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Schur power convexity of Stolarsky means

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Abstract. In this paper, the Schur convexity is generalized to Schur f-convexity, which contains the Schur geometrical convexity, Schur harmonic convexity and so on. When $f : \mathbb{R}_+ \to \mathbb{R}$ is defined by $f(x) = (x^m - 1)/m$ if $m \neq 0$ and $f(x) = \ln x$ if $m = 0$, the necessary and sufficient conditions for f-convexity (is called Schur m-power convexity) of Stolarsky means are given, which generalized and unified certain known results.

1. Introduction and main results

Let $p, q \in \mathbb{R}$ and $a, b \in \mathbb{R}_+ := (0, \infty)$ with $a \neq b$. The so-called Stolarsky means $S_{p,q}(a,b)$ are defined by

$$S_{p,q}(a,b) = \begin{cases} \left(\dfrac{q(a^p - b^p)}{p(a^q - b^q)}\right)^{\frac{1}{p-q}} & \text{if } \ pq(p-q) \neq 0, \\ \left(\dfrac{a^p - b^p}{p(\ln a - \ln b)}\right)^{\frac{1}{p}} & \text{if } \ p \neq 0, \ q = 0, \\ \left(\dfrac{a^q - b^q}{q(\ln a - \ln b)}\right)^{\frac{1}{q}} & \text{if } \ q \neq 0, \ p = 0, \\ \exp\left(\dfrac{a^p \ln a - b^p \ln b}{a^p - b^p} - \dfrac{1}{p}\right) & \text{if } \ p = q \neq 0, \\ \sqrt{ab} & \text{if } \ p = q = 0. \end{cases} \tag{1.1}$$

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Also, $S_{p,q}(a,a) = a$. It is known that the Stolarsky means $S_{p,q}(a,b)$ are C^∞ function on the domain $\{(p,q,a,b) : p,q \in \mathbb{R},\ a,b \in \mathbb{R}_+\}$ (see [19, Lemma 1]), and obviously symmetric with respect to a, b and p, q.

Most of the classical two variable means are special cases of $S_{p,q}(a,b)$, for example, $S_{1,2} = A$ is the arithmetic means, $S_{0,0} = G$ is the geometric mean, $S_{-1,-2} = H$ is the harmonic mean, $S_{1,0} = L$ is the logarithmic mean, $S_{1,1} = I$ is the identric mean (exponential mean), and more generally, the r-th power mean is equal to $S_{r,2r}$. The basic properties of Stolarsky means, as well as their comparison theorems, log-convexity, and inequalities were studied in papers [3], [8], [12], [15], [16], [18], [19], [20], [23], [24], [25], [26], [27], [28], [36], [37], [42], [43], [44], [46], [47].

Schur convexity was introduced by Schur in 1923 [21], and it has many important applications in analytic inequalities [2], [11], [49], linear regression [35], graphs and matrices [7], combinatorial optimization [14], information-theoretic topics [9], Gamma functions [22], stochastic orderings [32], reliability [13], and other related fields.

In recent years, the Schur convexity and Schur geometrical convexity of $S_{p,q}(a,b)$ have attracted the attention of a considerable number of mathematicians [4], [5, 17], [29], [30], [31], [33]. Qi [30] first proved that the Stolarsky means $S_{p,q}(a,b)$ are Schur convex on $(-\infty,0] \times (-\infty,0]$ and Schur concave on $[0,\infty) \times [0,\infty)$ with respect to (p,q) for fixed $a,b > 0$ with $a \neq b$. Yang [45] improved Qi's result and proved that Stolarsky means $S_{p,q}(a,b)$ are Schur convex with respect to (p,q) for fixed $a,b > 0$ with $a \neq b$ if and only if $p+q < 0$ and Schur concave if and only if $p+q > 0$.

Qi *et al.* [29] tried to obtain the Schur convexity of $S_{p,q}(a,b)$ with respect to (a,b) for fixed (p,q) and declared an incorrect conclusion. Shi *et al.* [33] observed that the above conclusion is wrong and obtained a sufficient condition for the Schur convexity of $S_{p,q}(a,b)$ with respect to (a,b). Chu and Zhang [5] improved Shi's results and gave a necessary and sufficient condition. This perfectly solved the Schur convexity of Stolarsky means with respect to (a,b).

The Schur geometrical convexity was introduced by Zhang [50], and there has many interesting results [10], [34], [39], [40]. For the Schur geometrical convexity of Stolarsky means $S_{p,q}(a,b)$, Chu and Zhang [4] proved that they are Schur geometrically convex with respect to $(a,b) \in \mathbb{R}_+^2$ if $p+q \geq 0$ and Schur geometrically concave if $p+q \leq 0$. Li *et al.* [17] also investigated the Schur geometrical convexity of generalized exponent mean $I_p(a,b)$. In 2010, a necessary and sufficient condition for Schur geometrical convexity of the four-parameter means with respeto to a pair of parameters was given in [48]. This give a unified treatment for Schur geometrical convexity of Stolarsky and Gini means.

Recently, Anderson *et al.* [1] discussed an attractive class of inequalities, which arise from the notation of harmonic convexity. And then it was started to research for *Schur harmonic convexity.* Chu *et al.* [6] showed that the Hamy symmetric function is Schur harmonic convex and obtained some analytic inequalities including the well-known Weierstrass inequalities. Xiao [41] proved that the Lehmer mean values $L_p(a, b)$ are Schur harmonic convex (Schur harmonic concave) with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if $p \geq (\leq)0$.

The purpose of this paper is to generalize the notion of Schur convexity and investigated the so-called *Schur power convexity* of Stolarsky means $S_{p,q}(a, b)$. Our main results are as follows.

Theorem 1. *For $m > 0$ and fixed $(p, q) \in \mathbb{R}^2$, Stolarsky mean $S_{p,q}(a, b)$ is Schur m-power convex with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if $p + q \geq 3m$ and $\min(p, q) \geq m$.*

Theorem 2. *For $m > 0$ and fixed $(p, q) \in \mathbb{R}^2$, Stolarsky mean $S_{p,q}(a, b)$ is Schur m-power concave with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if $p + q \leq 3m$ and $\min(p, q) \leq m$.*

Theorem 3. *For $m < 0$ and fixed $(p, q) \in \mathbb{R}^2$, Stolarsky mean $S_{p,q}(a, b)$ is Schur m-power convex with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if $p + q \geq 3m$ and $\max(p, q) \geq m$.*

Theorem 4. *For $m < 0$ and fixed $(p, q) \in \mathbb{R}^2$, Stolarsky mean $S_{p,q}(a, b)$ is Schur m-power concave with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if $p + q \leq 3m$ and $\max(p, q) \leq m$.*

Theorem 5. *For $m = 0$ and fixed $(p, q) \in \mathbb{R}^2$, Stolarsky mean $S_{p,q}(a, b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if $p + q \geq (\leq)0$.*

The organization of the paper is as follows. In Section 2, based on the notion and lemmas of Schur convexity, we introduce the definition of Schur f-convex and Schur f-concave function, and prove decision theorem for Schur f-convexity. As a special case, the definition and decision theorem of Schur power convexity are deduced. In Section 3, some lemmas are given. In Section 4, our main results are proved.

2. Schur f-convexity and Schur power convexity

For convenience of readers, we recall some definitions as follows.

