

The D numbers and the central factorial numbers

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Abstract. In this paper the author establishes some formulas for the D numbers $D_{2n}^{(k)}$ and $d_{2n}^{(k)}$. Several identities involving the D numbers, the Bernoulli numbers and the central factorial numbers are also presented.

1. Introduction

The Bernoulli polynomials $B_n^{(k)}(x)$ of order k , for any integer k , may be defined by (see [1], [3], [6], [10])

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \quad (1.1)$$

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k , $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers (see [3], [5]). By (1.1), we can get (see [10, p. 145])

$$\frac{d}{dx} B_n^{(k)}(x) = n B_{n-1}^{(k)}(x), \quad (1.2)$$

$$B_n^{(k+1)}(x) = \frac{k-n}{k} B_n^{(k)}(x) + (x-k) \frac{n}{k} B_{n-1}^{(k)}(x) \quad (1.3)$$

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and

$$B_n^{(k+1)}(x+1) = \frac{nx}{k} B_{n-1}^{(k)}(x) - \frac{n-k}{k} B_n^{(k)}(x), \quad (1.4)$$

where $n \in \mathbb{N}$, $\mathbb{N}$ being the set of positive integers.

The numbers $B_n^{(n)}$ are called the Nörlund numbers (see [2]–[4], [10]). A generating function for the Nörlund numbers $B_n^{(n)}$ is (see [10, p. 150])

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}. \quad (1.5)$$

The D numbers $D_{2n}^{(k)}$ may be defined by (see [8]–[11])

$$(t \csc t)^k = \sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!}, \quad |t| < \pi. \quad (1.6)$$

By (1.1), (1.6), and note that $\csc t = \frac{2i}{e^{it} - e^{-it}}$ (where $i^2 = -1$), we can get

$$\sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!} = \left(\frac{2it}{e^{it} - e^{-it}} \right)^k = \left(\frac{2it}{e^{2it} - 1} \right)^k e^{kit} = \sum_{n=0}^{\infty} B_n^{(k)} \left(\frac{k}{2} \right) \frac{(2it)^n}{n!}.$$

Therefore,

$$D_{2n}^{(k)} = 4^n B_{2n}^{(k)} \left(\frac{k}{2} \right). \quad (1.7)$$

Taking $k = 1, 2$ in (1.7), and noting that $B_{2n}^{(1)}(\frac{1}{2}) = (2^{1-2n} - 1)B_{2n}$, $B_{2n}^{(2)}(1) = (1 - 2n)B_{2n}$ (see [10, p. 22, p. 145]), we have

$$D_{2n}^{(1)} = (2 - 2^{2n})B_{2n} \quad \text{and} \quad D_{2n}^{(2)} = 4^n(1 - 2n)B_{2n}. \quad (1.8)$$

The D numbers $D_{2n}^{(k)}$ satisfy the recurrence relation (see [8])

$$D_{2n}^{(k)} = \frac{(2n - k + 2)(2n - k + 1)}{(k - 2)(k - 1)} D_{2n}^{(k-2)} - \frac{2n(2n - 1)(k - 2)}{k - 1} D_{2n-2}^{(k-2)}. \quad (1.9)$$

By (1.9), we can immediately deduce the following (see [10, p. 147]):

$$D_{2n}^{(2n+1)} = \frac{(-1)^n (2n)!}{4^n} \binom{2n}{n}, \quad D_{2n}^{(2n+2)} = \frac{(-1)^n 4^n}{2n + 1} (n!)^2 \quad (1.10)$$

and

$$D_{2n}^{(2n+3)} = \frac{(-1)^n (2n)!}{2 \cdot 4^{2n}} \binom{2n+2}{n+1} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} \right). \quad (1.11)$$

The numbers $D_{2n}^{(2n)}$ satisfy the recurrence relation (see [8])

$$\sum_{j=0}^n \frac{(-1)^j}{4^j(2j+1)} \binom{2j}{j} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!} = \frac{(-1)^n}{4^n} \binom{2n}{n}, \quad (1.12)$$

so we find $D_0^{(0)} = 1$, $D_2^{(2)} = -\frac{2}{3}$, $D_4^{(4)} = \frac{88}{15}$, $D_6^{(6)} = -\frac{3056}{21}$, $D_8^{(8)} = \frac{319616}{45}$, $D_{10}^{(10)} = -\frac{18940160}{33}$, $\dots$

