

A correspondence theorem for L -functions and partial differential operators

By JERZY KACZOROWSKI (Poznań) and ALBERTO PERELLI (Genova)

*Dedicated to Professors K. Győry, A. Pethő, J. Pintz and A. Sárközy with
admiration to their works in number theory*

Abstract. Given an L -function $F(s)$ from the extended Selberg class, we associate a function $\Phi_F(x, y)$. We show that the functions $\Phi_F(x, y)$ are, in the general case, the analogs of the modular forms associated with the GL_2 L -functions. Indeed, we prove that every $\Phi_F(x, y)$ is eigenfunction of a certain partial differential operator. Moreover, we prove a general correspondence theorem for such L -functions involving the functions $\Phi_F(x, y)$.

Let $F(s)$ be a function in the extended Selberg class $\mathcal{S}^\sharp$. This means that $(s-1)^m F(s)$ is entire of finite order for some non-negative integer m , $F(s)$ is representable for $\sigma > 1$ as an absolutely convergent Dirichlet series with coefficients $a(n)$ and satisfies the functional equation

$$\gamma(s)F(s) = \omega\bar{\gamma}(1-s)\bar{F}(1-s)$$

with $|\omega| = 1$ and

$$\gamma(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j),$$

Mathematics Subject Classification: 11M41, 11F66, 11F25.

Key words and phrases: L -functions, Selberg class, correspondence theorems, modular forms, partial differential operators.

where $Q > 0$, $\Re \mu_j \geq 0$ and $\lambda_j > 0$. Here $\bar{f}(s) = \overline{f(\bar{s})}$. We also write $d_F = 2 \sum_{j=1}^r \lambda_j$ for the degree of $F(s)$ and

$$\mu = \frac{1}{2} + \sum_{j=1}^r \left(\mu_j - \frac{1}{2} \right).$$

Moreover, for $x, y \in \mathbb{R}$ with $y > 0$, we let

$$\Phi_F(x, y) = y^{-\mu} \sum_{n=1}^{\infty} a(n) \tilde{\gamma}(n\sqrt{q}y) e(nx)$$

where $q = q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j}$ is the conductor of $F(s)$, $e(x) = e^{2\pi i x}$ and $\tilde{\gamma}(\xi)$ is the inverse Mellin transform of the gamma-factor $\gamma(s)$, i.e. for $\xi > 0$

$$\tilde{\gamma}(\xi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s) \xi^{-s} ds.$$

Since $\tilde{\gamma}(\xi) \ll \xi^{-A}$ for every $A > 0$, the series defining $\Phi_F(x, y)$ has good convergence properties.

Examples of $\Phi_F(x, y)$. The function $\Phi_F(x, y)$ becomes a familiar object when $F(s)$ is a classical L -function of degree 2.

1. Holomorphic cusp forms. Let $f(z)$ be a holomorphic cusp form of weight k and level N

$$f(z) = \sum_{n=1}^{\infty} \alpha(n) e(nz), \quad z = x + iy;$$

see Ch. 7 of IWANIEC [3]. Writing $a(n) = \alpha(n)n^{-(k-1)/2}$, the normalized L -function associated with $f(z)$ is

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

hence $F(s) = L(s, f)$ is an entire function of degree 2 in $\mathcal{S}^\sharp$ with

$$\gamma(s) = \left(\frac{\sqrt{N}}{2\pi} \right)^s \Gamma\left(s + \frac{k-1}{2}\right), \quad \mu = \frac{k-1}{2}, \quad q = N. \quad (1)$$

We therefore have

$$\tilde{\gamma}(\xi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s + \mu) \left(\frac{2\pi\xi}{\sqrt{q}} \right)^{-s} ds = \left(\frac{\sqrt{q}}{2\pi\xi} \right)^{-\mu} e^{-2\pi\xi/\sqrt{q}}$$

and hence

$$\Phi_F(x, y) = (2\pi)^\mu \sum_{n=1}^{\infty} a(n) n^\mu e^{-2\pi n y + 2\pi i n x} = (2\pi)^\mu f(z). \quad (2)$$

2. Maass forms. Let $f(z)$ be a Maass form of level N and given parity

$$f(z) = \sqrt{y} \sum_{n \neq 0} a(n) n^\varepsilon K_{i\kappa}(2\pi|n|y) e(nx), \quad z = x + iy$$

where $1/4 + \kappa^2$ is the eigenvalue of $f(z)$, $K_{i\kappa}(z)$ is the Bessel K -function and $\varepsilon = 0$ if $f(z)$ is even, $\varepsilon = 1$ otherwise; see Ch. 3 of TERRAS [12]. The L -function associated with $f(z)$ is

