Publ. Math. Debrecen
79/3-4 (2011), 309–315
DOI: 10.5486/PMD.2011.5131

Generalized LCM matrices

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This paper is dedicated to Kálmán Győry, András Sárközy on the occsasion of their 70th birthday, Attila Pethő, János Pintz on the occasion of their 60th birthday

Abstract. Let f be an arithmetical function. The matrix $[f[i,j]]_{n\times n}$ given by the value of f in least common multiple of $[i,j]$, $f\big([i,j]\big)$ as its i, j entry is called the least common multiple (LCM) matrix. We consider the generalization of this matrix where the elements are in the form $f\big(n,[i,j]\big)$ and $f\big(n,i,j,[i,j]\big)$.

1. Introduction

The classical Smith determinant was introduced in 1875 by H. J. S. Smith [12] who also proved that

$$\det[(i,j)]_{n\times n}=\begin{vmatrix}(1,1) & (1,2) & \dots & (1,n)\\ (2,1) & (2,2) & \dots & (2,n)\\ \dots & \dots & \dots & \dots\\ (n,1) & (n,2) & \dots & (n,n)\end{vmatrix}=\varphi(1)\varphi(2)\dots\varphi(n), \tag{1}$$

where (i,j) represents the greatest common divisor of i and j, and $\varphi(n)$ denotes the Euler totient function.

Mathematics Subject Classification: 11C20, 11A25, 15A36.
Key words and phrases: LCM matrix, Smith determinant, arithmetical function.

The GCD matrix with respect to f is

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f((1,1)) & f((1,2)) & \dots & f((1,n)) \\ f((2,1)) & f((2,2)) & \dots & f((2,n)) \\ \dots & \dots & \dots & \dots \\ f((n,1)) & f((n,2)) & \dots & f((n,n)) \end{bmatrix}.$$

There are quite a few generalized forms of GCD matrices, which can be found in several references [1], [3], [7], [8], [11].

H. J. S. Smith [12] also evaluated the determinant of

$$\big[[i,j]\big]_{n\times n} = \begin{bmatrix} [1,1] & [1,2] & \dots & [1,n] \\ [2,1] & [2,2] & \dots & [2,n] \\ \dots & \dots & \dots & \dots \\ [n,1] & [n,2] & \dots & [n,n] \end{bmatrix},$$

and proved that

$$\det \big[[i,j]\big]_{n\times n} = (n!)^2 g(1)g(2)\dots g(n) = \prod_{k=1}^{N} \varphi(k) \prod_{p\mid k} (-p).$$

where $g(n) = \frac{1}{n}\sum_{d\mid n} d\mu(d)$, $\mu(n)$ being the classical Möbius function.
The structure of an LCM matrix $\big[[i,j]\big]_{n\times n}$ is the following (I. Korkee, P. Haukkanen [10])

$$\big[[i,j]\big]_{n\times n} = AA^T$$

where $A = [a_{ij}]_{n\times n}$,

$$a_{ij} = \begin{cases} \sqrt{g(j)}, & \text{if } \ j \mid i \\ 0, & \text{if } \ j \nmid i \end{cases}.$$

The LCM matrix with respect to f is

$$[f[i,j]]_{n\times n} = \begin{bmatrix} f([1,1]) & f([1,2]) & \dots & f([1,n]) \\ f([2,1]) & f([2,2]) & \dots & f([2,n]) \\ \dots & \dots & \dots & \dots \\ f([n,1]) & f([n,2]) & \dots & f([n,n]) \end{bmatrix}.$$

Results concerning LCM matrices appear in papers S. Beslin [2], K. Bourque, S. Ligh [4], W. Feng, S. Hong, J. Zhao [6] P. Haukkanen, J. Wang and J. Sillanpää [7].