By (1.14), we can get a generating function for $D_{2n}^{(2n)}$ (see [8]):

$$\frac{t}{\sqrt{1+t^2} \log(t + \sqrt{1+t^2})} = \sum_{n=0}^{\infty} D_{2n}^{(2n)} \frac{t^{2n}}{(2n)!}, \quad |t| < 1. \quad (1.13)$$

The numbers $D_{2n}^{(2n-1)}$ satisfy the recurrence relation (see [9]):

$$\sum_{j=1}^n \binom{2n}{2j} (-1)^{j-1} 4^{j-1} ((j-1)!)^2 D_{2n-2j}^{(2n-1-2j)} = \frac{(-1)^{n-1} 2(2n)!}{4^n} \binom{2n-2}{n-1}, \quad (1.14)$$

so we find $D_2^{(1)} = -\frac{1}{3}$, $D_4^{(3)} = \frac{17}{5}$, $D_6^{(5)} = -\frac{1835}{21}$, $D_8^{(7)} = \frac{195013}{45}$, $D_{10}^{(9)} = -\frac{3887409}{11}$, $\dots$

By (1.16), we can get a generating function for $D_{2n}^{(2n-1)}$ (see [9])

$$\sum_{n=0}^{\infty} D_{2n}^{(2n-1)} \frac{t^{2n}}{(2n)!} = \frac{1}{\sqrt{1+t^2}} \left(\frac{t}{\log(t + \sqrt{1+t^2})} \right)^2. \quad (1.15)$$

These numbers $D_{2n}^{(2n)}$ and $D_{2n}^{(2n-1)}$ have many important applications. For example (see [10, p. 246])

$$\int_0^{\frac{\pi}{2}} \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!}, \quad \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{2^{2n} (2n-1)(n!)^2}, \quad (1.16)$$

and

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{(2n-1)(2n)!}. \quad (1.17)$$

The D numbers d_{2n} of the second kind may be defined by

$$\frac{t}{\log(t + \sqrt{1+t^2})} = \sum_{n=0}^{\infty} d_{2n} t^{2n}. \quad (1.18)$$

By (1.18), we have $d_0 = 1$, $d_2 = \frac{1}{6}$, $d_4 = -\frac{17}{360}$, $d_6 = \frac{367}{15120}$, $d_8 = -\frac{195013}{27216000}$,
 $d_{10} = \frac{1295803}{252806400}$.

We now turn to the central factorial numbers $t(n, k)$ of the first kind, which are usually defined by (see [7], [12])

$$x \left(x + \frac{n}{2} - 1\right) \left(x + \frac{n}{2} - 2\right) \cdots \left(x + \frac{n}{2} - n + 1\right) = \sum_{k=0}^n t(n, k) x^k, \quad (1.19)$$

or by means of the following generating function:

$$\left(2 \log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}\right)\right)^k = k! \sum_{n=k}^{\infty} t(n, k) \frac{x^n}{n!}. \quad (1.20)$$

It follows from (1.19) or (1.20) that

$$t(n, k) = t(n-2, k-2) - \frac{1}{4}(n-2)^2 t(n-2, k), \quad (1.21)$$

and that

$$\begin{aligned} t(n, 0) &= \delta_{n,0} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), & t(n, n) &= 1 \quad (n \in \mathbb{N}), \\ t(n, k) &= 0 \quad (n+k \text{ odd}) \quad \text{and} \quad t(n, k) = 0 \quad (k > n \text{ or } k < 0), \end{aligned}$$

where (and in what follows) $\delta_{m,n}$ denotes the Kronecker symbol.

