, y_1 + \cdots + y_n) \\ \geq G_{p_1, q_1; m_k}(x_1, y_1) + \cdots + G_{p_n, q_n; m_k}(x_n, y_n) \end{aligned} \quad (19)$$

holds for all $x_1, y_1, \dots, x_n, y_n > 0$.

$$\begin{aligned} \text{(iii)} \quad & \text{(a)} \quad 1 \geq \max\{p_1, q_1, \dots, p_n, q_n\}, \\ & \text{(b)} \quad \max\{p_0, q_0\} \geq \max\{0, p_1, q_1, \dots, p_n, q_n\}, \\ & \text{(c)} \quad \min\{0, p_0, q_0\} \geq \min\{p_i, q_i\}, \quad (i = 1, \dots, n). \end{aligned} \quad (20)$$

PROOF OF THEOREM 5 AND THEOREM 6. The equivalence of conditions (i) and (ii) in both theorems is a consequence of the equivalence of conditions (i) and (ii) of Theorem 3. To elaborate the third equivalent condition of Theorem 3, observe that if f, g are defined by (1), then the function $\mathcal{D}_{f,g}^*$ is of the form

$$\mathcal{D}_{f,g}^*(x, y) = y\delta_{p,q}\left(\frac{x}{y}\right) \quad (x, y \in \mathbb{R}_+),$$

where

$$\delta_{p,q}(t) := \begin{cases} \frac{t^p - t^q}{p - q} & \text{if } p \neq q \\ t^p \ln t & \text{if } p = q \end{cases} \quad (t \in \mathbb{R}_+). \quad (21)$$

Thus, by Theorem 3, inequalities (15) and (18) are satisfied if and only if

$$(y_1 + \cdots + y_n)\delta_{p_0, q_0}\left(\frac{x_1 + \cdots + x_n}{y_1 + \cdots + y_n}\right) \begin{matrix} \leq \\ \geq \end{matrix} y_1\delta_{p_1, q_1}\left(\frac{x_1}{y_1}\right) + \cdots + y_n\delta_{p_n, q_n}\left(\frac{x_n}{y_n}\right)$$

holds for all $x_1, y_1, \dots, x_n, y_n > 0$. With the notation $u_i := x_i/y_i$ and $t_i := y_i/(y_1 + \cdots + y_n)$, the above inequality is satisfied if and only if

$$\delta_{p_0, q_0}(t_1 u_1 + \cdots + t_n u_n) \begin{matrix} \leq \\ \geq \end{matrix} t_1 \delta_{p_1, q_1}(u_1) + \cdots + t_n \delta_{p_n, q_n}(u_n) \quad (22)$$

for all $u_1, \dots, u_n, t_1, \dots, t_n > 0$ with $t_1 + \cdots + t_n = 1$.

The domain of parameters when (22) is valid on the indicated domain was characterized in the paper [Pál82]. As it was proved in [Pál82], (22) holds with $\leq$ and $\geq$ inequality signs if and only if condition (iii) of Theorem 5 and Theorem 6 holds, respectively. $\square$

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