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Additive and multiplicative functions on arithmetical semigroups

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1. Introduction

Many problems concerning additive and multiplicative functions defined on $\mathbb{N}$ can be posed and solved in a more abstract setting. One can investigate functions on an arithmetical semigroup which by definition is a commutative semigroup $\mathcal{G}$ with identity element 1, and which contains a countable subset $\mathcal{P}$ such that every element $a \neq 1$ in $\mathcal{G}$ admits unique factorization into a finite product of powers of elements of $\mathcal{P}$. The direct generalization of $\mathbb{N}$ is the arithmetical semigroup satisfying

Axiom A. *A completely multiplicative norm function $\|\cdot\|$ is defined on $\mathcal{G}$ so that $\|p\| > 1$ for each $p \in \mathcal{P}$, and there exist constants $A > 0$, $0 \leq \theta' < 0$ such that*

$$\#\{a \in \mathcal{G}; \|a\| \leq x\} = Ax^{\theta} + O(x^{\theta'}). \tag{1}$$

The development of analytic and probabilistic number theory in such semigroups is represented by J. KNOPFMACHER's monograph [8], papers quoted in it and more recent publications. The semigroup of primary polynomials over a finite field as well as that of the integral divisors in algebraic function fields and many other interesting arithmetical semigroups do not fall under the scope of Axiom A because the regularity of norms of elements has different character. These semigroups satisfy

Axiom A*. *A completely additive degree function ∂ is defined on $\mathcal{G}$ so that $\partial(p) \geq 1$ for each $p \in \mathcal{P}$ and*

$$\mathcal{G}(n) := \#\{a \in \mathcal{G}; \partial(a) = n\} = Aq^n + O(q^{\nu n})$$

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as $n \to \infty$ *for some constants* $A > 0$, $q > 1$, $0 \le \nu < 1$.

Note that if one defines $\|a\| = q^{\partial(a)}$, then the last asymptotic relation implies (1) only for the sequence $x = x_n := q^n \to \infty$.

Another nonequivalent axiom has been suggested by the first of the authors [5]. In [9] J. KNOPFMACHER gives an introductory analytic theory of semigroups satisfying Axiom A^*, deeper problems he leaves as open questions (Chapter 12). The purpose of the present paper is to answer some of them. Investigating the mean values of multiplicative functions we obtain an analogue of the HALÁSZ theorem [4]. Observe that the specific structure of the considered semigroups implies a new effect in the problem which is shown below by the example of the Möbius function. By some limit theorems for additive functions we demonstrate that the probabilistic number theory in $\mathbb{N}$, systematically represented by J. KUBILIUS [11] and P.D.T.A. ELLIOTT [3], has its analogue in semigroups satisfying Axiom A^*.

2. Analytic background

In the following let the arithmetical semigroup $\mathcal{G}$ satisfy Axiom A*. For $y \in \mathbb{C}$, $|y| < 1$, we put

$$Z(y) = \sum_{n=0}^{\infty} \mathcal{G}(n)(q^{-1}y)^n = \prod_{k=1}^{\infty}(1 - (q^{-1}y)^k)^{-\pi(k)}$$

where

$$\pi(k) := \#\{p \in \mathcal{P}; \partial(p) = k\}.$$

The function $Z(y)$ is analytic in the disc $|y| < 1$ and has an analytic continuation into the disc $|y| < q^{1-\nu}$ with a simple pole at $y = 1$ with the residue $-A$ (see [9], Chapter 2). As it was shown in [6], Lemma 8.5 in [9] is not correct. In general, Axiom A* does not imply that $Z(y) \neq 0$ for $|y| = 1$ though only one simple exceptional zero at $y = -1$ can occur. In the general case we have the prime number theorem analogue

$$\pi(k) = q^k k^{-1}(1 - (-1)^k\kappa) + O(q^{c_0 k}) \tag{2}$$

with some c_0, $\max\{\frac{1}{2}, \nu\} < c_0 < 1$ ([6], [7]). Here $\kappa = 1$ if $Z(-1) = 0$ and $\kappa = 0$ otherwise.