Definition 1 ([21], [38]). Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n (n \geq 2)$.

(i) x is said to by majorized by y (in symbol $\mathbf{x} \prec \mathbf{y}$) if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{for } 1 \leq k \leq n-1, \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \tag{2.1}$$

where $x_{[1]} \geq x_{[2]} \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \cdots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a decreasing order.

(ii) $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$. Let $\Omega \subseteq \mathbb{R}^n (n \geq 2)$. The function $\phi : \Omega \to \mathbb{R}$ is said to be increasing if $\mathbf{x} \geq \mathbf{y}$ implies that $\phi(\mathbf{x}) \geq \phi(\mathbf{y})$. ϕ is said to be decreasing if and only if $-\phi$ is increasing.

(iii) $\Omega \subseteq \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n) \in \Omega$ for all $\mathbf{x}$, $\mathbf{y}$ and all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iv) Let $\Omega \subseteq \mathbb{R}^n (n \geq 2)$ be a set with nonempty interior. Then $\phi : \Omega \to \mathbb{R}$ is said to be Schur convex if $\mathbf{x} \prec \mathbf{y}$ on Ω implies that $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$. ϕ is said to be Schur concave if $-\phi$ is Schur convex.

Definition 2 ([21]). (i) $\Omega \subseteq \mathbb{R}^n (n \geq 2)$ is called symmetric set, if $\mathbf{x} \in \Omega$ implies that $\mathbf{xP} \in \Omega$ for every $n \times n$ permutation matrix $\mathbf{P}$.

(ii) The function $\phi : \Omega \to \mathbb{R}^n$ is called symmetric if for every permutation matrix $\mathbf{P}$, $\phi(\mathbf{xP}) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.

For the Schur convexity, there is the following well-known result.

Lemma 1 ([21], [38]). *Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric set with nonempty interior Ω^0 and $\phi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω^0. Then ϕ is Schur convex (Schur concave) on Ω if and only if ϕ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2}\right) \geq (\leq) 0 \tag{2.2}$$

Next let us define the Schur f-convexity as follows.

Definition 3. Let $\Omega = \mathbb{U}^n (\mathbb{U} \subseteq \mathbb{R})$ and f be a strictly monotone function defined on $\mathbb{U}$. Denote by

$$f(\mathbf{x}) = (f(x_1), f(x_2), \dots, f(x_n)) \quad \text{and} \quad f(\mathbf{y}) = (f(y_1), f(y_2), \dots, f(y_n)).$$

(i) Ω is called a f-convex set if $(f^{-1}(\alpha f(x_1) + \beta f(y_1)), \dots, f^{-1}(\alpha f(x_n) + \beta f(y_n))) \in \Omega$ for all $\mathbf{x}, \mathbf{y} \in \Omega$ and all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

(ii) Let Ω be a set with nonempty interior. Then function $\phi : \Omega \to \mathbb{R}$ is said to be Schur f-convex on Ω if $f(\mathbf{x}) \prec f(\mathbf{y})$ on Ω implies that $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$.

ϕ is said to be Schur f-concave if $-\phi$ is Schur f-convex.

Remark 1. Let $\Omega = \mathbb{U}^n (\mathbb{U} \subseteq \mathbb{R})$ and f be a strictly monotone function defined on $\mathbb{U}$ and $f(\Omega) = \{f(\mathbf{x}) : \mathbf{x} \in \Omega\}$. Then function $\phi : \Omega \to \mathbb{R}$ is Schur f-convex (Schur f-concave) if and only if $\phi \circ f^{-1}$ is Schur convex (Schur concave) on $f(\Omega)$.

Indeed, if function $\phi : \Omega \to \mathbb{R}$ is Schur f-convex, then $\forall \mathbf{x}', \mathbf{y}' \in f(\Omega)$, there are $\mathbf{x}, \mathbf{y} \in \mathbf{\Omega}$ such that $\mathbf{x}' = f(\mathbf{x}), \mathbf{y}' = f(\mathbf{y})$. If $f(\mathbf{x}) \prec f(\mathbf{y})$, that is, $\mathbf{x}' \prec \mathbf{y}'$, then $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$, that is, $\phi((f^{-1}(\mathbf{x}')) \leq \phi((f^{-1}(\mathbf{y}'))$. This shows that $\phi \circ f^{-1}$ is Schur convex on $f(\Omega)$. Conversely, if $\phi \circ f^{-1}$ is Schur convex on $f(\Omega)$, then $\forall \mathbf{x}, \mathbf{y} \in \Omega$ such that $f(\mathbf{x}) \prec f(\mathbf{y})$, we have $\phi((f^{-1}(f(\mathbf{x}))) \leq \phi((f^{-1}(f(\mathbf{y})))$, that is, $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$. This indicates ϕ is Schur f-convex on Ω.

In the same way, we can show that ϕ is Schur f-concave on Ω if and only if $\phi \circ f^{-1}$ is Schur concave on $f(\Omega)$.

Remark 2. Let $\Omega \subseteq \mathbb{R}^n (n \geq 2)$ be a symmetric set and the function $\phi : \Omega \to \mathbb{R}$ be Schur f-convex (Schur f-concave). Then ϕ is symmetric on Ω.

In fact, for any $\mathbf{x} \in \Omega$ and every permutation matrix $\mathbf{P}$, we have $\mathbf{xP} \in \Omega$. Note $\mathbf{xP}$ is another permutation of $\mathbf{x}$, hence $f(\mathbf{x}) \prec f(\mathbf{xP}) \prec f(\mathbf{x})$. Since ϕ is Schur f-convex (Schur f-concave), we have $\phi(\mathbf{x}) \leq (\geq) \phi(\mathbf{xP}) \leq (\geq) \phi(\mathbf{x})$, that is, $\phi(\mathbf{xP}) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega$. This shows that ϕ is symmetric on Ω.

By Lemma 1 and Remark 1, 2, we have the following

Theorem 6. *Assume that $\Omega = \mathbb{U}^n (\mathbb{U} \subseteq \mathbb{R})$ is a symmetric set with nonempty interior Ω^0, f is a strictly monotone and derivable function defined on $\mathbb{U}$, and $\phi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then ϕ is Schur f-convex (Schur f-concave) on Ω if and only if ϕ is symmetric on Ω and*

$$(f(x_1) - f(x_2)) \left(\frac{1}{f'(x_1)} \frac{\partial \phi}{\partial x_1} - \frac{1}{f'(x_2)} \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0 \tag{2.3}$$

holds for any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \Omega^0$ with $x_1 \neq x_2$.

Proof. We easily check that $\phi \circ f^{-1}$ is symmetric on $f(\Omega)$ if and only if ϕ is symmetric on Ω.

By Remark 1 and Lemma 1, $\phi \circ f^{-1}$ is Schur convex (Schur concave) if and only if $\phi \circ f^{-1}$ is symmetric on $f(\Omega)$ and

$$(y_1 - y_2) \left(\frac{\partial (\phi \circ f^{-1})}{\partial y_1} - \frac{\partial (\phi \circ f^{-1})}{\partial y_2} \right) \geq (\leq) 0$$

holds for any $\mathbf{y} \in f(\Omega)^0$ with $y_1 \neq y_2$. Substituting $f^{-1}(y) = x$ yields (2.3), where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$ with $x_1 \neq x_2$.

