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On the Diophantine equation $L_n = \binom{x}{5}$

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Dedicated to Professors K. Győry and A. Sárközy on their 70th birthdays and Professors A. Pethő and J. Pintz on their 60th birthdays

Abstract. In this paper we determine all integral solutions (n, x) of the Diophantine equation $L_n = \binom{x}{5}$, where L_n is the n-th Lucas number which is defined as follows, $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n > 1$. We follow ideas described in [10], that is we combine Baker's method and the so-called Mordell–Weil sieve to show that the only positive solution is $(n, x) = (1, 5)$.

1. Introduction

There are many articles concerning the Diophantine equation

$$R_n = P(x),$$

where R_n is a linear recursive sequence and $P \in \mathbb{Z}[X]$ is a polynomial. Several papers have been published identifying perfect powers, products of consecutive integers, binomial coefficients, figurate numbers and power sums in the Fibonacci, Lucas, Pell and associated Pell sequences. These binary recurrence sequences are defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2,$$

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \qquad \text{for } n \geq 2,$$

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$$P_0 = 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} \quad \text{for } n \geq 2,$$

$$Q_0 = 1, \quad Q_1 = 1, \quad Q_n = 2Q_{n-1} + Q_{n-2} \quad \text{for } n \geq 2.$$

It follows from a result by Ljunggren [24] that the only squares in the Fibonacci sequence are $F_0 = 0$, $F_1 = F_2 = 1$, $F_{12} = 144$. Later it was rediscovered by Cohn [16], [17] and Wyler [46]. Alfred [1] and Cohn [18] determined the perfect squares in the Lucas sequence. In case of the Pell sequence Pethő [32] and independently Cohn [19] obtained the complete list of perfect squares. London and Finkelstein [25] and Pethő [30] proved that the only cubes in the Fibonacci sequence are $F_0 = 0$, $F_1 = F_2 = 1$ and $F_6 = 8$. London and Finkelstein [25] also showed the the only cube in the Lucas sequence is 1. Higher powers were determined by Pethő [31]. Bugeaud, Mignotte and Siksek [13] applied a combination of Baker's method, modular approach and some classical techniques to show that the perfect powers in the Fibonacci sequence are 0,1,8 and 144, and the perfect powers in the Lucas sequence are 1 and 4.

Another interesting problem is to determine triangular numbers, numbers of the form $T_x = \frac{x(x+1)}{2}$ in binary recurrence sequences. Ming [26] proved that the only triangular numbers in the Fibonacci sequence are $F_0 = 0$, $F_1 = F_2 = 1$, $F_4 = 3$, $F_8 = 21$ and $F_{10} = 55$. It was shown by Ming [27] that $L_1 = 1$, $L_2 = 3$ and $L_{18} = 5778$ are the triangular numbers in the Lucas sequence. In case of the Pell sequence McDaniel [29] proved that the only triangular number is 1. Since $T_x = \binom{x}{2}$, it was a natural question to ask for all solutions of the Diophantine equations

$$F_n = \binom{x}{k}, \qquad L_n = \binom{x}{k}, \qquad P_n = \binom{x}{k}, \qquad Q_n = \binom{x}{k}.$$

It was Szalay [42] who solved the equations $F_n, L_n, P_n = \binom{x}{3}$. Later Szalay [41] also treated the equations $F_n, L_n = \binom{x}{3}$ and $F_n, L_n, P_n = \sum_{i=1}^{x} i^3$. Kovács [22] solved completely some related combinatorial Diophantine equations, e.g.

$$P_n = \binom{x}{4}$$

and

$$F_n = \Pi_4(x) = x(x+1)(x+2)(x+3).$$

Tengely [43] determined the g-gonal numbers in the Fibonacci, Lucas, Pell and associated Pell sequences for $g \leq 20$. In this paper we consider the Diophantine equation

$$L_n = \binom{x}{5}. \tag{1}$$

We prove the following result.

Theorem 1. *The only positive solution of equation* (1) *is* $(n, x) = (1, 5)$.