Let $f : \mathcal{G} \to \mathbb{C}$ be a multiplicative function which may depend on n or other parameters. We shall investigate the asymptotic behaviour of

$$M_n(f) := A^{-1}q^{-n} \sum_{\partial(a)=n} f(a)$$

as $n \to \infty$ where $|f(a)| \le 1$ for each $a \in \mathcal{G}$. Put

$$F(y) = \sum_{n=1}^{\infty} \sum_{\partial(a)=n} f(a)(q^{-1}y)^n =$$

$$= \prod_{p \in \mathcal{P}} (1 + f(p)(q^{-1}y)^{\partial(p)} + f(p^2)(q^{-1}y)^{2\partial(p)} + \dots) :=$$

$$:= \prod_{p} \chi_p(y) := \prod_{p} \chi_p(y; f).$$

For $|y| < 1$ the function $F(y)$ is analytic and $F(y) \neq 0$ in this disc. Due to the estimate (2) there is only a finite number of prime elements satisfying the inequality

$$\partial(p) \le (\log 2)/\log q := c.$$

For the other primes we have

$$|\chi_p(y) - 1| \le (q^{\partial(p)} - 1)^{-1} \le 1 - c_1,$$

uniformly in $|y| \le 1$ with some $c_1 > 0$ depending only on $\mathcal{G}$. Consider the equality

$$F(y) = \Big[\prod_{\partial(p) \le c} \chi_p(y) \exp\Big\{ - \sum_{\partial(p) \le c} f(p)(q^{-1}y)^{\partial(p)} +$$

$$+ \sum_{\partial(p) > c} (\log \chi_p(y) - f(p)(q^{-1}y)^{\partial(p)}) \Big\} \Big] \times \exp\Big\{ \sum_{k=1}^{\infty} \sum_{\partial(p)=k} f(p)(q^{-1}y)^k \Big\}$$

and denote by $H(y)$ the function in the square brackets. If further

$$L(y) = \sum_{k=1}^{\infty} \Big(q^{-k} \sum_{\partial(p)=k} f(p) \Big) y^k := \sum_{k=1}^{\infty} \mathcal{F}(k) y^k \,,$$

then

$$F(y) = H(y) \exp\{L(y)\}.$$

The routine considerations [3], [4], [13] show that $H(y)$ is analytic in the disc $|y| < 1 + c_2$ with some $c_2 > 0$ and $|H(y)| + |H'(y)| \ll 1$ uniformly with respect to all parameters of the function f in this region. The same we have for $Z(y)$, namely,

$$Z(y) = H_0(y) \exp\{L_0(y)\} \tag{3}$$

with

$$H_0(y) = \exp\Big\{\sum_{k=1}^{\infty}(-\pi(k)\log(1-q^{-k}y^k) - \mathcal{P}(k)y^k)\Big\}, L_0(y) = \sum_{k=1}^{\infty}\mathcal{P}(k)y^k$$

and $\mathcal{P}(k) = q^{-k}\pi(k)$. Here $H_0(y)$ is analytic and $|H_0(y)| + |H_0'(y)| \ll 1$ in the disc $|y| < 1 + c_2$. At last we can use the following representation

$$Z(y) = \tilde{Z}(y)/(1-y) \tag{4}$$

with a function $\tilde{Z}(y)$, analytic in $|y| < 1 + c_2$ such that $\tilde{Z}(1) = A$. Note that $\tilde{Z}(y) \neq 0$ for $|y| \leq 1$ except, maybe, at the point $y = -1$. In [6] we jointly with R. Warlimont showed that semigroups with $Z(-1) = 0$ really exist. We will see further that the asymptotic behaviour of $M_n(f)$ as $n \to \infty$ can be considerably different in both alternative cases when $\kappa = 0$ or $\kappa = 1$. The analogy with the corresponding results for multiplicative functions defined on $\mathbb{N}$ or semigroups satisfying Axiom A can brake down. This effect makes the problem more interesting.

3. Results

The dependence of f on n plays the essential role. When f does not depend on n we obtain an analogue of the Halász theorem [4].

Theorem 1. *Let $f : \mathcal{G} \to \mathbb{C}$ be a multiplicative function, $|f(a)| \leq 1$. Then there exist a real constant $t_0 \in (-\pi, \pi]$ and a complex constant D such that*

$$M_n(f) = D\exp\{it_0 n + i\sum_{k=1}^{n}\mathrm{Im}(\mathcal{F}(k)e^{-it_0 k})\} + o(1). \tag{5}$$

Here and in the following where it is supposed that $n \to \infty$ we do not indicate it.

Theorem 2. *In order that $M_n(f) = o(1)$ it is both necessary and sufficient that one of the following two conditions is satisfied:*

(I) *for each $t \in (-\pi, \pi]$ the series*

$$\sum_{k=1}^{\infty} q^{-k}\sum_{\partial(p)=k}(1 - \mathrm{Re}(f(p)e^{-itk})) \tag{6}$$

diverges;