This proof is finished. □

Putting $f(x) = 1, \ln x, x^{-1}$ in Definition 3 yield the Schur convexity, Schur geometrical convexity and Schur harmonic convexity. It is clear that the Schur f-convexity is a generalization of the Schur convexity mentioned above. In general, we have

Definition 4. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by $f(x) = (x^m - 1)/m$ if $m \neq 0$ and $f(x) = \ln x$ if $m = 0$. Then function $\phi : \Omega(\subseteq \mathbb{R}_+^n) \to \mathbb{R}$ is said to be Schur m-power convex on Ω if $f(\mathbf{x}) \prec f(\mathbf{y})$ on Ω implies that $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$.

ϕ is said to be Schur m-power concave if $-\phi$ is Schur m-power convex.

For Schur power convexity, by Theorem 6 we have

Corollary 1. *Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric set with nonempty interior Ω^0 and $\phi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω^0. Then ϕ is Schur m-power convex (Schur m-power concave) on Ω if and only if ϕ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\frac{\partial \phi}{\partial x_1} - x_2^{1-m}\frac{\partial \phi}{\partial x_2}\right) \geq (\leq) 0 \quad \text{if} \quad m \neq 0, \tag{2.4}$$

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial \phi}{\partial x_1} - x_2\frac{\partial \phi}{\partial x_2}\right) \geq (\leq) 0 \quad \text{if} \quad m = 0 \tag{2.5}$$

holds for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$ with $x_1 \neq x_2$.

3. Lemmas

Lemma 2. *For fixed $(p, q) \in \mathbb{R}^2$, Stolarsky mean $S_{p,q}(a, b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a, b) \in \mathbb{R}_+^2$ if and only if $g(t) \geq (\leq) 0$ for all $t > 0$, where*

$$g(t) = g_{p,q}(t) = \begin{cases} \dfrac{(p-q)\sinh At - p\sinh Bt - q\sinh Ct}{pq(p-q)} & \text{if } pq(p-q) \neq 0, \\ \dfrac{\sinh(p-m)t + \sinh(p+m)t - 2pt\cosh(p-m)t}{-p^2} & \text{if } p \neq 0,\ q = 0, \\ \dfrac{\sinh(q-m)t + \sinh(q+m)t - 2qt\cosh(q-m)t}{-q^2} & \text{if } q \neq 0,\ p = 0, \\ \dfrac{\sinh(2p-m)t + \sinh mt - 2pt\cosh mt}{p^2} & \text{if } p = q \neq 0, \\ -2t^2\sinh mt & \text{if } p = q = 0, \end{cases} \tag{3.1}$$

and

$$A = p + q - m, \quad B = p - q - m \quad C = p - q + m, \tag{3.2}$$

PROOF. Let $m \neq 0$ and $S = S_{p,q} := S_{p,q}(a,b)$ defined by (1.1).

In the case of $pq(p-q) \neq 0$. We have

$$\frac{\partial \ln S}{\partial a} = \frac{1}{S}\frac{\partial S}{\partial a} = \frac{1}{p-q}\left(\frac{pa^{p-1}}{a^p - b^p} - \frac{qa^{q-1}}{a^q - b^q}\right),$$
$$\frac{\partial \ln S}{\partial b} = \frac{1}{S}\frac{\partial S}{\partial b} = \frac{1}{p-q}\left(\frac{-pb^{p-1}}{a^p - b^p} - \frac{-qb^{q-1}}{a^q - b^q}\right),$$

hence,

$$a^{1-m}\frac{\partial S}{\partial a} - b^{1-m}\frac{\partial S}{\partial b} = \frac{S}{p-q}\left(p\frac{a^{p-m} + b^{p-m}}{a^p - b^p} - q\frac{a^{q-m} + b^{q-m}}{a^q - b^q}\right).$$

Substituting $\ln\sqrt{a/b} = t$ and using $\sinh x = \frac{1}{2}(e^x - e^{-x}), \cosh x = \frac{1}{2}(e^x + e^{-x})$, the right hand side above can be written as

$$\begin{aligned} a^{1-m}\frac{\partial S}{\partial a} - b^{1-m}\frac{\partial S}{\partial b} &= \frac{S(ab)^{-m/2}}{p-q}\left(p\frac{\cosh(p-m)t}{\sinh pt} - q\frac{\cosh(q-m)t}{\sinh qt}\right) \\ &= \frac{S}{2(ab)^{m/2}}\frac{p}{\sinh pt}\frac{q}{\sinh qt} \\ &\quad \cdot 2\frac{p\cosh(p-m)t\sinh qt - q\cosh(q-m)t\sinh pt}{pq(p-q)}. \end{aligned}$$

Using the “product into sum” formula for hyperbolic functions and (3.2), we have

$$\begin{aligned} \Delta &:= \frac{a^m - b^m}{m}\left(a^{1-m}\frac{\partial S_{p,q}}{\partial a} - b^{1-m}\frac{\partial S_{p,q}}{\partial b}\right) \\ &= \frac{a^m - b^m}{m(a-b)}\frac{(a-b)S_{p,q}}{2\,(ab)^{m/2}}\frac{p}{\sinh pt}\frac{q}{\sinh qt}\frac{(p-q)\sinh At - p\sinh Bt - q\sinh Ct}{pq(p-q)} \\ &= d_{p,q}(t)\cdot g_{p,q}(t), \end{aligned}$$

where

$$d_{p,q}(t) = \frac{a^m - b^m}{m(a-b)}\frac{(a-b)S_{p,q}}{2\,(ab)^{m/2}}\frac{p}{\sinh pt}\frac{q}{\sinh qt} \quad (pq(p-q) \neq 0) \tag{3.3}$$

and $g_{p,q}(t)$ is defined by (3.1).

In the case of $p \neq q = 0$. Since $S_{p,q}(a,b) \in C^\infty$ we have

$$\frac{\partial S_{p,0}}{\partial a} = \lim_{q\to 0}\frac{\partial S_{p,q}}{\partial a}, \qquad \frac{\partial S_{p,0}}{\partial b} = \lim_{q\to 0}\frac{\partial S_{p,q}}{\partial b},$$

$$\frac{\partial S_{p,p}}{\partial a}=\lim_{q\to p}\frac{\partial S_{p,q}}{\partial a},\qquad \frac{\partial S_{p,p}}{\partial b}=\lim_{q\to p}\frac{\partial S_{p,q}}{\partial b},$$
$$\frac{\partial S_{0,0}}{\partial a}=\lim_{p\to 0}\frac{\partial S_{p,p}}{\partial a},\qquad \frac{\partial S_{0,0}}{\partial b}=\lim_{p\to 0}\frac{\partial S_{p,p}}{\partial b}.$$

It follows that

$$\begin{aligned}\Delta &= \frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial S_{p,0}}{\partial a}-b^{1-m}\frac{\partial S_{p,0}}{\partial b}\right)\\ &=\lim_{q\to 0}\left(\frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial S_{p,q}}{\partial a}-b^{1-m}\frac{\partial S_{p,q}}{\partial b}\right)\right)\\ &=\lim_{q\to 0}\left(d_{p,q}(t)g_{p,q}(t)\right)=g_{p,0}(t)\lim_{q\to 0}d_{p,q}(t).\end{aligned}$$