In this paper we study matrices which have as variables the least common multiple and the indices

$$\big[f(n,[i,j])\big]_{n\times n}=\begin{bmatrix} f(n,[1,1]) & f(n,[1,2]) & \dots & f(n,[1,n])\\ f(n,[2,1]) & f(n,[2,2]) & \dots & f(n,[2,n])\\ \dots & \dots & \dots & \dots\\ f(n,[n,1]) & f(n,[n,2]) & \dots & f(n,[n,n])\end{bmatrix}$$

and the more general form matrices

$$\big[f(n,i,j,[i,j])\big]_{n\times n}=\begin{bmatrix} f(n,1,1,[1,1]) & f(n,1,2,[1,2]) & \dots & f(n,1,n,[1,n])\\ f(n,2,1,[2,1]) & f(n,2,2,[2,2]) & \dots & f(n,2,n,[2,n])\\ \dots & \dots & \dots & \dots\\ f(n,n,1,[n,1]) & f(n,n,2,[n,2]) & \dots & f(n,n,n,[n,n])\end{bmatrix}$$

2. Generalized LCM matrices

Theorem 2.1. *For a given totally multiplicative arithmetical function $g(n)$ let*

$$f(n,[i,j])=g([i,j])\sum_{k\le\frac{n}{[i,j]}}g(k).$$

Then

$$\big[f(n,[i,j])\big]_{n\times n}=C_n^T\operatorname{diag}\big(g(1),\ g(2),\dots,g(n)\big)C_n, \tag{2}$$

where $C_n=[c_{ij}]_{n\times n}$

$$c_{ij}=\begin{cases}1, & \text{if } j\mid i\\ 0, & \text{if } j\nmid i\end{cases}.$$

For a determinant we have

$$\det\big[f(n,[i,j])\big]_{n\times n}=g(1)g(2)\dots g(n). \tag{3}$$

Proof. After multiplication, the general element of $A=(a_{ij})_{n\times n}$,

$$A=C_n^T\operatorname{diag}\big(g(1),\ g(2),\dots,g(n)\big)C_n$$

is

$$a_{ij}=\sum_{k=1}^{n}c_{ki}g(k)c_{kj}=\sum_{\substack{i\mid k\\ j\mid k\\ k\le n}}g(k)=\sum_{\substack{[i,j]\mid k\\ k\le n}}g(k)=\sum_{l\le\frac{n}{[i,j]}}g\big([i,j]l\big).$$

Because $g(n)$ is totally multiplicative

$$a_{ij} = g\big([i,j]\big) \sum_{\ell \le \frac{n}{[i,j]}} g(\ell) = f\big([i,j]\big).$$

If we calculate the determinant of both parts of (2) we have (3). □

Particular cases

Example 1. If $g(n) = 1$, then

$$f\big(n, [i,j]\big) = \left\lfloor \frac{n}{[i,j]} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x.
From Theorem 2.1 we have

$$\left[\left\lfloor \frac{n}{[i,j]} \right\rfloor\right]_{n\times n} = C_n^T \operatorname{diag}\big(1, 1, \ldots, 1\big) C_n, \qquad \det \left[\left\lfloor \frac{n}{[i,j]} \right\rfloor\right]_{n\times n} = 1.$$

Example 2. If $g(n) = n$, then

$$f\big(n, [i,j]\big) = \frac{\left\lfloor \frac{n}{[i,j]} \right\rfloor \left\lfloor \frac{n}{[i,j]} + 1 \right\rfloor}{2}.$$

The decomposition of generalized LCM matrix is

$$\big[f(n, [i,j])\big]_{n\times n} = C_n^T \operatorname{diag}\big(1,\ 2, \ldots, n\big) C_n,$$

and the determinant

$$\det \big[f(n, [i,j])\big]_{n\times n} = n!.$$

Example 3. If $g(n) = (-1)^{\Omega(n)}$ is a Liouville function, then

$$f\big(n, [i,j]\big) = (-1)^{\Omega\left([i,j]\right)} \sum_{k \le \frac{n}{[i,j]}} (-1)^{\Omega(k)}$$

and

$$\big[f(n, [i,j])\big]_{n\times n} = C_n^T \operatorname{diag}\big(1,\ -1, \ldots, (-1)^{\Omega(n)}\big) C_n,$$

$$\det \big[f(n, [i,j])\big]_{n\times n} = (-1)^{\sum_{k=1}^{n} \Omega(k)}.$$

We remark that matrices related to the greatest integer function appeared in [9], [5].