(II) *there exists a unique* $t_0 \in (-\pi, \pi]$ *such that the series* (6) *converges for* $t = t_0$ *and*

$$\prod_{\partial(p)\le c} \chi_p(e^{-it_0}; f) = 0.$$

In the formulation of Theorems 1 and 2 it is difficult to notice the role which plays the exceptional zero of $Z(y)$ at $y = -1$. Therefore we consider the classical example of the Möbius function $\mu : \mathcal{G} \to \{0, 1, -1\}$ defined as a multiplicative function taking values $\mu(p) = -1$ and $\mu(p^k) = 0$ for each prime p and $k \ge 2$. Due to the prime number theorem (2) the convergence of the series (6) at $t_0 \in (-\pi, \pi]$ for $f = \mu$ is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} \frac{1 + \cos t_0 k}{k}(1 - (-1)^k \kappa). \tag{7}$$

If $\kappa = 0$, such a t_0 does not exist, hence $M_n(\mu) = o(1)$. If $\kappa = 1$, evidently, the series (7) converges for $t_0 = \pi$. Now Theorem 1 with the calculated constant D implies

$$M_n(\mu) = (-1)^n \prod_{k=1}^{\infty} \left(1 - \frac{1 + (-1)^k}{q^k} + \frac{(-1)^k}{q^{2k}}\right)^{\pi(k)} + o(1).$$

The formula (2) shows that in this case the last infinite product converges.

The following theorem is an analogue of the Delange result [1], [2].

Theorem 3. *In order that*

$$\lim_{n\to\infty} M_n(f) = M(f)$$

exists and $M(f) \ne 0$ *it is both necessary and sufficient that the series*

$$\sum_{k=1}^{\infty} q^{-k} \sum_{\partial(p)=k} (1 - f(p))$$

converges and

$$\prod_{\partial(p)\le c} \chi_p(1; f) \ne 0.$$

We remark here that Theorem 3 will be proved by applying Theorem 1. The first of the authors together with P. G. Slattery has proved it by more simple considerations (see [15]).

As in [12] for the semigroup $\mathbb{N}$, we obtain only partial results when f depends on n. Let us define the class $\mathcal{M}$ of multiplicative functions $f = f_n : \mathcal{G} \to \mathbb{C}$, $|f(a)| \le 1$, characterized by these two conditions:

$$\sup_{f\in\mathcal{M}} n^{-1} \sum_{k=1}^{n} kq^{-k} \sum_{\partial(p)=k} (1 - \operatorname{Re} f(p)) := \mu_n = o(1), \tag{8}$$

$$\sup_{f\in\mathcal{M}} \sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} (1 - \operatorname{Re} f(p)) \le M < \infty. \tag{9}$$

Let in the following $H_n(1)$ be derived from $H(1)$ defined in Section 2, by putting $f(p) = 1$ for $\partial(p) > n$.

Theorem 4. *For arbitrary δ, $0 < \delta < \frac{1}{3}$, and uniformly in $f \in \mathcal{M}$ we have*

$$M_n(f) = H_n(1)H_0^{-1}(1) \times \exp\left\{\sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} (f(p) - 1)\right\} + O_{M,\delta}\left(\mu_n^{\frac{1}{6}-\delta} + n^{-\frac{1}{2}}\right). \tag{10}$$

The above Theorems supply a possibility to prove probabilistic results about the distribution of additive functions defined on $\mathcal{G}$. We present here only some of them. Put

$$\nu_n(\dots) = A^{-1}q^{-n}\#\{a \in \mathcal{G}, \partial(a) = n, \dots\}.$$

Then one of the main problems can be formulated as follows:

Let $h_n : \mathcal{G} \to \mathbb{R}$ be a sequence of additive functions, and let $\alpha(n)$ be a sequence of real constants. When does the sequence

$$\nu_n(x) := \nu_n(h_n(a) - \alpha(n) < x)$$

weakly converge to a limit distribution function?

In the following the weak convergence will be denoted by $\Longrightarrow$.
Put $u^* = \min\{|u|, 1\}\operatorname{sgn} u$, $u \in \mathbb{R}$. As in the paper [12] we have

Theorem 5. *Let $h_n : \mathcal{G} \to \mathbb{R}$ be a sequence of additive functions, $\alpha(n) \in \mathbb{R}$. Suppose in advance that $h_n(p^r) \to 0$ for each fixed $p \in \mathcal{P}$, $r \ge 1$ and*

$$\sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} h_n^{*2}(p) = O(1), \quad \frac{1}{n}\sum_{k=1}^{n} kq^{-k} \sum_{\partial(p)=k} h_n^{*2}(p) = o(1). \tag{11}$$

In order that $\nu_n(x) \implies \Phi(x)$ *where* $\Phi(x)$ *is a distribution function it is both necessary and sufficient that the following two conditions are satisfied:*
(i) *there exists a non-decreasing bounded function* $\Psi(u)$ *defined on* $\bar{\mathbb{R}}$ *such that*

$$\Psi_n(u) := \sum_{k=1}^{n} q^{-k} \sum_{\substack{\partial(p)=k \\ h_n(p)<u}} h_n^{*2}(p) \implies \Psi(u),$$

$$\Psi_n(\pm\infty) \to \Psi(\pm\infty);$$

(ii) *for some constant* $\alpha \in \mathbb{R}$

$$\alpha(n) = \sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} h_n^*(p) + \alpha + o(1).$$