Likewise, in the case of $q\neq p=0$, we have

$$\Delta=\frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial S_{0,q}}{\partial a}-b^{1-m}\frac{\partial S_{0,q}}{\partial b}\right)=g_{0,q}(t)\lim_{p\to 0}d_{p,q}(t).$$

In the case of $p=q\neq 0$, we have

$$\Delta=\frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial S_{p,p}}{\partial a}-b^{1-m}\frac{\partial S_{p,p}}{\partial b}\right)=g_{p,p}(t)\lim_{q\to p}d_{p,q}(t),$$

In the case of $p=q=0$, we have

$$\Delta=\frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial S_{0,0}}{\partial a}-b^{1-m}\frac{\partial S_{0,0}}{\partial b}\right)=g_{0,0}(t)\lim_{p\to 0}d_{p,p}(t),$$

Summarizing all cases above yield

$$\Delta=\frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial S}{\partial a}-b^{1-m}\frac{\partial S}{\partial b}\right)\tag{3.4}$$

$$=\begin{cases} g_{p,q}(t)\cdot d_{p,q}(t) & \text{if }\ pq(p-q)\neq 0,\\ g_{p,0}(t)\lim_{q\to 0}d_{p,q}(t) & \text{if }\ p\neq 0, q=0,\\ g_{0,q}(t)\lim_{p\to 0}d_{p,q}(t) & \text{if }\ q\neq 0, p=0,\\ g_{p,p}(t)\lim_{q\to p}d_{p,q}(t) & \text{if }\ p=q\neq 0,\\ g_{0,0}(t)\lim_{p\to 0}d_{p,0}(t) & \text{if }\ p=q=0.\end{cases}\tag{3.5}$$

Since Δ is symmetric with respect to a and b, without loss of generality we assume that $a > b$. It is easy to verify that $\frac{a^m - b^m}{m(a-b)} > 0$, $\frac{(a-b)S_{p,q}}{2(ab)^{m/2}} > 0$, $\frac{p}{\sinh pt}, \frac{q}{\sinh qt} > 0$ if $pq(p-q) \neq 0$ for $t = \ln\sqrt{a/b} > 0$, which implies that $d_{p,q}(t)$ and its limits at $(p,q) \in \{(p,q) : pq(p-q) = 0\}$ are all positive. Thus by Corollary 1 Stolarsky mean $S_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $\Delta \geq (\leq) 0$ if and only if $g(t) = g_{p,q}(t) \geq (\leq) 0$ for all $t > 0$.

It is easy to check that for $m = 0$ this lemma is also true.

This Lemma is proved. □

Lemma 3. *Both the $g(t) = g_{p,q}(t)$ defined by (3.1) and $g'(t) := \partial g_{p,q}(t)/\partial t$ are symmetric with respect to p and q, and continuous with respect to (p,q) on $\mathbb{R}^2$.*

Proof. Firstly, it is easy to check $g_{p,q}(t)$ is symmetric with respect to p an q. Hence, $\partial g_{q,p}(t)/\partial t = \partial g_{p,q}(t)/\partial t$, which implies that $\partial g_{p,q}(t)/\partial t$ is also symmetric with respect to p and q.

Secondly, by the proof of Lemma 2, it is easy to see that $g(t) = g_{p,q}(t)$ is continuous with respect to (p,q) on $\mathbb{R}^2$.

Lastly, we prove $g'(t) = \partial g_{p,q}(t)/\partial t$ is also continuous with respect to (p,q) on $\mathbb{R}^2$.

A simple calculation yields

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t}$$

$$= \begin{cases} \dfrac{(p-q)A\cosh At - pB\cosh Bt - qC\cosh Ct}{pq(p-q)} & \text{if } pq(p-q) \neq 0, \\ \dfrac{(p+m)\cosh(p+m)t - (p+m)\cosh(p-m)t - 2p(p-m)t\sinh(p-m)t}{-p^2} & \text{if } p \neq 0, q = 0, \\ \dfrac{(q+m)\cosh(q+m)t - (q+m)\cosh(q-m)t - 2q(q-m)t\sinh(q-m)t}{-q^2} & \text{if } q \neq 0, p = 0, \\ \dfrac{(2p-m)\cosh(2p-m)t - (2p-m)\cosh mt - 2pmt\sinh mt}{p^2} & \text{if } p = q \neq 0, \\ -4t\sinh mt - 2mt^2\cosh mt & \text{if } p = q = 0. \end{cases} \tag{3.6}$$

It is obvious that $\partial g_{p,q}(t)/\partial t$ is continuous with respect to $(p,q) \in \{(p,q) : p, q \in \mathbb{R}, pq(p-q) \neq 0\}$. We have also to verify that $\partial g_{p,q}(t)/\partial t$ is continuous on $(p,q) \in$

$\{(p,0) : p \in \mathbb{R},\ p \neq 0\}$, $\{(0,q) : q \in \mathbb{R},\ q \neq 0\}$, $\{(p,p) : p \in \mathbb{R},\ p \neq 0\}$, $\{(0,0)\}$. In fact, some simple calculations yield

$$\lim_{q\to 0} \frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{p,0}(t)}{\partial t}, \quad \lim_{p\to 0} \frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{0,q}(t)}{\partial t},$$
$$\lim_{q\to p} \frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{p,p}(t)}{\partial t}, \quad \lim_{p\to 0} \frac{\partial g_{p,p}(t)}{\partial t} = \frac{\partial g_{0,0}(t)}{\partial t},$$

which completes this proof. □

Lemma 4. *We have*

$$\lim_{t\to 0,\, t>0} \frac{3g(t)}{2t^3} = p + q - 3m. \tag{3.7}$$

Proof. It is easy to check that $g(0) = g'(0) = g''(0) = 0$.

In the case of $pq(p-q) \neq 0$. Applying L'Hospital's rule (three times) we have

$$\begin{aligned}\lim_{t\to 0,\, t>0} \frac{3g(t)}{2t^3} &= \lim_{t\to 0,\, t>0} \frac{g'(t)}{2t^2} = \dots \\ &= \frac{(p-q)A^3 - pB^3 - qC^3}{4pq(p-q)} = p + q - 3m.\end{aligned} \tag{3.8}$$

In the case of $pq(p-q) = 0$. Likewise, some simple calculations also lead to (3.7).