Theorem 2.2. *For a given totally multiplicative function g let*

$$f(n,i,j,[i,j]) = \sum_{k\le n} g(k) - g(i)\sum_{l\le \frac{n}{i}} g(l) - g(j)\sum_{l\le \frac{n}{j}} g(l) + g([i,j])\sum_{k\le \frac{n}{[i,j]}} g(k).$$

Then

$$\big[f(n,i,j,[i,j])\big]_{n\times n} = D_n^T \operatorname{diag}[g(1),g(2),\ldots,g(n)]D_n,$$

where$D_n = [d_{ij}]_{n\times n}$,

$$d_{ij} = \begin{cases} 1, & \text{if } j\nmid i \\ 0, & \text{if } j\mid i \end{cases}.$$

Proof. After multiplication the general element of the matrix

$$A = [a_{ij}]_{n\times n} = D_n^T \operatorname{diag}[g(1),g(2),\ldots,g(n)]D_n$$

is

$$\begin{aligned} a_{ij} &= \sum_{\substack{i\nmid k\\ j\nmid k\\ k\le n}} g(k) = \sum_{k\le n} g(k) - \sum_{i\mid k} g(k) - \sum_{j\mid k} g(k) + \sum_{\substack{i\mid k\\ j\mid k\\ k\le n}} g(k) \\ &= \sum_{k\le n} g(k) - \sum_{il\le n} g(il) - \sum_{jl\le n} g(jl) + \sum_{\substack{[i,j]\mid k\\ k\le n}} g(k) \end{aligned}$$

The total multiplicativity of g implies,

$$\begin{aligned} a_{ij} &= \sum_{k\le n} g(k) - g(i)\sum_{l\le \frac{n}{i}} g(l) - g(j)\sum_{l\le \frac{n}{j}} g(l) + g([i,j])\sum_{k\le \frac{n}{[i,j]}} g(k) \\ &= f(n,i,j,[i,j]). \end{aligned}$$ □

Particular cases

Example 4. If $g(n)=1$, then

$$f\big(n,i,j,[i,j]\big) = \tau(n) - \tau\left(\left\lfloor \frac{n}{i}\right\rfloor\right) - \tau\left(\left\lfloor \frac{n}{j}\right\rfloor\right) + \left\lfloor \frac{n}{[i,j]}\right\rfloor,$$

where $\tau(n) = \sum_{d\mid n} 1$. By Theorem 2.2

$$\left[f\left(n,i,j,\left\lfloor \frac{n}{[i,j]}\right\rfloor\right)\right]_{n\times n} = D_n^T \operatorname{diag}\big(1,\ 1,\ldots,1\big)D_n.$$

Example 5. If $g(n) = n$, then

$$f\big(n,i,j,[i,j]\big) = \sigma(n) - \sigma\left(\left\lfloor \frac{n}{i} \right\rfloor\right) - \sigma\left(\left\lfloor \frac{n}{j} \right\rfloor\right) + \frac{\left\lfloor \frac{n}{[i,j]} \right\rfloor \left\lfloor \frac{n}{[i,j]} + 1 \right\rfloor}{2},$$

where $\sigma(n) = \sum_{d|n} d$.
The general form of a generalized LCM matrix is

$$\big[f(n,i,j,[i,j])\big]_{n\times n} = D_n^T \operatorname{diag}\big(1,2,\ldots,n\big) D_n.$$

Example 6. If $g(n) = (-1)^{\Omega(n)}$ is the Liouville function then

$$f\big(n,i,j,[i,j]\big) = \sum_{k\le n} (-1)^{\Omega(k)} - (-1)^{\Omega(i)} \sum_{l\le \frac{n}{i}} (-1)^{\Omega(l)} - (-1)^{\Omega(j)} \sum_{l\le \frac{n}{j}} (-1)^{\Omega(l)} g$$
$$+ (-1)^{\Omega\left([i,j]\right)} \sum_{k\le \frac{n}{[i,j]}} (-1)^{\Omega(k)}$$

and

$$\big[f(n,i,j,[i,j]\big]_{n\times n} = D_n^T \operatorname{diag}\big(1,-1,\ldots,(-1)^{\Omega(n)}\big) D_n.$$

Remark 2.1. Due to the fact that the first line of the matrix $[f(n,i,j,[i,j])]_{n\times n}$ contains only 0-s, the determinant of the matrix will always be 0.

Acknowledgement. This research was supported by the grant of Sapientia Foundation, Institute of Scientific Research.

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(Received February 9, 2011; revised August 26, 2011)