By (1.21), we have

$$\begin{aligned} t(2n+1, 1) &= \frac{(-1)^n 1^2 \cdot 3^2 \cdots (2n-1)^2}{4^n}, \\ t(2n+2, 2) &= (-1)^n (n!)^2 \quad (n \in \mathbb{N}_0) \end{aligned} \quad (1.22)$$

and

$$t(2n+2, 4) = (-1)^{n+1} (n!)^2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) \quad (n \in \mathbb{N}). \quad (1.23)$$

By (1.9) and (1.21), we have

$$D_{2n-2k}^{(2n+1)} = \frac{4^{n-k}}{\binom{2n}{2k}} t(2n+1, 2k+1) \quad (n \geq k \geq 0). \quad (1.24)$$

The central factorial numbers $T(n, k)$ of the second kind can be defined by (see [7], [12])

$$x^n = \sum_{k=0}^n T(n, k) x \left(x + \frac{k}{2} - 1\right) \left(x + \frac{k}{2} - 2\right) \cdots \left(x + \frac{k}{2} - k + 1\right) \quad (1.25)$$

or by the generating function

$$\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^k = k! \sum_{n=k}^{\infty} T(n, k) \frac{x^n}{n!}. \quad (1.26)$$

It follows from (1.25) or (1.26) that

$$T(n, k) = T(n-2, k-2) + \frac{1}{4}k^2 T(n-2, k), \quad (1.27)$$

and that

$$T(n, 0) = \delta_{n,0} \quad (n \in \mathbb{N}_0), \quad T(n, n) = 1 \quad (n \in \mathbb{N}),$$

$$T(n, k) = 0 \quad (n+k \text{ odd}), \quad \text{and} \quad T(n, k) = 0 \quad (k > n \text{ or } k < 0).$$

The main purpose of this paper is to prove several expressions formulas for the D numbers $D_{2n}^{(k)}$ and $d_{2n}^{(k)}$. We also obtain some identities involving the D numbers, the Bernoulli numbers and the central factorial numbers.

2. Main results

Theorem 1. *Let $n, k \in \mathbb{N}$. Then*

$$(i) \quad D_{2n}^{(2k)} = 2k \binom{2n}{2k} \sum_{j=0}^{k-1} \frac{4^j t(2k, 2k-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \quad (n \geq k); \quad (2.1)$$

$$(ii) \quad D_{2n}^{(2k+1)} = (2k+1) \binom{2n}{2k+1} \times \sum_{j=0}^k \frac{4^j t(2k+1, 2k+1-2j)}{2n-2j} D_{2n-2j}^{(1)} \quad (n \geq k+1). \quad (2.2)$$

By (2.1), (2.2) and (1.8), we may immediately deduce the following Corollary 1:

Corrolary 1. *Let $n, k \in \mathbb{N}$. Then*

$$(i) \quad D_{2n}^{(2k)} = -2k \binom{2n}{2k} \sum_{j=0}^{k-1} \frac{4^j t(2k, 2k-2j)}{2n-2j} B_{2n-2j} \quad (n \geq k); \quad (2.3)$$

$$(ii) \quad D_{2n}^{(2k+1)} = (2k+1) \binom{2n}{2k+1} \times \sum_{j=0}^k \frac{4^j (2-2^{2n-2j}) t(2k+1, 2k+1-2j)}{2n-2j} B_{2n-2j} \quad (n \geq k+1). \quad (2.4)$$

Theorem 2. Let $n, k \in \mathbb{N}$. Then

$$(i) \quad D_{2n}^{(2k)} = 4^n \sum_{j=0}^n \frac{(2k)!(2j)!}{(2k+2j)!} t(2k+2j, 2k) T(2n, 2j); \quad (2.5)$$

$$(ii) \quad D_{2n}^{(2k+1)} = 4^n \sum_{j=0}^n \frac{(2k+1)!(2j)!}{(2k+1+2j)!} t(2k+1+2j, 2k+1) T(2n, 2j). \quad (2.6)$$

By (2.3), (2.4), (2.5) and (2.6), we may immediately deduce the following Corollary 2:

Corollary 2. Let $n, k \in \mathbb{N}$. Then

$$(i) \quad - \binom{2n}{2k} \sum_{j=0}^{k-1} \frac{t(2k, 2k-2j)}{2n-2j} B_{2n-2j} \\ = \sum_{j=0}^n \frac{(2k-1)!(2j)!}{(2k+2j)!} t(2k+2j, 2k) T(2n, 2j) \quad (n \geq k); \quad (2.7)$$

$$(ii) \quad \binom{2n}{2k+1} \sum_{j=0}^k \frac{(2^{2j+1-2n}-1)t(2k+1, 2k+1-2j)}{2n-2j} B_{2n-2j} \\ = \sum_{j=0}^n \frac{(2k)!(2j)!}{(2k+1+2j)!} t(2k+1+2j, 2k+1) T(2n, 2j) \quad (n \geq k+1). \quad (2.8)$$