This completes the proof. □

Lemma 5. *Let $m > 0$ and $\beta = \max(|A|, |B|, |C|)$ where A, B, C are defined by* (3.2). *Then*

(i) *If $pq(p-q) \neq 0$ and $p > q$, then*

$$\lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = \begin{cases} \dfrac{p+q-m}{pq} & \text{if } p > q > m \text{ or } q < p < 0, \\ \dfrac{p-q-m}{q(p-q)} & \text{if } p > q = m, \\ -\dfrac{p-q+m}{p(p-q)} & \text{if } p > 0,\ q < m,\ p > q. \end{cases} \tag{3.9}$$

(ii) *If $p \neq q = 0$, then*

$$\lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = \begin{cases} -\infty & \text{if } p < 0, \\ -(p+m)p^{-2} & \text{if } p > 0. \end{cases} \tag{3.10}$$

(iii) *If* $p = q \neq 0$, *then*

$$\lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = \begin{cases} (2p-m)p^{-2} & \textit{if } \ p > m \text{ or } p < 0, \\ -\infty & \textit{if } \ 0 < p \leq m. \end{cases} \tag{3.11}$$

(iv) *If* $p = q = 0$, *then*

$$\lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = -\infty. \tag{3.12}$$

Proof. (3.9)–(3.12) easily follow from the following limit relations:

$$\lim_{t\to\infty} \frac{2\cosh \alpha t}{e^{\beta t}} = \begin{cases} 1 & \text{if } \ \beta = |\alpha|, \\ 0 & \text{if } \ \beta > |\alpha|; \end{cases} \tag{3.13}$$

$$\lim_{t\to\infty} \frac{2\alpha t \sinh \alpha t}{e^{\beta t}} = \begin{cases} \infty & \text{if } \ \beta = |\alpha|, \\ 0 & \text{if } \ \beta > |\alpha|. \end{cases} \tag{3.14}$$

(i) If $pq(p-q) \neq 0$ and $p > q$, then $\beta = \max(|A|, |B|, |C|) = \max(|A|, |C|)$ because $|C|^2 - |B|^2 = 4m(p-q) > 0$. By (3.6) and (3.13) we have

$$\begin{aligned} pq(p-q) \lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} &= pq(p-q) \lim_{t\to\infty} \frac{2g'(t)}{e^{\beta t}} \\ &= \lim_{t\to\infty} 2\frac{(p-q)A\cosh At - pB\cosh Bt - qC\cosh Ct}{e^{\beta t}} \\ &= \begin{cases} (p-q)A & \text{if } \ |A| > |C|, \text{ i.e. } p(q-m) > 0, \\ (p-q)A - qC & \text{if } \ |A| = |C|, \text{ i.e. } p(q-m) = 0, \\ -qC & \text{if } \ |A| < |C|, \text{ i.e. } p(q-m) < 0. \end{cases} \end{aligned} \tag{3.15}$$

Taking into account $pq(p-q) \neq 0$ and $p > q$, we obtain

$$pq(p-q) \lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = \begin{cases} (p-q)(p+q-m) & \text{if } \ p > q > m \text{ or } q < p < 0, \\ p(p-q-m) & \text{if } \ p > q = m, \\ -q(p-q+m) & \text{if } \ p > 0, \ q < m, \ p > q. \end{cases}$$

Divided by $pq(p-q)$ in the above limit relation yields (3.9).

(ii) If $p \neq q = 0$, then $\beta = \max(|A|, |B|, |C|) = \max(|p-m|, |p+m|)$. By (3.6) and (3.13), (3.14) we have

$$\lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = \lim_{t\to\infty} \frac{2g'(t)}{e^{\beta t}}$$

$$= \lim_{t\to\infty} 2\frac{(p+m)\cosh(p+m)t - (p+m)\cosh(p-m)t - 2p(p-m)t\sinh(p-m)t}{-p^2e^{\beta t}}$$

$$= \begin{cases} -\infty & \text{if } |p-m| > |p+m|, \text{ i.e. } p<0, \\ -(p+m)p^{-2} > 0 & \text{if } |p-m| < |p+m|, \text{ i.e. } p>0. \end{cases}$$

(iii) If $p = q \neq 0$, then $\beta = \max(|A|, |B|, |C|) = \max(|2p-m|, m)$. By (3.6) and (3.13), (3.14) we have

$$\lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = \lim_{t\to\infty} \frac{2g'(t)}{e^{\beta t}}$$

$$= \lim_{t\to\infty} 2\frac{(2p-m)\cosh(2p-m)t - (2p-m)\cosh mt - 2pmt\sinh mt}{p^2e^{\beta t}}$$

$$= \begin{cases} (2p-m)p^{-2} & \text{if } |2p-m| > m, \text{ i.e. } p>m \text{ or } p<0, \\ -\infty & \text{if } |2p-m| = m, \text{ i.e. } p=m, \\ -\infty & \text{if } |2p-m| < m, \text{ i.e. } 0<p<m. \end{cases}$$

(iv) If $p = q = 0$, then $\beta = \max(|A|, |B|, |C|) = \max(m, m, m) = m$. By (3.6) and (3.13), (3.14) we have

$$\lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = \lim_{t\to\infty} 2\frac{-2mt^2\sinh mt}{e^{mt}} = -\infty.$$

This proof is complete. □

Lemma 6. *Suppose that $|t_1|, |t_2|, |t_3|$ are pairwise distinct numbers. Then the following identities*

$$\operatorname{sgn}\left(u(t_1,t_2,t_3)\right) = \operatorname{sgn}(\cosh t_1 - \cosh t_3) = \operatorname{sgn}(|t_1| - |t_3|) \tag{3.16}$$

hold, where

$$u(t_1,t_2,t_3) = \frac{t_1\sinh t_1 - t_2\sinh t_2}{\cosh t_1 - \cosh t_2} - \frac{t_2\sinh t_2 - t_3\sinh t_3}{\cosh t_2 - \cosh t_3} \tag{3.17}$$

Proof. To prove the first identity of (3.16), we note that both the function $t \to \cosh t$ and $t \to t\sinh t$ are even on $\mathbb{R}$, and so we have

$$u(t_1,t_2,t_3) = u(|t_1|, |t_2|, |t_3|).$$

Put $\cosh|t_i| = x_i$, $i = 1,2,3$, then $x_1, x_2, x_3 > 1$ and are also pairwise distinct, and

$$|t_i| = \ln\left(x_i + \sqrt{x_i^2 - 1}\right), \ \sinh|t_i| = \sqrt{x_i^2-1}.$$

Thus, the first identity of (3.16) is equivalent to

$$\operatorname{sgn}\left(\frac{f(x_1)-f(x_2)}{x_1-x_2}-\frac{f(x_2)-f(x_3)}{x_2-x_3}\right)=\operatorname{sgn}(x_1-x_3), \tag{3.18}$$

where

$$f(x)=\sqrt{x^2-1}\ln\left(x+\sqrt{x^2-1}\right),\ x>1.$$

By simple calculations, we get

$$f'(x)=1+\frac{x}{\sqrt{x^2-1}}\ln\left(x+\sqrt{x^2-1}\right),$$
$$f''(x)=\frac{x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)}{\left(\sqrt{x^2-1}\right)^3}:=\frac{h(x)}{\left(\sqrt{x^2-1}\right)^3}.$$

Since $h'(x)=2\left(\sqrt{x^2-1}\right)^{-1}>0$, we have $h(x)>h(1)=0$, which yields $f''(x)>0$, and so f is convex on $(1,\infty)$. From the properties of convex functions it follows that

$$\frac{1}{x_1-x_3}\left(\frac{f(x_1)-f(x_2)}{x_1-x_2}-\frac{f(x_2)-f(x_3)}{x_2-x_3}\right)>0,$$

which implies that the first identity of (3.16) holds.

Next we show that the second identity of (3.16) holds. Since the function $t\to\cosh t$ is even on $\mathbb{R}$ and strictly increasing on $\mathbb{R}_+$, we have

$$\cosh t_1-\cosh t_3=(|t_1|-|t_3|)\frac{\cosh|t_1|-\cosh|t_3|}{|t_1|-|t_3|},$$

from which the second identity of (3.16) follows.