If these conditions are satisfied the characteristic function of the distribution $\Phi(x)$ *is equal to*

$$\exp\Big\{it\alpha + \int_{\mathbb{R}} (e^{itu} - 1 - itu^*)u^{*-2}d\Psi(u)\Big\}.$$

The famous Kubilius class H (see [11]) has its analogue too. Denote

$$A(n) = \sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} h(p), \quad B^2(n) = \sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} h^2(p).$$

Definition. We say that an additive function $h : \mathcal{G} \to \mathbb{R}$ belongs to the class H if $B(n) \to \infty$ and $B(un) \sim B(n)$ for each u, $0 < u < 1$.

Theorem 6. *Let* $h \in H$. *In order that* $\nu_n(h(a) - A(n) < xB(n))$ *weakly converges to a distribution function with variance one it is both necessary and sufficient that there exists a distribution function* $K(u)$ *such that*

$$K_n(u) := B^{-2}(n) \sum_{k=1}^{n} q^{-k} \sum_{\substack{\partial(p)=k \\ h(p)<uB(n)}} h^2(p) \implies K(u).$$

If this condition is satisfied the characteristic function $\varphi(t)$ *of the limiting distribution can be expressed by the Kolmogorov formula*

$$\varphi(t) = \exp\Big\{\int_{\mathbb{R}} (e^{itu} - 1 - itu)u^{-2}dK(u)\Big\}.$$

The function

$$\sum_{p|a} \log \partial(p)$$

defined on the semigroup of primary polynomials over a finite constant field evidently belongs to the class H and a remainder term estimate in its limit theorem has been obtained by J. L. Nicolas [14].

At last we demonstrate that limit theorems with normalizing constants $\beta(n) \sim n^{\alpha}$, $\alpha > 0$, are available, too.

Theorem 7. *Suppose $\beta(n) \to \infty$, and assume that there exists a nondecreasing bounded function $\Psi_0(u)$ such that*

$$\Psi_n^{\sim}(u) := \sum_{k=1}^{n} q^{-k} \sum_{\substack{\partial(p)=k \\ h(p)<u\beta(n)}} \left(\frac{h(p)}{\beta(n)}\right)^{*2} \implies \Psi_0(u), \Psi_n^{\sim}(\pm\infty) \to \Psi_0(\pm\infty).$$

Let further for each $\ell \in \mathbb{N}$ the sequences

$$\frac{\ell}{n^{\ell}} \sum_{k=1}^{n} q^{-k} k^{\ell} \sum_{\substack{\partial(p)=k \\ h(p)<u\beta(n)}} 1$$

weakly converge to some limiting distribution function $V_{\ell}(u)$. Then with

$$\alpha(n) = \beta(n) \sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} \left(\frac{h(p)}{\beta(n)}\right)^{*}$$

the sequence $\nu_n(h(a) - \alpha(n) < x\beta(n))$ weakly converges to a limiting distribution.

Its rather complicated characteristic function is given in the proof.

Corollary. *For each $\rho > 0$ the sequence*

$$\nu_n\Big(\sum_{p|a} \partial^{\rho}(p) < xn^{\varrho}\Big)$$

weakly converges to the distribution with the characteristic function

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^z}{z} \exp\Big\{\int_0^1 \frac{e^{itu^{\rho}}-1}{u} e^{-zu} du\Big\} dz.$$

Comparing our probabilistic results with that obtained for additive functions defined on $\mathbb{N}$ (see [3], [11], [12], [13]) one can notice a great analogy. Principal differences appear in the problem of distribution of real valued multiplicative functions. To this topic we shall devote the next paper.

4. Auxilliary lemmas

Our method is based on the information about the function $L(y)$ near the boundary $|y| = 1$. Let in the following $r = \exp\{-\frac{1}{n}\}$. The symbol $\ll$ will be used when the constant implied depends on $\mathcal{G}$ only. All the lemmas of this paragraph hold even for sequences of multiplicative functions $f = f_n$ such that $|f(a)| \le 1$.