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

hence $F(s) = L(s, f)$ is an entire function of degree 2 in $\mathcal{S}^\sharp$ with

$$\gamma(s) = \left(\frac{\sqrt{N}}{\pi} \right)^s \Gamma\left(\frac{s + \varepsilon + i\kappa}{2} \right) \Gamma\left(\frac{s + \varepsilon - i\kappa}{2} \right), \quad \mu = \varepsilon - \frac{1}{2}, \quad q = N.$$

In this case we have

$$\tilde{\gamma}(\xi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma\left(\frac{s + \varepsilon + i\kappa}{2} \right) \Gamma\left(\frac{s + \varepsilon - i\kappa}{2} \right) \left(\frac{\pi\xi}{\sqrt{q}} \right)^{-s} ds.$$

Making the substitution $s + \varepsilon + i\kappa = 2w$ and using the following integral representation (obtained by the inverse Mellin transform of formula 11.1 of p. 115 of OBERHETTINGER [10], choosing $a = 2$)

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2} \right)^\nu \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(w) \Gamma(w - \nu) \left(\frac{z}{2} \right)^{-2w} dw$$

with $c > \max(0, \Re \nu)$, we obtain

$$\tilde{\gamma}(\xi) = \left(\frac{\pi\xi}{\sqrt{q}} \right)^\varepsilon K_{i\kappa} \left(\frac{2\pi\xi}{\sqrt{q}} \right)$$

and hence

$$\Phi_F(x, y) = \pi^\varepsilon \sqrt{y} \sum_{n=1}^{\infty} a(n) n^\varepsilon K_{i\kappa}(2\pi n y) e(nx) = \frac{\pi^\varepsilon}{2} f(z). \quad (3)$$

Therefore, in both cases $\Phi_F(x, y)$ reduces (essentially) to the modular form to which $F(s)$ is associated. $\square$

In Theorem 1 below we assume that $\lambda_j \in \mathbb{Q}$ for every j . Hence, without loss of generality, we may assume that the λ_j are all equal and of the form

$$\lambda_j = \frac{1}{k} \quad k \in \mathbb{N}, \quad j = 1, \dots, r \quad (4)$$

(this can be seen by means of the multiplication formula for the Γ function, see [6]). In particular, the degree d_F is a rational number; note that it is expected that $d_F \in \mathbb{N}$ for every $F \in \mathcal{S}^\sharp$. For $F \in \mathcal{S}^\sharp$ satisfying (4) we consider the partial differential operator

$$\mathcal{D} = \prod_{j=1}^r \left(-\frac{1}{k} y \frac{\partial}{\partial y} + \mu_j - \frac{\mu}{k} \right) - \frac{(2\pi)^{r-k}}{k^r i^k} y^k \frac{\partial^k}{\partial x^k},$$

where multiplication means composition of differential operators. Note that $\Phi_F(x, y)$ depends strongly on $F(s)$, while $\mathcal{D}$ depends only on the functional equation satisfied by $F(s)$. However, the operator $\mathcal{D}$ is not invariant, in the sense that it depends on the shape of the functional equation of $F(s)$, which may be changed by applications of the multiplication formula for the Γ function. We have

Theorem 1. *Let $F \in \mathcal{S}^\sharp$ satisfy (4). Then*

$$\mathcal{D}\Phi_F(x, y) = 0.$$

Remark. Let $\mathcal{D}_0$ be a partial differential operator of the form

$$\mathcal{D}_0 = \prod_{j=1}^r \left(-\lambda y \frac{\partial}{\partial y} + \nu_j \right).$$

It is not difficult to detect the structure of such operators. Indeed, it can be proved that $\mathcal{D}_0$ can be written as

$$\mathcal{D}_0 = \sum_{j=0}^r W_{j,r}(\lambda, \nu_j) y^j \frac{\partial^j}{\partial y^j}$$

where the $W_{j,r}$ are polynomials satisfying

$$W_{j,r+1} = (\nu_{r+1} - j\lambda)W_{j,r} - \lambda W_{j-1,r}, \quad W_{0,r} = \prod_{j=1}^r \nu_j, \quad W_{r,r} = (-1)^r \lambda^r.$$

Moreover, the ring generated by such operators is the polynomial ring $\mathbb{C}[y \frac{\partial}{\partial y}]$. $\square$