This proof is ended. □

Lemma 7. *Let*

$$g'(t)=\frac{\partial g_{p,q}(t)}{\partial t}=g_1(t)\cdot g_2(t)\quad \textit{for } pq(p-q)\neq 0, \tag{3.19}$$

where

$$g_1(t)=\frac{\cos Bt-\cos Ct}{p-q}, \tag{3.20}$$

$$g_2(t)=\frac{(p-q)A\frac{\cosh At-\cosh Ct}{\cos Bt-\cos Ct}-pB}{pq} \tag{3.21}$$

and A,B,C are defined by (3.2)*. Then for all $t>0$, we have*

(i) $\operatorname{sgn}(g_1(t)) = -\operatorname{sgn}(m)$.

(ii) $\operatorname{sgn}(g_2(t)) = -\operatorname{sgn}(m)\operatorname{sgn}(g'(t))$.

(iii) $g_2(t)$ *is monotone with* $t > 0$.

Proof. (i) By the second identity of (3.16) we have

$$\operatorname{sgn}(g_1(t)) = \frac{\operatorname{sgn}(|Bt| - |Ct|)}{\operatorname{sgn}(p-q)} = -\operatorname{sgn}(m)$$

for all $t > 0$.

(ii) Using (3.19) and the first result of this lemma yield

$$\operatorname{sgn}(g_2(t)) = \frac{\operatorname{sgn}(g'(t))}{\operatorname{sgn}(g_1(t))} = \frac{\operatorname{sgn}(g'(t))}{-\operatorname{sgn}(m)} = -\operatorname{sgn}(m)\operatorname{sgn}(g'(t)).$$

(iii) To prove that $g_2(t)$ is monotone with $t > 0$, it is enough to show that $\operatorname{sgn}(g_2'(t))$ does not depend on all $t > 0$. In fact, we have

$$\operatorname{sgn}(g_2'(t)) = -\operatorname{sgn}(m)\operatorname{sgn}(p-m)\operatorname{sgn}(q-m)\operatorname{sgn}(p+q-m) \tag{3.22}$$

holds for $pq(p-q) \neq 0$.

A simple derivative computation yields

$$\begin{aligned} pqg_2'(t) &= (p-q)A\frac{\cosh At - \cosh Ct}{\cos Bt - \cos Ct} \\ &\quad\times \left(\frac{A\sinh At - C\sinh Ct}{\cosh At - \cosh Ct} - \frac{B\sinh Bt - C\sinh Ct}{\cos Bt - \cos Ct}\right) \\ &= t^{-1}(p-q)A\frac{\cosh At - \cosh Ct}{\cos Bt - \cos Ct}u(At, Ct, Bt), \end{aligned}$$

where $u(t_1, t_2, t_3)$ is defined by (3.17). From (3.16) and $t > 0$ it follows that

$$\begin{aligned} \operatorname{sgn}(pqg_2'(t)) &= \operatorname{sgn}\left(t^{-1}(p-q)\right)\operatorname{sgn}(A)\frac{\operatorname{sgn}(|At| - |Ct|)}{\operatorname{sgn}(|Bt| - |Ct|)}\operatorname{sgn}(\cosh|At| - \cosh|Bt|) \\ &= \operatorname{sgn}(p-q)\operatorname{sgn}(p+q-m)\frac{\operatorname{sgn}(p(q-m))}{\operatorname{sgn}(-m(p-q))}\operatorname{sgn}(q(p-m)) \\ &= -\operatorname{sgn}(m)\operatorname{sgn}(p)\operatorname{sgn}(q)\operatorname{sgn}(p-m)\operatorname{sgn}(q-m)\operatorname{sgn}(p+q-m), \end{aligned}$$

which is equivalent to (3.22) for $pq(p-q) \neq 0$.

This accomplishes the proof. □

4. Proofs of main results

Proof of Theorem 1. Denote by

$$D = \{(p,q) : p+q-3m \geq 0,\ \min(p,q) \geq m\} \quad (m>0).$$

By Lemma 3.1, to prove Theorem 1, it suffices to prove that $g_{p,q}(t) \geq 0$ for all $t>0$ if and only if $(p,q) \in D$.

Necessity. We prove that $(p,q) \in D$ is the necessary conditions for $g(t) = g_{p,q}(t) \geq 0$ for all $t>0$. It is obvious that

$$\lim_{t\to 0,\, t>0} \frac{3g(t)}{2t^3} \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} \geq 0. \tag{4.1}$$

The necessary conditions will be obtained from (4.1) together with (3.7) and (3.9)–(3.12). We divide the proof of necessity into six cases.

(i) *Case 1*: $pq(p-q) \neq 0$ and $p>q$.

Subcase 1:

$$\begin{cases} p+q-3m \geq 0, \\ \dfrac{p+q-m}{pq} \geq 0, \\ p>q>m \text{ or } q<p<0 \end{cases} \quad \Longrightarrow \begin{cases} p+q-3m \geq 0, \\ p>q>m, \end{cases}$$

which implies that $(p,q) \in \{(p,q) : p>q>m\} = D_{11}$.

Subcase 2:

$$\begin{cases} p+q-3m \geq 0, \\ \dfrac{p-q-m}{q(p-q)} \geq 0, \\ p>q=m \end{cases} \quad \Longrightarrow \begin{cases} p \geq 2m, \\ q=m, \end{cases}$$

which implies that $(p,q) \in \{(p,q) : p \geq 2m,\ q=m\} = D_{12}$.

Subcase 3:

$$\begin{cases} p+q-3m \geq 0, \\ -\dfrac{p-q+m}{p(p-q)} \geq 0, \\ p>0, \\ q<m, \\ p>q, \end{cases} \quad \Longrightarrow \text{which is impossible.}$$

(i') *Case 1'*: $pq(p-q) \neq 0$ and $p < q$.

Since $g_{p,q}(t)$ is symmetric with respect to p and q, so $(p,q) \in D'_{11} \cup D'_{12}$ if (4.1) holds, where

$$D'_{11} = \{(p,q) : q > p > m\}, \quad D'_{12} = \{(p,q) : q \geq 2m,\ p = m\}.$$

(ii) *Case 2*: $p \neq q = 0$.

Subcase 1:

$$\begin{cases} p+q-3m \geq 0, \\ -\infty \geq 0, \\ p < 0 = q, \end{cases} \implies \text{which is impossible.}$$

Subcase 2:

$$\begin{cases} p+q-3m \geq 0, \\ -(p+m)p^{-2} \geq 0, \\ p > 0 = q, \end{cases} \implies \text{which is impossible.}$$

(ii') *Case 2'*: $q \neq p = 0$.

Since $g_{p,q}(t)$ is symmetric with respect to p and q, so this case is also impossible if (4.1) holds.

(iii) *Case 3*: $p = q \neq 0$.

Subcase 1:

$$\begin{cases} p+q-3m \geq 0, \\ (2p-m)p^{-2} \geq 0, \\ p > m \text{ or } p < 0, \\ p = q \neq 0 \end{cases} \implies \begin{cases} p+q-3m \geq 0, \\ p = q > m, \end{cases}$$

which implies that $(p,q) \in \{(p,q) : p+q-3m \geq 0,\ p = q > m\} = D_{31}$.