Lemma 1. *We have*

$$\max_{|t|\le\pi} \exp\{\operatorname{Re} L(re^{it})\} \ll n \exp\Big\{-\min_{|t|\le\pi} \sum_{k=1}^{n} (\mathcal{P}(k) - \operatorname{Re}(\mathcal{F}(k)e^{itk}))\Big\}.$$

Proof. At first we note that

$$\sum_{k>n} r^k/k + \sum_{k\le n} (r^k - 1)/k \ll 1 \tag{12}$$

and by (3) and (4) we have $\exp\{L_0(r)\} \ll Z(r) \ll n$. Now remembering the asymptotic formula (2) and estimating $\exp\{\operatorname{Re} L(re^{it}) - L_0(r)\}$ we obtain the desired result.

Lemma 2. *Suppose $K > 1$ and*

$$\Psi_n := \sum_{k=1}^{n} (\mathcal{P}(k) - \operatorname{Re}\mathcal{F}(k)) \le M_1 < \infty. \tag{13}$$

Then for $\delta_1 > 0$ there exists a positive constant C_1, depending on $\mathcal{G}$, M_1 and δ_1 only, such that

$$\max_{K/n\le|t|\le\pi} \exp\{\operatorname{Re} L(re^{it})\} \le C_1 n K^{-1+\delta_1}.$$

Proof. As previously we have

$$\exp\{\operatorname{Re} L(re^{it})\} \ll n \exp\Big\{-\sum_{k=1}^{\infty} r^k (\mathcal{P}(k) - \operatorname{Re}(\mathcal{F}(k)e^{itk}))\Big\}. \tag{14}$$

For the difference in the brackets we will apply the inequality

$$1 - \operatorname{Re}(z_1 z_2) \ge 1 - \operatorname{Re} z_1 - (1 - \operatorname{Re} z_2) - 2\sqrt{1 - \operatorname{Re} z_1}\sqrt{1 - \operatorname{Re} z_2}$$

where $z_1, z_2 \in \mathbb{C}$ and $|z_1| \le 1$, $|z_2| \le 1$. Hence

$$\mathcal{P}(k) - \operatorname{Re}(\mathcal{F}(k)e^{itk}) \ge q^{-k}\pi(k)(1 - \cos tk) - q^{-k} \sum_{\partial(p)=k} (1 - \operatorname{Re} f(p)) -$$

$$-2q^{-k}\sqrt{1-\cos tk}\sum_{\partial(p)=k}\sqrt{1-\operatorname{Re} f(p)}.$$

Applying twice the Cauchy inequality we have from (12) and (14)

$$\exp\{\operatorname{Re} L(re^{it})\} \ll n\exp\Big\{-\sum_{k=1}^{\infty}q^{-k}r^k\pi(k)(1-\cos tk)+\Psi_n+$$

$$+O((\sum_{k=1}^{\infty}q^{-k}r^k\pi(k)(1-\cos tk))^{\frac12}(\Psi_n+1)^{\frac12})\Big\}.$$

Now appealing to (3) and (4) we see that

$$\exp\{\operatorname{Re} L(re^{it})\} \le C_2 n\left(\frac{|Z(re^{it})|^{1-\delta_1}}{Z(r)}\right) \le C_3 nK^{-1+\delta_1}$$

if $K/n \le |t| \le \pi$ and $C_i = C_i(\delta_1, M_1)$, $i = 2, 3$, are constants. Lemma 2 is proved.

Lemma 3. *We have*

$$J_1 := \int_{-\pi}^{\pi}|L'(re^{it})|^2dt \ll n.$$

Proof. We will apply the Parseval equality for the Fourier series. It yields

$$J_1 = \int_{-\pi}^{\pi}\Big|\sum_{k=1}^{\infty}kr^{k-1}\mathcal{F}(k)e^{it(k-1)}\Big|^2dt =$$

$$=\sum_{k=1}^{\infty}k^2r^{2(k-1)}|\mathcal{F}(k)|^2 \le \sum_{k=1}^{\infty}k^2r^{2(k-1)}\mathcal{P}(k)^2 = \int_{-\pi}^{\pi}|L_0'(re^{it})|^2dt\,.$$

But by (3) and (4) the last integral does not exceed

$$\int_{-\pi}^{\pi}\left|\frac{Z'}{Z}(re^{it})\right|^2dt + O(1) \ll \int_{-\pi}^{\pi}\frac{dt}{|1\pm re^{it}|^2}+1 \ll n+\int_{\frac1n}^{\pi}\frac{dt}{t^2} \ll n.$$

Thus the Lemma is proved.