2. Auxiliary results

Consider the hyperelliptic curve

$$\mathcal{C}: \quad y^2 = F(x) := x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0, \tag{2}$$

where $b_i \in \mathbb{Z}$. Let α be a root of F and $J(\mathbb{Q})$ be the Jacobian of the curve $\mathcal{C}$. We have that

$$x - \alpha = \kappa\xi^2$$

where κ, $\xi \in K = \mathbb{Q}(\alpha)$ and κ comes from a finite set. By knowing the Mordell–Weil group of the curve $\mathcal{C}$ it is possible to provide a method to compute such a finite set. To each coset representative $\sum_{i=1}^{m}(P_i - \infty)$ of $J(\mathbb{Q})/2J(\mathbb{Q})$ we associate

$$\kappa = \prod_{i=1}^{m} \left(\gamma_i - \alpha d_i^2\right),$$

where the set $\{P_1, \ldots, P_m\}$ is stable under the action of Galois, all $y(P_i)$ are non-zero and $x(P_i) = \gamma_i/d_i^2$ where γ_i is an algebraic integer and $d_i \in \mathbb{Z}_{\geq 1}$. If P_i, P_j are conjugate then we may suppose that $d_i = d_j$ and so γ_i, γ_j are conjugate. We have the following lemma (Lemma 3.1 in [10]).

Lemma 1. *Let $\mathcal{K}$ be a set of κ values associated as above to a complete set of coset representatives of $J(\mathbb{Q})/2J(\mathbb{Q})$. Then $\mathcal{K}$ is a finite subset of $\mathcal{O}_K$ and if (x, y) is an integral point on the curve* (2) *then $x - \alpha = \kappa\xi^2$ for some $\kappa \in \mathcal{K}$ and $\xi \in K$.*

As an application of his theory of lower bounds for linear forms in logarithms, Baker [2] gave an explicit upper bound for the size of integral solutions of hyperelliptic curves. This result has been improved by many authors (see e.g. [3], [4], [6], [11], [34], [36], [37] and [45]).

In [10] an improved completely explicit upper bound were proved combining ideas from [11], [12], [13], [23], [28], [33], [45], [44]. Now we will state the theorem which gives the improved bound. We introduce some notation. Let K be a number field of degree d and let r be its unit rank and R its regulator. For $\alpha \in K$ we denote by $\mathrm{h}(\alpha)$ the logarithmic height of the element α. Let

$$\partial_K = \begin{cases} \dfrac{\log 2}{d} & \text{if } \ d = 1, 2, \\ \dfrac{1}{4}\left(\dfrac{\log\log d}{\log d}\right)^3 & \text{if } \ d \geq 3 \end{cases}$$

and

$$\partial'_K = \left(1 + \frac{\pi^2}{\partial_K^2}\right)^{1/2}.$$

Define the constants

$$c_1(K) = \frac{(r\,!)^2}{2^{r-1}d^r}, \qquad c_2(K) = c_1(K)\left(\frac{d}{\partial_K}\right)^{r-1},$$

$$c_3(K) = c_1(K)\frac{d^r}{\partial_K}, \qquad c_4(K) = rdc_3(K), \qquad c_5(K) = \frac{r^{r+1}}{2\partial_K^{r-1}}.$$

Let

$$\partial_{L/K} = \max\left\{[L:\mathbb{Q}]\ \ [K:\mathbb{Q}]\partial'_K,\ \frac{0.16[K:\mathbb{Q}]}{\partial_K}\right\},$$

where $K \subseteq L$ are number fields. Define

$$C(K,n) := 3 \cdot 30^{n+4} \cdot (n+1)^{5.5}\, d^2\, (1 + \log d).$$

The following theorem will be used to get an upper bound for the size of the integral solutions of our equation. It is Theorem 3 in [10].

Theorem 2. *Let α be an algebraic integer of degree at least 3 and κ be an integer belonging to K. Denote by α_1, α_2, α_3 distinct conjugates of α and by κ_1, κ_2, κ_3 the corresponding conjugates of κ. Let*

$$K_1 = \mathbb{Q}(\alpha_1, \alpha_2, \sqrt{\kappa_1\kappa_2}), \quad K_2 = \mathbb{Q}(\alpha_1, \alpha_3, \sqrt{\kappa_1\kappa_3}), \quad K_3 = \mathbb{Q}(\alpha_2, \alpha_3, \sqrt{\kappa_2\kappa_3}),$$

and

$$L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \sqrt{\kappa_1\kappa_2}, \sqrt{\kappa_1\kappa_3}).$$

In what follows R stands for an upper bound for the regulators of K_1, K_2 and K_3 and r denotes the maximum of the unit ranks of K_1, K_2, K_3. Let

$$c_j^* = \max_{1\le i\le 3} c_j(K_i)$$

and

$$N = \max_{1\le i,j\le 3} \left|\mathrm{Norm}_{\mathbb{Q}(\alpha_i,\alpha_j)/\mathbb{Q}}(\kappa_i(\alpha_i - \alpha_j))\right|^2$$

and

$$H^* = c_5^* R + \frac{\log N}{\min_{1\le i\le 3}[K_i:\mathbb{Q}]} + \mathrm{h}(\kappa).$$