Remark 1. Setting $k = 1$ in (2.7), and noting that $t(2j+2, 2) = (-1)^j (j!)^2$, we obtain

$$B_{2n} = \frac{1}{1-2n} \sum_{j=0}^n \frac{(-1)^j (j!)^2}{(j+1)(2j+1)} T(2n, 2j).$$

Remark 2. Setting $k = 2$ in (2.7), and noting that $t(2j+4, 4) = (-1)^j ((j+1)!)^2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(j+1)^2}\right)$, we obtain

$$(1-n)B_{2n} + nB_{2n-2} = \frac{72}{(2n-1)(2n-3)} \\ \times \sum_{j=0}^n \frac{(-1)^j ((j+1)!)^2 (2j)!}{(4+2j)!} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(j+1)^2}\right) T(2n, 2j).$$

Theorem 3. Let $n, k \in \mathbb{N}_0$. Then

$$\sum_{n=0}^{\infty} D_{2n}^{(2n-k)} \frac{t^{2n}}{(2n)!} = \frac{1}{\sqrt{1+t^2}} \left(\frac{t}{\log(t + \sqrt{1+t^2})} \right)^{k+1} \quad (|t| < 1). \quad (2.9)$$

Theorem 4. Let $n, k \in \mathbb{N}_0$ and

$$d_{2n}^{(k)} := \sum_{\substack{v_1, \dots, v_k \in \mathbb{N}_0 \\ v_1 + \dots + v_k = n}} d_{2v_1} d_{2v_2} \dots d_{2v_k}. \quad (2.10)$$

Then

$$(2n)! d_{2n}^{(k)} = \frac{k}{k-2n} D_{2n}^{(2n-k)}. \quad (2.11)$$

Theorem 5. Let $n, k \in \mathbb{N}_0$. Then

$$(2n)! d_{2n}^{(k)} = 4^n \sum_{j=0}^n \frac{k!(2j)!}{(k+2j)!} T(k+2j, k) t(2n, 2j). \quad (2.12)$$

3. Proofs of Theorems 1–5

PROOF OF THEOREM 1. (i) We prove (2.1) by using mathematical induction.

1. When $k = 1, 2$, (2.1) is true by (1.9).

2. Suppose (2.1) is true for some natural number k . By the superposition of (1.9) and (1.21), we have

$$\begin{aligned} D_{2n}^{(2k+2)} &= \frac{(2n-2k)(2n-2k-1)}{(2k)(2k+1)} D_{2n}^{(2k)} - \frac{2n(2n-1)(2k)}{2k+1} D_{2n-2}^{(2k)} \\ &= \frac{(2n-2k)(2n-2k-1)}{(2k)(2k+1)} 2k \binom{2n}{2k} \sum_{j=0}^{k-1} \frac{4^j t(2k, 2k-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \\ &\quad - \frac{2n(2n-1)(2k)}{2k+1} 2k \binom{2n-2}{2k} \sum_{j=0}^{k-1} \frac{4^j t(2k, 2k-2j)}{(2n-2j-2)(2n-2j-3)} D_{2n-2j-2}^{(2)} \\ &= (2k+2) \binom{2n}{2k+2} \sum_{j=0}^{k-1} \frac{4^j t(2k, 2k-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \\ &\quad - (2k+2) \binom{2n}{2k+2} (2k)^2 \sum_{j=1}^k \frac{4^{j-1} t(2k, 2k+2-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \\ &= (2k+2) \binom{2n}{2k+2} \sum_{j=0}^k \frac{4^j \left(t(2k, 2k-2j) - \frac{1}{4} (2k)^2 t(2k, 2k+2-2j) \right)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \\ &= (2k+2) \binom{2n}{2k+2} \sum_{j=0}^k \frac{4^j t(2k+2, 2k+2-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)}, \end{aligned} \quad (3.1)$$

and (3.1) shows that (2.1) is also true for the natural number $k+1$. From 1 and 2, we know that (2.1) is true.