Lemma 4. *For each $\gamma > 1$ there exists a positive constant C_4 depending on $\mathcal{G}$ and γ only such that*

$$J_2 = J_2(\gamma) := \int_{-\pi}^{\pi}\exp\{\gamma\operatorname{Re} L(re^{it})\}dt \le C_4 n^{\gamma-1}.$$

The PROOF is based on the Parseval equality, too. At first we expand the integrand into a Fourier series. By (12) it is enough to consider the function

$$Q(t) := \prod_{k=1}^{n} \exp\left\{\frac{\gamma}{2} r^k \mathcal{F}(k) e^{itk}\right\} = \prod_{k=1}^{n} \sum_{l=0}^{\infty} \left(\frac{\gamma}{2}\mathcal{F}(k)\right)^{\ell} \frac{r^{k\ell} e^{itk\ell}}{\ell!} =$$

$$= \sum_{m=0}^{\infty} r^m e^{itm} \sum_{\substack{\ell_1,\dots,\ell_n \geq 0 \\ \ell_1+2\ell_2+\dots+n\ell_n=m}} \frac{\mathcal{F}^{\ell_1}(1)\dots\mathcal{F}^{\ell_n}(n)}{\ell_1!\dots\ell_n!} \left(\frac{\gamma}{2}\right)^{\ell_1+\dots+\ell_n} :=$$

$$:= \sum_{m=0}^{\infty} r^m e^{itm} b_m \ .$$

But

$$|b_m| \leq \sum_{\substack{\ell_1,\dots,\ell_n \geq 0 \\ \ell_1+2\ell_2+\dots+n\ell_n=m}} \frac{\mathcal{P}^{\ell_1}(1)\dots\mathcal{P}^{\ell_n}(n)}{\ell_1!\dots\ell_n!} \left(\frac{\gamma}{2}\right)^{\ell_1+\dots+\ell_n} := d_m \ .$$

Now the Parseval equality yields

$$J_2 = \sum_{m=0}^{\infty} r^{2m}|b_m|^2 \leq \sum_{m=0}^{\infty} r^{2m} d_m^2 = \int_{-\pi}^{\pi} \left|\exp\left\{\frac{\gamma}{2}\sum_{k=1}^{n} r^k \mathcal{P}(k) e^{itk}\right\}\right|^2 dt$$

which by (3) and (12) does not exceed

$$C_5 \int_{-\pi}^{\pi} |Z(re^{it})|^{\gamma} dt \ll \int_{-\pi}^{\pi} \frac{dt}{|1-re^{it}|^{\gamma}} \leq C_6 n^{\gamma-1}.$$

This ends the proof.

The Main Lemma. *Suppose the condition* (13) *is satisfied. Then for each* $K > 1$, $0 < \delta_2 < 1$ *the following formula holds:*

$$M_n(f) = \frac{H(1)}{H_0(1)2\pi i} \int_{1-iK}^{1+iK} \frac{e^z}{z} \exp\left\{\sum_{k=1}^{\infty} (\mathcal{F}(k) - \mathcal{P}(k)) e^{-\frac{zk}{n}}\right\} dz \times$$

$$\times \left(1 + O\left(\frac{K}{n}\right)\right) + O(K^{-\frac{1}{2}+\delta_2}) + O(n^{-\frac{1}{2}}).$$

The functions $H(y)$ *and* $H_0(y)$ *are defined in the second section and the constant in the symbol* O *depends at most on* $\mathcal{G}$, δ_2 *and* M_1, *given by the condition* (13).

PROOF. By the Cauchy formula we have

$$M_n(f) = \frac{1}{A2\pi i}\int_{|y|=r} F(y)y^{-n-1}dy \tag{15}$$

where $r = \exp\left\{-\frac{1}{n}\right\}$. Let J denotes the part of the integral where $K/n \le \arg y = t \le \pi$, $K > 1$. Integrating by parts we obtain

$$\begin{aligned} J = ie\int_{K/n}^{\pi} F(re^{it})e^{-itn}dt = O(n^{-1}\max_{K/n\le t\le\pi}|F(re^{it})|)+ \\ + O(n^{-1}\int_{K/n}^{\pi}|H'(re^{it})|\exp\{\operatorname{Re}L(re^{it})\}dt) + \\ + O(n^{-1}\int_{K/n}^{\pi}|H(re^{it})|\exp\{\operatorname{Re}L(re^{it})\}|L'(re^{it})|dt). \end{aligned}$$

Now Lemmas 2, 3, 4 and the estimates discussed in the second part show that for $1 < \gamma < 2$

$$\begin{aligned} J = O(K^{-1+\delta_1}) + O(n^{-1}J_2^{\frac{1}{2}}(2))+ \\ +O(n^{-1}J_1^{\frac{1}{2}}(\max_{K/n<t\le\pi}\exp\{\operatorname{Re}L(re^{it})\})^{\frac{2-\gamma}{2}}J_2^{\frac{1}{2}}(\gamma) = \\ = O(K^{-1+\delta_1}) + O(n^{-\frac{1}{2}}) + O(K^{-(1-\delta_1)(1-\frac{\delta}{2})}) = O(K^{-\frac{1}{2}+\delta_2} + n^{-\frac{1}{2}}). \end{aligned} \tag{16}$$