Define

$$A_1^* = 2H^* \cdot C(L, 2r+1) \cdot (c_1^*)^2 \partial_{L/L} \cdot \left(\max_{1\le i\le 3} \partial_{L/K_i}\right)^{2r} \cdot R^2,$$

and

$$A_2^* = 2H^* + A_1^* + A_1^* \log\{(2r+1) \cdot \max\{c_4^*, 1\}\}.$$

If $x \in \mathbb{Z}\backslash\{0\}$ *satisfies* $x - \alpha = \kappa\xi^2$ *for some* $\xi \in K$ *then*

$$\log|x| \le 8A_1^* \log(4A_1^*) + 8A_2^* + H^* + 20\log 2 + 13\,\mathrm{h}(\kappa) + 19\,\mathrm{h}(\alpha).$$

To obtain a lower bound for the possible unknown integer solutions we are going to use the so-called Mordell–Weil sieve. The Mordell–Weil sieve has been successfully applied to prove the non-existence of rational points on curves (see e.g. [7], [9], [21] and [35]).

Let $C/\mathbb{Q}$ be a smooth projective curve (in our case a hyperelliptic curve) of genus $g \ge 2$. Let J be its Jacobian. We assume the knowledge of some rational point on C,so let D be a fixed rational point on C and let $\jmath$ be the corresponding Abel–Jacobi map:

$$\jmath : C \to J, \qquad P \mapsto [P - D].$$

Let W be the image in J of the known rational points on C and $D_1, \dots, D_r$ generators for the free part of $J(\mathbb{Q})$. By using the Mordell–Weil sieve we are going to obtain a very large and smooth integer B such that

$$\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q}).$$

Let

$$\phi : \mathbb{Z}^r \to J(\mathbb{Q}), \qquad \phi(a_1, \dots, a_r) = \sum a_i D_i,$$

so that the image of ϕ is the free part of $J(\mathbb{Q})$. The variant of the Mordell–Weil sieve explained in [10] provides a method to obtain a very long decreasing sequence of lattices in $\mathbb{Z}^r$

$$B\mathbb{Z}^r = L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_k$$

such that

$$\jmath(C(\mathbb{Q})) \subset W + \phi(L_j)$$

for $j = 1, \dots, k$.

The next lemma [10, Lemma 12.1] gives a lower bound for the size of rational points whose image are not in the set W.

Lemma 2. *Let* W *be a finite subset of* $J(\mathbb{Q})$ *and* L *be a sublattice of* $\mathbb{Z}^r$. *Suppose that* $\jmath(C(\mathbb{Q})) \subset W + \phi(L)$. *Let* μ_1 *be a lower bound for* $h - \hat{h}$ *and*

$$\mu_2 = \max\left\{\sqrt{\hat{h}(w)} : w \in W\right\}.$$

Denote by M the height-pairing matrix for the Mordell–Weil basis $D_1, \dots, D_r$ and let $\lambda_1, \dots, \lambda_r$ be its eigenvalues. Let

$$\mu_3 = \min\left\{\sqrt{\lambda_j} : j = 1, \dots, r\right\}$$

and $m(L)$ the Euclidean norm of the shortest non-zero vector of L. Then, for any $P \in C(\mathbb{Q})$, either $\jmath(P) \in W$ or

$$h(\jmath(P)) \geq (\mu_3 m(L) - \mu_2)^2 + \mu_1.$$

3. Proof of Theorem 1

In this section first we prove a lemma and than we use it to prove Theorem 1.

Lemma 3. (a) *The integral solutions of the equation*

$$\mathcal{C}^+ : Y^2 = X^2(X+15)^2(X+20) + 180000000 \tag{3}$$

are

$$(X, Y) \in \{(25, -15000), (25, 15000)\}.$$

(b) *There are no integral solution of the equation*

$$\mathcal{C}^- : Y^2 = X^2(X+15)^2(X+20) - 180000000. \tag{4}$$

Proof of Lemma 3. We start with the proof of part (a). Let $J(\mathbb{Q})^+$ be the Jacobian of the genus two curve (3). Using MAGMA [5] we obtain that $J(\mathbb{Q})^+$ is free of rank 1 with Mordell–Weil basis given by

$$D = (25, 15000) - \infty.$$

The MAGMA programs used to compute these data are based on Stoll's papers [38], [39], [40]. The rank of the Jacobian of $\mathcal{C}^+$ is 1, so classical Chabauty's method (see e.g. [14], [15], [20]) can be applied. The `Chabauty` procedure of MAGMA provides an upper bound for the number of rational points on the curve and in this case it is equal to the number of known points. Therefore