(ii) Use the same method as (i). This completes the proof of Theorem 1. $\square$

PROOF OF THEOREM 2. By (1.1), (1.20), (1.26) and noting the identity

$$t = 2 \log \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2} + \sqrt{1 + \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2} \right)^2} \right),$$

we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{(k)} \left(\frac{k}{2} \right) \frac{t^n}{n!} &= \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \right)^k = \left(\frac{2 \log \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2} + \sqrt{1 + \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2} \right)^2} \right)}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \right)^k \\ &= k! \sum_{j=k}^{\infty} \frac{t(j, k)}{j!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^{j-k} = k! \sum_{j=k}^{\infty} \frac{t(j, k)}{(2j)!} (j-k)! \sum_{n=j-k}^{\infty} T(n, j-k) \frac{t^n}{n!} \\ &= k! \sum_{j=0}^{\infty} \frac{t(k+j, k)}{(k+j)!} j! \sum_{n=j}^{\infty} T(n, j) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n \frac{k!j!}{(k+j)!} t(k+j, k) T(n, j) \frac{t^n}{n!}. \end{aligned} \quad (3.2)$$

Comparing the coefficient $\frac{t^n}{n!}$ on both sides of (3.2), we have

$$B_n^{(k)} \left(\frac{k}{2} \right) = \sum_{j=0}^n \frac{k!j!}{(k+j)!} t(k+j, k) T(n, j). \quad (3.3)$$

By (3.3), (1.7), and noting that $t(n, k) = T(n, k) = 0$ ($n+k$ odd), we immediately obtain Theorem 2. This completes the proof of Theorem 2. $\square$

PROOF OF THEOREM 3. Note the identity (see [10, p. 203])

$$B_{2n}^{(k)} \left(x + \frac{k}{2} \right) = \sum_{j=0}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} x^2 (x^2 - 1^2) (x^2 - 2^2) \dots (x^2 - (j-1)^2), \quad (3.4)$$

we have

$$\begin{aligned} &\frac{B_{2n}^{(k)} \left(x + \frac{k}{2} \right) - B_{2n}^{(k)} \left(\frac{k}{2} \right)}{x^2} \\ &= \sum_{j=1}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (x^2 - 1^2) (x^2 - 2^2) \dots (x^2 - (j-1)^2). \end{aligned} \quad (3.5)$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{B_{2n}^{(k)} \left(x + \frac{k}{2}\right) - B_{2n}^{(k)} \left(\frac{k}{2}\right)}{x^2} = \sum_{j=1}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (-1)^{j-1} ((j-1)!)^2. \quad (3.6)$$

By (3.6) and (1.2), we have

$$\lim_{x \rightarrow 0} \frac{2n(2n-1)B_{2n-2}^{(k)} \left(x + \frac{k}{2}\right)}{2} = \sum_{j=1}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (-1)^{j-1} ((j-1)!)^2, \quad (3.7)$$

i.e.,

$$n(2n-1)B_{2n-2}^{(k)} \left(\frac{k}{2}\right) = \sum_{j=1}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (-1)^{j-1} ((j-1)!)^2. \quad (3.8)$$

By (3.8) and (1.7), we have

$$n(2n-1)D_{2n-2}^{(k)} = \sum_{j=1}^n \binom{2n}{2j} (-1)^{j-1} 4^{j-1} ((j-1)!)^2 D_{2n-2j}^{(k-2j)}. \quad (3.9)$$

Thus

$$\frac{1}{2} \sum_{n=1}^{\infty} D_{2n-2}^{(2n-k)} \frac{t^{2n}}{(2n-2)!} = \sum_{n=1}^{\infty} \sum_{j=1}^n \binom{2n}{2j} (-1)^{j-1} 4^{j-1} ((j-1)!)^2 D_{2n-2j}^{(2n-k-2j)} \frac{t^{2n}}{(2n)!}.$$

i.e.,

$$\begin{aligned} & \frac{t^2}{2} \sum_{n=0}^{\infty} D_{2n}^{(2n-(k-2))} \frac{t^{2n}}{(2n)!} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} 4^{n-1} ((n-1)!)^2 \frac{t^{2n}}{(2n)!} \sum_{n=1}^{\infty} D_{2n}^{(2n-k)} \frac{t^{2n}}{(2n)!}. \end{aligned} \quad (3.10)$$