Here $0 < \delta_2 < \frac{1}{2}$ is arbitrary and γ, δ_1 γ are chosen to satisfy

$$(1-\delta_1)(1-\frac{\gamma}{2}) > \frac{1}{2} - \delta_2.$$

The same estimate holds for $-\pi \le t \le -\frac{K}{n}$. In the remaining interval $|t| \le \frac{K}{n}$ we compair $F(y)$ with $Z(y)$ to obtain

$$F(re^{it}) = \frac{\tilde{Z}(re^{it})H(re^{it})}{(1-re^{it})H_0(re^{it})}\exp\Big\{\sum_{k=1}^{\infty}(\mathcal{F}(k)-\mathcal{P}(k))e^{-\frac{(1-itn)k}{n}}\Big\}.$$

Further we have for $|t| \le \frac{K}{n}$

$$\tilde{Z}(re^{it}) = A + O\left(\frac{K}{n}\right);$$

$$H_i(re^{it}) = H_i(1) + O\left(\frac{K}{n}\right), \quad i = 0, 1, \quad H = H_1;$$

$$(n(1-re^{it}))^{-1} = \left(n\left(\frac{1-itn}{n} + O\left(\left(\frac{|1+itn|}{n}\right)^2\right)\right)\right)^{-1}$$

$$= (1-itn)^{-1}\left(1 + O\left(\frac{K}{n}\right)\right).$$

Using these estimates and (16) in the formula (15) we obtain

$$M_n(f) = \frac{H(1)}{H_0(1)}\left(1 + O\left(\frac{K}{n}\right)\right)\frac{1}{2\pi}\int_{|t|\le\frac{K}{n}} \frac{e^{1-itn}}{1-itn} \times$$

$$\times \exp\left\{\sum_{k=1}^{\infty}(\mathcal{F}(k) - \mathcal{P}(k))e^{-\frac{(1-itn)k}{n}}\right\} dt + O(K^{-\frac{1}{2}+\delta_2} + n^{-\frac{1}{2}}).$$

Thus the Main Lemma is proved.

5. Proofs of the Theorems

Proof of Theorem 4. The values $f(p^k)$ for $\partial(p) > n$ have no influence to $M_n(f)$ therefore we take $f(p^k) = 1$ when $\partial(p) > n$ and $k \ge 1$. It is easy to check that (8) implies

$$\sup_{f\in\mathcal{M}} n^{-1}\Big|\sum_{k=1}^{n} kq^{-k} \sum_{\partial(p)=k} \operatorname{Im} f(p)\Big| = O(\sqrt{\mu_n}).$$

Hence for $z = 1 + it$, $|t| \le K$, we have

$$\exp\Big\{\sum_{k=1}^{n}(\mathcal{F}(k) - \mathcal{P}(k))e^{-zk/n}\Big\} =$$

$$= \exp\Big\{\sum_{k=1}^{n}(\mathcal{F}(k) - \mathcal{P}(k))(1 + O(|z|\sqrt{\mu_n}))\Big\} =$$

$$= \exp\Big\{\sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} (f(p) - 1)\Big\}(1 + O(|z|\sqrt{\mu_n})).$$

The integral in the formula for $M_n(f)$ in the Main Lemma is equal to

$$\exp\Big\{\sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} (f(p)-1)\Big\} \int_{1-iK}^{1+iK} \frac{e^z(1+O(|z|\sqrt{\mu_n}))}{z} dz =$$

$$= \exp\Big\{\sum_{k=1}^{N} q^{-k} \sum_{\partial(p)=k} (f(p)-1)\Big\}(1+O(K^{-1}+K\sqrt{\mu_n})).$$

The choice $K = \mu_n^{-\frac{1}{3}}$ and the Main Lemma yield Theorem 4.

Proof of Theorem 1. Suppose at first that the series (6) converges at $t = t_0$. Such a $t_0 \in (-\pi, \pi]$ can only be unique. For, if (6) converges for $t = t_0^1$, too, then the inequality

$$1 - \operatorname{Re}(z_1 z_2) \ll (1 - \operatorname{Re} z_1) + (1 - \operatorname{Re} z_2),\ |z_1| \le 1,\ |z_2| \le 1$$

and (2) yield

$$\infty > \sum_{k=1}^{\infty} q^{-k} \sum_{\partial(p)=k} (1 - \cos(t_0 - t_0^1)k) \gg \sum_{\ell=1}^{\infty} \frac{1 - \cos(t_0 - t_0^1)(2\ell+1)}{2\ell+1}$$

which is impossible if $t_0 \neq t_0^1$.