Subcase 2:

$$\begin{cases} p+q-3m \geq 0, \\ -\infty \geq 0, \\ 0 < p \leq m, \\ p = q \neq 0, \end{cases} \implies \text{which is impossible.}$$

(iv) *Case 4*: $p = q = 0$.

$$\begin{cases} p + q - 3m \geq 0, \\ -\infty \geq 0, \\ p = q = 0, \end{cases} \qquad \Longrightarrow \text{which is impossible.}$$

Summarizing all the cases yield

$$(p, q) \in (D_{11} \cup D_{12}) \cup (D'_{11} \cup D'_{12}) \cup D_{31} = D.$$

Sufficiency. We prove the condition $(p, q) \in D$ is sufficient for $g(t) = g_{p,q}(t) \geq 0$ for all $t > 0$. Since $g(0) = 0$, it is enough to prove $g'(t) \geq 0$ if $(p, q) \in D$.

(i) In the case of $(p, q) \in D$ with $pq(p - q) \neq 0$. By (3.8) and (3.15), we see that

$$\operatorname{sgn}(g'(0)) = \operatorname{sgn}(g(0)) \geq 0 \quad \text{and} \quad \operatorname{sgn}(g'(\infty)) = \operatorname{sgn}(g(\infty)) \geq 0$$

if $(p, q) \in D$ with $pq(p - q) \neq 0$.

On the other hand, noting $m > 0$ and by (ii) and (iii) of Lemma 7, we have

$$\operatorname{sgn}(g_2(0)) = -\operatorname{sgn}(m)\operatorname{sgn}(g'(0)) \leq 0,$$
$$\operatorname{sgn}(g_2(\infty)) = -\operatorname{sgn}(m)\operatorname{sgn}(g'(\infty)) \leq 0$$

and $g_2(t)$ is monotone with $t > 0$, which mean that $g_2(t) \leq 0$ for all $t > 0$. Taking into account $\operatorname{sgn}(g_1(t)) = -\operatorname{sgn}(m) < 0$, we obtain that $g'(t) = g_1(t)g_2(t) \geq 0$ for all $t > 0$.

(ii) In the case of $(p, q) \in D$ with $pq(p - q) = 0$. Form Lemma 3 it follows that

$$g'(t) = \frac{\partial g_{p,0}(t)}{\partial t} = \lim_{q \to 0} \frac{\partial g_{p,q}(t)}{\partial t} \geq 0 \quad \text{if } (p, q) \in D \text{ with } p \neq q = 0.$$

Similarly, we have

$$g'(t) = \frac{\partial g_{0,q}(t)}{\partial t} \geq 0 \quad \text{if } (p, q) \in D \text{ with } q \neq p = 0,$$

$$g'(t) = \frac{\partial g_{p,p}(t)}{\partial t} \geq 0 \quad \text{if } (p, q) \in D \text{ with } p = q \neq 0,$$

$$g'(t) = \frac{\partial g_{0,0}(t)}{\partial t} \geq 0 \quad \text{if } (p, q) \in D \text{ with } p = q = 0.$$

Therefore, $g'(t) = \partial g_{p,q}(t)/\partial t \geq 0$ if $(p, q) \in D$.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Denote by

$$E=\{(p,q\}:p+q-3m\leq 0,\ p\geq q,\ q\leq m\}\quad (m>0),$$

$$E'=\{(p,q\}:p+q-3m\leq 0,\ q\geq p,\ p\leq m\}\quad (m>0).$$

Then

$$E\cup E'=\{(p,q\}:p+q-3m\leq 0 \text{ and } \min(p,q)\leq m\}\quad (m>0).$$

By Lemma 3.1, to prove Theorem 2, it suffices to show that $g_{p,q}(t)\leq 0$ for all $t>0$ if and only if $(p,q)\in E\cup E'$.

Necessity. We prove $(p,q)\in E\cup E'$ is the necessary conditions for $g(t)=g_{p,q}(t)\leq 0$ for all $t>0$. It is clear that

$$\lim_{t\to 0,\,t>0}\frac{3g(t)}{2t^3}\leq 0\quad\text{and}\quad \lim_{t\to\infty}\frac{2\beta g(t)}{e^{\beta t}}\leq 0. \tag{4.2}$$

We derive the necessary conditions from (4.2) together with (3.7) and (3.9)–(3.12). To this aim, we divide the proof of necessity into six cases.

(i) *Case 1*: $pq(p-q)\neq 0$ and $p>q$.

Subcase 1:

$$\begin{cases} p+q-3m\leq 0, \\ \dfrac{p+q-m}{pq}\leq 0, \\ p>q>m \text{ or } q<p<0 \end{cases} \Longrightarrow 0>p>q,$$

which implies that $(p,q)\in\{(p,q):0>p>q\}=E_{11}$.

Subcase 2:

$$\begin{cases} p+q-3m\leq 0, \\ \dfrac{p-q-m}{q(p-q)}\leq 0, \\ p>q=m \end{cases} \Longrightarrow \begin{cases} p\leq 2m, \\ q=m, \\ p>q, \end{cases}$$

which implies that $(p,q)\in\{(p,q):q=m,p\leq 2m\}=E_{12}$.

Subcase 3:

$$\begin{cases} p+q-3m\leq 0, \\ -\dfrac{p-q+m}{p(p-q)}\leq 0, \\ p>0, \\ q<m, \\ p>q \end{cases} \Longrightarrow \begin{cases} p+q-3m\leq 0, \\ p>0, \\ q<m, \\ p>q, \end{cases}$$

which implies that $(p,q)\in\{(p,q): p+q-3m\le 0,\ p>0,\ q<m,\ p>q\}=E_{13}$.

(i') *Case 1'*: $pq(p-q)\neq 0$ and $p<q$.

Since $g_{p,q}(t)$ is symmetric with respect to p and q, so $(p,q)\in E'_{11}\cup E'_{12}\cup E'_{13}$ if (4.2) holds, where

$$\begin{aligned}
E'_{11} &= \{(p,q): 0>q>p\},\\
E'_{12} &= \{(p,q): p=m,\ q\le 2m,\ q>p\},\\
E'_{13} &= \{(p,q): p+q-3m\le 0,\ q>0,\ p<m,\ q>p\}.
\end{aligned}$$

(ii) *Case 2*: $p\neq q=0$.

Subcase 1:

$$\begin{cases} p+q-3m\le 0,\\ -\infty\le 0,\\ p<0=q \end{cases} \Longrightarrow \begin{cases} p+q-3m\le 0,\\ p<0=q, \end{cases}$$

which implies that $(p,q)\in\{(p,q): p+q-3m\le 0,\ p<0=q\}=E_{21}$.

Subcase 2:

$$\begin{cases} p+q-3m\le 0,\\ -(p+m)p^{-2}\le 0,\\ p>0=q \end{cases} \Longrightarrow \begin{cases} p+q-3m\le 0,\\ p>0=q, \end{cases}$$

which implies that $(p,q)\in\{(p,q): p+q-3m\le 0,\ p>0=q\}=E_{22}$.

(ii') *Case 2'*: $q\neq p=0$.