PROOF OF THEOREM 1. From the definition of $\tilde{\gamma}(\xi)$ and $\Phi_F(x, y)$ we have

$$\begin{aligned} \frac{1}{k} y \frac{\partial}{\partial y} \Phi_F(x, y) &= -\frac{\mu}{k} \Phi_F(x, y) - y^{-\mu} \sum_{n=1}^{\infty} a(n) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s) (n\sqrt{q}y)^{-s} \frac{s}{k} ds e(nx) \\ &= \left(\mu_j - \frac{\mu}{k}\right) \Phi_F(x, y) - y^{-\mu} \sum_{n=1}^{\infty} a(n) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s) \left(\frac{s}{k} + \mu_j\right) (n\sqrt{q}y)^{-s} ds e(nx), \end{aligned}$$

hence

$$\begin{aligned} \left(-\frac{1}{k} y \frac{\partial}{\partial y} + \mu_j - \frac{\mu}{k}\right) \Phi_F(x, y) \\ = y^{-\mu} \sum_{n=1}^{\infty} a(n) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s) \left(\frac{s}{k} + \mu_j\right) (n\sqrt{q}y)^{-s} ds e(nx). \end{aligned}$$

Therefore

$$\begin{aligned} \prod_{j=1}^r \left(-\frac{1}{k} y \frac{\partial}{\partial y} + \mu_j - \frac{\mu}{k}\right) \Phi_F(x, y) \\ = y^{-\mu} \sum_{n=1}^{\infty} a(n) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s) \prod_{j=1}^r \left(\frac{s}{k} + \mu_j\right) (n\sqrt{q}y)^{-s} ds e(nx). \quad (5) \end{aligned}$$

By (4) and the factorial formula for the Γ function we have

$$\gamma(s) \prod_{j=1}^r \left(\frac{s}{k} + \mu_j\right) = Q^s \prod_{j=1}^r \Gamma\left(\frac{s}{k} + \mu_j + 1\right) = Q^{-k} \gamma(s+k),$$

hence, by the substitution $s+k=w$ and using Cauchy's theorem, the right hand side of (5) becomes

$$\begin{aligned} Q^{-k} y^{-\mu} \sum_{n=1}^{\infty} a(n) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s+k) (n\sqrt{q}y)^{-s} ds e(nx) \\ = Q^{-k} y^{-\mu} (\sqrt{q}y)^k \sum_{n=1}^{\infty} a(n) n^k \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(w) (n\sqrt{q}y)^{-w} dw e(nx) \\ = Q^{-k} y^{-\mu} (\sqrt{q}y)^k \sum_{n=1}^{\infty} a(n) n^k \tilde{\gamma}(n\sqrt{q}y) e(nx) \\ = \left(\frac{\sqrt{q}}{2\pi i Q}\right)^k y^k \frac{\partial^k}{\partial x^k} \Phi_F(x, y). \quad (6) \end{aligned}$$

The result follows now from (5), (6) and the definition of q . $\square$

SPECIAL CASES OF THEOREM 1. Let us examine again two classical cases of Theorem 1.

1. Holomorphic cusp forms. In this case by (1) we have

$$\mathcal{D} = -y \frac{\partial}{\partial y} + iy \frac{\partial}{\partial x} = iy \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial x} \right),$$

hence $\mathcal{D}\Phi_F(x, y) = 0$ means

$$\frac{\partial \Phi_F(x, y)}{\partial x} = -i \frac{\partial \Phi_F(x, y)}{\partial y}$$

i.e. the Cauchy–Riemann equations. But, see (2), $\Phi_F(x, y) = (2\pi)^\mu f(z)$, hence $\mathcal{D}\Phi_F(x, y) = 0$ is equivalent to the fact that $f(z)$ is holomorphic.

2. Maass forms. In this case we have

$$\begin{aligned} \mathcal{D} &= \left(-\frac{y}{2} \frac{\partial}{\partial y} + \frac{\varepsilon + i\kappa}{2} - \frac{\varepsilon - 1/2}{2} \right) \left(-\frac{y}{2} \frac{\partial}{\partial y} + \frac{\varepsilon - i\kappa}{2} - \frac{\varepsilon - 1/2}{2} \right) + \left(\frac{y}{2} \right)^2 \frac{\partial^2}{\partial x^2} \\ &= \frac{y^2}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{4} \left(\frac{1}{4} + \kappa^2 \right). \end{aligned}$$

Hence $\mathcal{D}\Phi_F(x, y) = 0$ becomes

$$-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi_F(x, y) = \left(\frac{1}{4} + \kappa^2 \right) \Phi_F(x, y)$$

i.e. $\Phi_F(x, y)$ is an eigenfunction of the hyperbolic laplacian, with eigenvalue $\frac{1}{4} + \kappa^2$, as expected by (3). $\square$