Note the identity (see [6])

$$\sum_{n=0}^{\infty} (-1)^n 4^n (n!)^2 \frac{t^{2n}}{(2n)!} = \frac{1}{1+t^2} \left(1 - \frac{t}{\sqrt{1+t^2}} \log(t + \sqrt{1+t^2}) \right), \quad (3.11)$$

where $|t| < 1$. We have

$$\sum_{n=0}^{\infty} (-1)^n 4^n (n!)^2 \frac{t^{2n+2}}{(2n+2)!} = \frac{1}{2} \left(\log(t + \sqrt{1+t^2}) \right)^2, \quad (3.12)$$

i.e.

$$\sum_{n=1}^{\infty} (-1)^{n-1} 4^{n-1} ((n-1)!)^2 \frac{t^{2n}}{(2n)!} = \frac{1}{2} \left(\log(t + \sqrt{1+t^2}) \right)^2. \quad (3.13)$$

Thus, by (3.10) and (3.13), we have

$$\sum_{n=1}^{\infty} D_{2n}^{(2n-k)} \frac{t^{2n}}{(2n)!} = \left(\frac{t}{\log(t + \sqrt{1+t^2})} \right)^2 \sum_{n=0}^{\infty} D_{2n}^{(2n-(k-2))} \frac{t^{2n}}{(2n)!}. \quad (3.14)$$

By (3.14), and note that

$$\lim_{t \rightarrow 0} \frac{t}{\log(t + \sqrt{1+t^2})} = 1 \quad \text{and} \quad D_0^{(-k)} = 1, \quad (3.15)$$

we have

$$\sum_{n=0}^{\infty} D_{2n}^{(2n-k)} \frac{t^{2n}}{(2n)!} = \left(\frac{t}{\log(t + \sqrt{1+t^2})} \right)^2 \sum_{n=0}^{\infty} D_{2n}^{(2n-(k-2))} \frac{t^{2n}}{(2n)!}. \quad (3.16)$$

(3.16), (1.13) and (1.15), we immediately obtain (2.9). This completes the proof of Theorem 3. $\square$

PROOF OF THEOREM 4. By Theorem 3, we have

$$\begin{aligned} \frac{d}{dt} \sum_{n=0}^{\infty} D_{2n}^{(2n-k+2)} \frac{t^{2n}}{(2n)!} &= \frac{d}{dt} \frac{1}{\sqrt{1+t^2}} \left(\frac{t}{\log(t + \sqrt{1+t^2})} \right)^{k-1} \\ &= -\frac{t}{1+t^2} \sum_{n=0}^{\infty} D_{2n}^{(2n-k+2)} \frac{t^{2n}}{(2n)!} + \frac{k-1}{t} \sum_{n=0}^{\infty} D_{2n}^{(2n-k+2)} \frac{t^{2n}}{(2n)!} \\ &\quad - \frac{k-1}{t(1+t^2)} \left(\frac{t}{\log(t + \sqrt{1+t^2})} \right)^k, \end{aligned}$$

i.e.

$$\begin{aligned} (k-1) \left(\frac{t}{\log(t + \sqrt{1+t^2})} \right)^k &= (k-1) \sum_{n=0}^{\infty} D_{2n}^{(2n-k+2)} \frac{t^{2n}}{(2n)!} \\ &\quad + (k-2)t^2 \sum_{n=0}^{\infty} D_{2n}^{(2n-k+2)} \frac{t^{2n}}{(2n)!} - (1+t^2) \sum_{n=1}^{\infty} D_{2n}^{(2n-k+2)} \frac{t^{2n}}{(2n-1)!}. \end{aligned} \quad (3.17)$$

By (3.17), (1.18), (2.9) and (2.10), we have

$$(k-1)(2n)! d_{2n}^{(k)} = (k-1-2n) D_{2n}^{(2n-k+2)} + 2n(2n-1)(k-2n) D_{2n-2}^{(2n-k)}. \quad (3.18)$$

By (3.18) and (1.9), we immediately obtain Theorem 4. This completes the proof of Theorem 4. $\square$