Since $g_{p,q}(t)$ is symmetric with respect to p and q, so $(p,q)\in E'_{21}\cup E'_{22}$ if (4.2) holds, where

$$\begin{aligned}
E'_{21} &= \{(p,q): p+q-3m\le 0,\ q<0=p\},\\
E'_{22} &= \{(p,q): p+q-3m\le 0,\ q>0=p\}.
\end{aligned}$$

(iii) *Case 3*: $p=q\neq 0$.

Subcase 1:

$$\begin{cases} p+q-3m\le 0,\\ (2p-m)p^{-2}\le 0,\\ p>m \text{ or } p<0,\\ p=q\neq 0 \end{cases} \Longrightarrow \begin{cases} p+q-3m\le 0,\\ p=q<0, \end{cases}$$

which implies that $(p,q)\in\{(p,q): p+q-3m\le 0,\ p=q<0\}=E_{31}$.

Subcase 2:

$$\begin{cases} p+q-3m \le 0, \\ -\infty \le 0, \\ 0 < p \le m, \\ p = q \ne 0 \end{cases} \implies \begin{cases} p+q-3m \le 0, \\ 0 < p = q \le m, \end{cases}$$

which implies that $(p,q) \in \{(p,q) : p+q-3m \le 0,\ 0 < p = q \le m\} = E_{32}$.

(iv) *Case 4*: $p = q = 0$.

$$\begin{cases} p+q-3m \le 0, \\ -\infty \le 0, \\ p = q = 0, \end{cases} \implies \text{which implies that } (p,q) \in \{(0,0)\} = E_4.$$

Summarizing all the cases yield

$$\begin{aligned} (p,q) \in {} & (E_{11} \cup E_{12} \cup E_{13}) \cup (E'_{11} \cup E'_{12} \cup E'_{13}) \\ & \cup (E_{21} \cup E_{22}) \cup (E'_{21} \cup E'_{22}) \cup (E_{31} \cup E_{32}) \cup E_{24} = E \cup E'. \end{aligned}$$

Sufficiency. Similarly to proof of sufficiency of Theorem 1, we can prove $g'(t) \le 0$ if $(p,q) \in E \cup E'$. Hence $g(t) = g_{p,q}(t) \le g(0) = 0$ for all $t > 0$.

The proof of Theorem 2 is completed. □

Proof of Theorem 3. Let $g_{p,q,m}(t) := g_{p,q}(t)$ defined by (3.1) and

$$p' = -p,\ q' = -q,\ m' = -m.$$

We easily verify that, for $p, q, p', q', m, m' \in \mathbb{R}$,

$$g_{p,q,m}(t) = -g_{p',q',m'}(t).$$

From this and Lemma 2, for $m < 0$ Stolarsky mean $S_{p,q}(a,b)$ is Schur m-power convex if and only if $S_{p',q'}(a,b)$ is Schur m'-power concave with respect to $(a,b) \in \mathbb{R}^2_+$, which, by Theorem 2, if and only if

$$p' + q' \le 3m' \quad \text{and} \quad \min(p', q') \le m',$$

that is,

$$p + q \ge 3m \quad \text{and} \quad \max(p,q) \ge m.$$

Theorem 3 follows. □

Proof of Theorem 4. Similarly to the proof of Theorem 3, we have that for $m < 0$ Stolarsky mean $S_{p,q}(a,b)$ is Schur m-power concave if and only if $S_{p',q'}(a,b)$ is Schur m'-power convex with respect to $(a,b) \in \mathbb{R}^2_+$, which, by Theorem 1, if and only if

$$p' + q' \geq 3m' \quad \text{and} \quad \min(p', q') \geq m',$$

that is,

$$p + q \leq 3m \quad \text{and} \quad \max(p, q) \leq m,$$

The proof of Theorem 4 ends. □

Proof of Theorem 5. By Lemma 3.1, to prove Theorem 5, it is enough to prove that $g_{p,q}(t) \geq (\leq) 0$ for all $t > 0$ if and only if $p + q \geq (\leq) 0$ for $m = 0$. For this end, we divide the proof into four cases.

(i) *Case 1*: $pq(p - q) \neq 0$. By (3.1) we have

$$\begin{aligned} g_{p,q}(t) &= \frac{(p-q)\sinh(p+q)t - (p+q)\sinh(p-q)t}{pq(p-q)} \\ &= t(p+q)\frac{k((p+q)t) - k((p-q)t)}{pq}. \end{aligned}$$

Denote by $k(x) = (\sinh x)/x$ if $x \neq 0$ and $k(0) = 1$. We easily check that $k(-x) = k(x)$ and $k'(x) > (<) 0$ for $x > (<) 0$. In fact, $k'(x) = x^{-2}w(x)$, $w(x) = x\cosh x - \sinh x > (<) 0$ for $x > (<) 0$ because $w'(x) = x\sinh x > 0$ for $x \neq 0$. Thus,

$$\begin{aligned} &\operatorname{sgn}\left(\frac{k((p+q)t) - k((p-q)t)}{pq}\right) \\ &= \operatorname{sgn}\left(\frac{|(p+q)t| - |(p-q)t|}{pq}\right)\operatorname{sgn}\left(\frac{k(|(p+q)t|) - k(|(p-q)t|}{|(p+q)t|) - |(p-q)t|}\right) \\ &= \operatorname{sgn}\left(\frac{t}{|p+q| + |p-q|}\,\frac{(p+q)^2 - (p-q)^2}{pq}\right) = 1, \end{aligned}$$

it follows that

$$\operatorname{sgn}(g_{p,q}(t)) = \operatorname{sgn}(t(p+q))\operatorname{sgn}\left(\frac{k((p+q)t) - k((p-q)t)}{pq}\right) = \operatorname{sgn}(p+q).$$

This shows that $g_{p,q}(t) \geq (\leq) 0$ for all $t > 0$ if and only if $p + q \geq (\leq) 0$.

(ii) *Case 2*: $pq = 0$, $p \neq q$. By (3.1) we have

$$g_{p,0}(t) = \frac{2}{p^2}(pt\cosh(pt) - \sinh(pt)) \quad (p \neq 0).$$

Since $w(x) = x \cosh x - \sinh x > (<)0$ for $x > (<)0$, $g_{p,0}(t) \geq (\leq)0$ $(p \neq 0)$ for all $t > 0$ if and only if $pt > (<)0$, that is, $p > (<)0$.

In the same way, we can prove that $g_{0,q}(t) \geq (\leq)0$ $(q \neq 0)$ for all $t > 0$ if and only if $q > (<)0$.

(iii) *Case 3*: $p = q \neq 0$. By (3.1) we have

$$g_{p,p}(t) = \frac{\sinh(2pt) - 2pt}{p^2} = \frac{2t}{p}\left(\frac{\sinh(2pt)}{2pt} - 1\right) = \frac{2t}{p}\left(k(2pt) - k(0)\right).$$

Since $k'(x) > (<)0$ for $x > (<)0$, we get $k(2pt) > k(0)$. It follows that $g_{p,p}(t) \geq (\leq)0$ $(p \neq 0)$ for all $t > 0$ if and only if $2t/p > (<)0$, that is, $p > (<)0$.

(iv) *Case 4*: $p = q = 0$. Clearly, $g_{0,0}(t) = 0$.

To sum up, for $m = 0$, $g_{p,q}(t) \geq (\leq)0$ for all $t > 0$ if and only if $p + q \geq (\leq)0$.

The proof of Theorem 5 is completed. □

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