The classification of the degree 2 L -functions in the Selberg class (i.e., roughly, the Dirichlet series with functional equation of degree 2 and Euler product) is a well known open problem, see SELBERG [11], CONREY–GHOSH [2] and our survey papers [5], [4], [7], [8], [9]. Roughly speaking, it is expected that such functions are the L -functions associated with the holomorphic and non-holomorphic modular forms. The classification of the degree 2 functions in the extended Selberg class (where no Euler product is assumed) is more difficult, and as far as we know there isn't even a clear expectation about it. The above special cases suggest that the functions $\Phi_F(x, y)$ are, in the general case, the analogs of the classical modular forms. Such analogy is supported also by the following extension of the Hecke correspondence theorem; we refer e.g. to BERNDT–KNOPP [1]

for an account of Hecke's correspondence theorem. Suppose that the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

is absolutely convergent for $\sigma > 1$, and let $\mu \in \mathbb{C}$, $q > 0$, $Q > 0$, $\lambda_j > 0$ and $\Re \mu_j \geq 0$. We write $\gamma(s)$, $\tilde{\gamma}(\xi)$ and $\Phi_F(x, y)$ as above, and

$$\tilde{\gamma}^*(\xi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \tilde{\gamma}(s) \xi^{-s} ds \quad \Phi_F^*(x, y) = y^{-\bar{\mu}} \sum_{n=1}^{\infty} \overline{a(n)} \tilde{\gamma}^*(n\sqrt{q}y) e(nx).$$

With this notation our general correspondence theorem is

Theorem 2. *Let $|\omega| = 1$. With the above notation the following statements are equivalent.*

- (i) $F(s)$ extends to an entire function of finite order and satisfies the functional equation

$$\gamma(s)F(s) = \omega \bar{\gamma}(1-s) \bar{F}(1-s).$$

- (ii) For $y > 0$ we have

$$\Phi_F(0, y) = \omega q^{-\bar{\mu}-1/2} y^{-\bar{\mu}-\mu-1} \Phi_F^* \left(0, \frac{1}{qy} \right).$$

Remarks. 1. Results of type of Theorem 2 already exist in the literature, mainly due to S. Bochner and his collaborators and followers. We refer to Chapters 7 and 8 of BERNDT–KNOPP [1] and to the literature quoted there for several variants of the principle underlying Theorem 2.

2. The condition that $F(s)$ is entire is not crucial in Theorem 2. Indeed, the same argument proves a version of Theorem 2 where $F(s)$ in (i) has a pole at $s = 1$ and the modular relation in (ii) is slightly modified by adding terms corresponding to the residue of $F(s)$ at $s = 1$. $\square$

PROOF OF THEOREM 2. We first show that (i) implies (ii). Since $F(s)$ has polynomial growth, the integrals below have good convergence properties, justifying our formal manipulations. Note first that, thanks to the functional equation, $F(s)\gamma(s)$ is an entire function since $F(s)$ is entire and $\gamma(s)$ is holomorphic for $\sigma > 0$. Hence by Cauchy's theorem we have

$$\Phi_F(0, y) = y^{-\mu} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s) F(s) (\sqrt{q}y)^{-s} ds$$

$$\begin{aligned}
&= \omega y^{-\mu} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \bar{\gamma}(1-s) \bar{F}(1-s) (\sqrt{q}y)^{-s} ds \\
&= \omega y^{-\mu} \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} \bar{\gamma}(w) \bar{F}(w) (\sqrt{q}y)^{w-1} dw \\
&= \omega y^{-\mu} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \bar{\gamma}(w) \bar{F}(w) (\sqrt{q}y)^{w-1} dw \\
&= \omega q^{-1/2} y^{-\mu-1} \sum_{n=1}^{\infty} \overline{a(n)} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \bar{\gamma}(w) \left(\frac{n\sqrt{q}}{qy} \right)^{-w} dw,
\end{aligned}$$

hence

$$\Phi_F(0, y) = \omega q^{-\bar{\mu}-1/2} y^{-\bar{\mu}-\mu-1} \Phi_F^* \left(0, \frac{1}{qy} \right)$$

and the first statement follows. Now we prove that (ii) implies (i). We have

$$\left(\frac{y}{\sqrt{q}} \right)^{\mu} \Phi_F \left(0, \frac{y}{\sqrt{q}} \right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s) F(s) y^{-s} ds,$$

hence by the inversion formula of Mellin's transform we get

$$\gamma(s) F(s) = q^{-\mu/2} \int_0^{\infty} \Phi_F \left(0, \frac{y}{\sqrt{q}} \right) y^{s+\mu-1} dy.$$