PROOF OF THEOREM 5. By (2.10), (1.18), (1.20) and (1.26), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} d_{2n}^{(k)} t^{2n} &= \left(\frac{t}{\log(t + \sqrt{1+t^2})} \right)^k = \left(\frac{e^{\log(t + \sqrt{1+t^2})} - e^{-\log(t + \sqrt{1+t^2})}}{2 \log(t + \sqrt{1+t^2})} \right)^k \\
&= k! \sum_{j=k}^{\infty} \frac{T(j, k)}{j!} \left(2 \log(t + \sqrt{1+t^2}) \right)^{j-k} \\
&= k! \sum_{j=k}^{\infty} \frac{T(j, k)}{j!} (j-k)! \sum_{n=j-k}^{\infty} 2^n t(n, j-k) \frac{t^n}{n!} \\
&= k! \sum_{j=0}^{\infty} \frac{T(k+j, k)}{(k+j)!} j! \sum_{n=j}^{\infty} 2^n t(n, j) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} 2^n \sum_{j=0}^n \frac{k! j!}{(k+j)!} T(k+j, k) t(n, j) \frac{t^n}{n!}. \tag{3.19}
\end{aligned}$$

Comparing the coefficient $\frac{t^{2n}}{(2n)!}$ on both sides of (3.19), we have

$$(2n)! d_{2n}^{(k)} = 4^n \sum_{j=0}^n \frac{k! (2j)!}{(k+2j)!} T(k+2j, k) t(2n, 2j). \tag{3.20}$$

This completes the proof of Theorem 5. $\square$

Remark 3. By (1.19) and noting that $t(n, k) = 0$ ($n+k$ odd), we have

$$x^2(x^2 - 1^2) \dots (x^2 - (n-1)^2) = \sum_{k=0}^n t(2n, 2k) x^{2k}.$$

Therefore,

$$\int_0^{\frac{1}{2}} x^2(x^2 - 1^2) \dots (x^2 - (n-1)^2) dx = \sum_{k=0}^n \frac{1}{(2k+1)2^{2k+1}} t(2n, 2k), \tag{3.21}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{(2k+1)2^{2k+1}} t(2n, 2k) \frac{x^{2n}}{(2n)!} &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)2^{2k+1}} \sum_{n=k}^{\infty} t(2n, 2k) \frac{x^{2n}}{(2n)!} \\
&= \sum_{k=0}^{\infty} \frac{1}{(2k+1)! 2^{2k+1}} \left(2 \log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right) \right)^{2k} \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right) \right)^{2k}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4 \log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right)} \left(\sum_{k=0}^{\infty} \frac{1 - (-1)^k}{k!} \left(\log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right) \right)^k \right) \\
&= \frac{1}{4 \log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right)} \left(e^{\log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right)} - e^{-\log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right)} \right) \\
&= \frac{x}{4 \log \left(\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \right)} = \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} d_{2n} x^{2n}.
\end{aligned}$$

Hence,

$$\sum_{k=0}^n \frac{1}{(2k+1)2^{2k+1}} t(2n, 2k) = \frac{(2n)!}{2^{2n+1}} d_{2n}. \quad (3.22)$$

By (3.21) and (3.22), we have

$$\int_0^{\frac{1}{2}} x^2(x^2 - 1^2) \dots (x^2 - (n-1)^2) dx = \frac{(2n)!}{2^{2n+1}} d_{2n}. \quad (3.23)$$

By (3.23) and noting that (see [10, p.211])

$$\frac{\sin \pi x}{\pi x} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} x^2(x^2 - 1^2) \dots (x^2 - (n-1)^2), \quad (3.24)$$

we have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx &= \pi \int_0^{\frac{1}{2}} \frac{\sin \pi x}{\pi x} dx \\
&= \frac{\pi}{2} + \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \int_0^{\frac{1}{2}} x^2(x^2 - 1^2) \dots (x^2 - (n-1)^2) dx \\
&= \frac{\pi}{2} + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} d_{2n} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} d_{2n}.
\end{aligned} \quad (3.25)$$

By (3.25) and (2.11), we immediately obtain (see (1.16))

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{2^{2n} (2n-1) (n!)^2}. \quad (3.26)$$

On the other hand, by (1.13) we have

$$\sum_{n=0}^{\infty} (-1)^n D_{2n}^{(2n)} \frac{t^{2n}}{(2n)!} = \frac{it}{\sqrt{1-t^2} \log(it + \sqrt{1-t^2})}$$

therefore (see (1.16)),

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!} &= \int_0^1 \frac{it}{\sqrt{1-t^2} \log(it + \sqrt{1-t^2})} dt \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\cos x} \frac{i \sin x}{\log(i \sin x + \cos x)} d \sin x = \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx, \quad (3.27) \end{aligned}$$

where $i^2 = -1$.

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