$$\mathcal{C}^+(\mathbb{Q}) = \{\infty, (25, \pm 15000)\}.$$

Now we deal with part (b). Let $J(\mathbb{Q})^-$ be the Jacobian of the genus two curve (4). Using MAGMA we determine a Mordell–Weil basis which is given by

$$D_1 = (\omega_1, -200\omega_1) + (\overline{\omega}_1, -200\overline{\omega}_1) - 2\infty,$$

$$D_2 = (\omega_2, 120000) + (\overline{\omega}_2, 120000) - 2\infty,$$

where ω_1 is a root of the polynomial $x^2 - 5x + 1500$ and ω_2 is a root of $x^2 + 195x + 13500$. Let $f = x^2(x+15)^2(x+20) - 180000000$ and α be a root of f. We will choose for coset representatives of $J(\mathbb{Q})^-/2J(\mathbb{Q})^-$ the linear combinations $\sum_{i=1}^{2} n_i D_i$, where $n_i \in \{0, 1\}$. Then

$$x - \alpha = \kappa\xi^2,$$

where $\kappa \in \mathcal{K}$ and $\mathcal{K}$ is constructed as described in Lemma 1. We have that $\mathcal{K} = \{1, \alpha^2 - 5\alpha + 1500, \alpha^2 + 195\alpha + 13500, \alpha^4 + 190\alpha^3 + 14025\alpha^2 + 225000\alpha + 20250000\}$. By local arguments it is possible to restrict the set $\mathcal{K}$ further (see e.g. [7], [8]). In our case one can eliminate

$$\alpha^2 - 5\alpha + 1500, \quad \alpha^2 + 195\alpha + 13500$$

by local computations in $\mathbb{Q}_2$ and

$$\alpha^4 + 190\alpha^3 + 14025\alpha^2 + 225000\alpha + 20250000$$

by local computations in $\mathbb{Q}_3$. It remains to deal with the case $\kappa = 1$. We apply Theorem 2 to get a large upper bound for $\log|x|$. A MAGMA code were written to obtain the bounds appeared in [10], it can be found at
`http://www.warwick.ac.uk/~maseap/progs/intpoint/bounds.m`. We used the above Magma functions to compute an upper bound corresponding to the case $\kappa = 1$. It turned out to be

$$1.58037 \times 10^{285}.$$

The set of known rational points on the curve (4) is $\{\infty\}$. Let W be the image of this set in $J(\mathbb{Q})^-$. Applying the Mordell–Weil implemented by Bruin and Stoll and explained in [10] we obtain that $\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q})^-$, where

$$B = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot \cdot 43 \cdot 47 \cdot 61 \cdot 67 \cdot 79 \cdot 83 \cdot 109 \cdot 113 \cdot 127,$$

that is

$$B = 678957252681082328769065398948800.$$

Now we use an extension of the Mordell–Weil sieve due to Samir Siksek to obtain a very long decreasing sequence of lattices in $\mathbb{Z}^2$. After that we apply Lemma 2 to obtain a lower bound for possible unknown rational points. We get that if (x, y) is an unknown integral point, then

$$\log|x| \geq 7.38833 \times 10^{1076}.$$

This contradicts the bound for $\log|x|$ we obtained by Baker's method. □

Finally we prove Theorem 1.

Proof of Theorem 1. We will use the following well known property of the sequences F_n and L_n:

$$L_n^2 - 5F_n^2 = 4(-1)^n.$$

We have that

$$\binom{x}{5}^2 \pm 4 = 5F_n^2.$$

The above equation can be reduced to two genus two curves given by (3) and (4), where $Y = 5^3 5! F_n$ and $X = 5x^2 - 20x$. By Lemma 3 we have that $X = 25$ and we also have that $X = 5x^2 - 20x$. That is it remains to solve a quadratic equation in x. We obtain that $x \in \{-1, 5\}$. Hence the only positive solution of equation (1) is $(n, x) = (1, 5)$, that is

$$1 = L_1 = \binom{5}{5}. \qquad \square$$

Remark. A similar method could be used to treat the equation $F_n = \binom{x}{5}$, where F_n is the n-th Fibonacci number. We obtain two genus two curves, one of them can be solved by using the ideas described in [10], but in case of the remaining curve the rank of the Jacobian is 4 and we could not obtain a good enough lower bound for the possible unknown integral solutions.

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