Now we apply the well known trick of splitting the integration over $(0, \infty)$ into $(0, 1) \cup (1, \infty)$ and then transforming the integral over $(0, 1)$. We get

$$\begin{aligned}
q^{-\mu/2} \int_0^1 \Phi_F \left(0, \frac{y}{\sqrt{q}} \right) y^{s+\mu-1} dy &= \omega q^{-\bar{\mu}/2} \int_0^1 \Phi_F^* \left(0, \frac{1}{\sqrt{q}y} \right) y^{s-\bar{\mu}-2} dy \\
&= \omega q^{-\bar{\mu}/2} \int_1^{\infty} \Phi_F^* \left(0, \frac{y}{\sqrt{q}} \right) y^{-s+\bar{\mu}} dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
\gamma(s) F(s) &= q^{-\mu/2} \int_1^{\infty} \Phi_F \left(0, \frac{y}{\sqrt{q}} \right) y^{s+\mu-1} dy \\
&\quad + \omega q^{-\bar{\mu}/2} \int_1^{\infty} \Phi_F^* \left(0, \frac{y}{\sqrt{q}} \right) y^{-s+\bar{\mu}} dy.
\end{aligned} \tag{7}$$

Thanks to the decay properties of $\tilde{\gamma}(\xi)$ and $\tilde{\gamma}^*(\xi)$ as $\xi \rightarrow \infty$, the right hand side of (7) provides the analytic continuation of $\gamma(s)F(s)$ to $\mathbb{C}$, as well as the functional equation. The other properties in (i) follow from this in a standard way. $\square$

ACKNOWLEDGMENTS. This research was partially supported by the Istituto Nazionale di Alta Matematica, by a grant PRIN2008 and by grant N N201 605940 of the National Science Centre.

References

- [1] B. C. BERNDT and M. I. KNOPP, Hecke's Theory of Modular Forms and Dirichlet Series, *World Scientific*, 2008.
- [2] J. B. CONREY and A. GHOSH, On the Selberg class of Dirichlet series: small degrees, *Duke Math. J.* **72** (1993), 673–693.
- [3] H. IWANIEC, Topics in classical automorphic forms, *Graduate Studies in Mathematics* **17**, American Mathematical Society, Providence, RI, 1997.
- [4] J. KACZOROWSKI, Axiomatic theory of L -functions: the Selberg class, In: Analytic Number Theory, C.I.M.E. Summer School, Cetraro (Italy) 2002, (A. Perelli and C. Viola, eds.), *Springer L.N. 1891*, 2006.
- [5] J. KACZOROWSKI and A. PERELLI, The Selberg class: a survey, In: Number Theory in Progress, Proc. Conf. in Honor of A. Schinzel, (K. Györy et al., eds.), *de Gruyter*, 1999.
- [6] J. KACZOROWSKI and A. PERELLI, On the structure of the Selberg class, II: invariants and conjectures, *J. Reine Angew. Math.* **524** (2000), 73–96.
- [7] A. PERELLI, A survey of the Selberg class of L -functions, part I, *Milan J. Math.* **73** (2005), 19–52.
- [8] A. PERELLI, A survey of the Selberg class of L -functions, part II, *Riv. Mat. Univ. Parma* (7) **3*** (2004), 83–118.
- [9] A. PERELLI, Non-linear twists of L -functions: a survey, *Milan J. Math.* **78** (2010), 117–134.
- [10] F. OBERHETTINGER, Tables of Mellin Transforms, *Springer Verlag*, 1974.
- [11] A. SELBERG, Old and new conjectures and results about a class of Dirichlet series, In: Proc. Amalfi Conf. Analytic Number Theory, (E. Bombieri et al., eds.), *Università di Salerno*, 1992, 367–385, *Collected Papers*, vol. II, 47–63, *Springer Verlag*, 1991.
- [12] A. TERRAS, Harmonic Analysis on Symmetric Spaces and Applications I, *Springer Verlag*, 1985.

JERZY KACZOROWSKI
FACULTY OF MATHEMATICS
AND COMPUTER SCIENCE
A. MICKIEWICZ UNIVERSITY
61-614 POZNAŃ
POLAND

AND

INSTITUTE OF MATHEMATICS
OF THE POLISH ACADEMY OF SCIENCES
00-956 WARSAW
POLAND

E-mail: kjerzy@amu.edu.pl

ALBERTO PERELLI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI GENOVA
VIA DODECANESO 35
16146 GENOVA
ITALY

E-mail: perelli@dima.unige.it

(Received February 14, 2011; revised September 7, 2011)