Put $\hat{f}(a) = f(a)\exp\{-it_0\partial(a)\}$. For $\mathcal{M} = \{\hat{f}\}$ the condition (9) is satisfied. Now $\hat{f}$ does not depend on n, so (9) implies the condition (8). Applying (10) we get the asymptotic relation

$$M_n(\hat{f}) = \frac{\hat{H}_n(1)}{H_0(1)} \exp\Big\{\sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} (\hat{f}(p)-1)\Big\} + o(1)$$

where $\hat{H}_n(1)$ is obtained from $H_n(1)$ by setting $\hat{f}(p^k)$ in place of $f(p^k)$. Since

$$M_n(f) = e^{it_0 n} M_n(\hat{f}),$$

we obtain (5) with

$$D = \frac{\lim\limits_{n\to\infty} \hat{H}_n(1)}{H_0(1)} \exp\Big\{\sum_{k=1}^{\infty} q^{-k} \sum_{\partial(p)=k} (\operatorname{Re}(f(p)e^{-it_0 k}) - 1)\Big\},$$

when the series (6) converges at $t = t_0$.

If the condition (I) of Theorem 2 is satisfied, then as in the proof of the Main Lemma we integrate in the complex domain. We take the

formula

$$M_n(f) = \frac{A}{2\pi i n} \int_{|y|=r} F'(y) y^{-n} dy$$

where, as before, $r = \exp\left\{-\frac{1}{n}\right\}$. Now by Lemma 1 we have

$$\max_{|t|\le\pi} \exp\{\operatorname{Re} L(re^{it})\} = o(n).$$

Then for arbitrary $\gamma,\ 1 < \gamma < 2$, applying Lemmas 3 and 4 we obtain

$$M_n(f) = O(n^{-1} J_2^{\frac{1}{2}}(2)) + O(n^{\frac{2-\gamma}{2}} J_1^{\frac{1}{2}} J_2^{\frac{1}{2}}(\gamma)) = o(1).$$

So in this case we have $D = 0$, and Theorem 1 is proved.

Theorems 2 and 3 are only corollaries of Theorem 1 if we analyse the formula (5) and possible values of the constant D.

Remark. When the function f depends on n, too, and

$$\min_{|t|\le\pi} \sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} (1 - \operatorname{Re}(f(p)e^{-itk})) \to \infty$$

then $M_n(f) = o(1)$ as $n \to \infty$. In order to see this one can repeat the second part of the proof of Theorem 1.

Proof of Theorem 5. Let $f_n(a) = f_n(a,t) = \exp\{ith_n(a)\}$, $t \in \mathbb{R}$ and let $T > 0$ be an arbitrary fixed number. We will apply Theorem 4 when $\mathcal{M} = \{f_n(a,t); |t| \le T\}$. The conditions (8) and (9) evidently follow from (11). The formulas given in the second part for H_ℓ, $\ell \ge 0$, yield that the condition $f_n(p^k, t) = o(1)$ which holds uniformly in $|t| \le T$ for each $p \in \mathcal{P}$ and $k \ge 1$ fixed, imply $H_n(1)/H_0(1) = 1 + o(1)$ with the same uniformity. The weak convergence of $\nu_n(x)$ is equivalent to the uniform convergence of $M_n(f_n)\exp\{-it\alpha(n)\}$ when $|t| \le T$. Due to the formula (10) we have that the last one is equivalent to the convergence of

$$\sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} (\exp\{ith_n(p)\} - 1) - it\alpha(n) =$$

$$= \int_{\mathbb{R}} (e^{itu} - 1 - itu^*) u^{*^{-2}} d\Psi_n(u) + it\Big(\sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} h_n^*(p) - \alpha(n)\Big) \tag{17}$$

with the same uniformity. Now as in [3] or [12] using the Helly theorem we obtain both the necessity and sufficiency of the conditions (i) and (ii) in Theorem 5 as well as the formula of the characteristic function of the limiting distribution.

Proof of Theorem 6 is in fact contained in the previous one. We note that for each $0 < \varepsilon < 1$

$$Q_n := \frac{1}{n}\sum_{k=1}^{n} kq^{-k} \sum_{\partial(p)=k} h^2(p) \le \varepsilon B^2(n) + (B^2(n) - B^2(\varepsilon n)).$$

Hence $Q_n = o(B^2(n))$ as $n \to \infty$ and the condition (11) even without the star notation for $h_n = h/B(n)$ is satisfied. Other details can be omitted.

Proof of Theorem 7. Let $f_n(p^k) = \exp\{ith(p^k)/\beta(n)\}$ when $\partial(p) \le n$ and $f_n(p^k) = 1$ for $\partial(p) > n$. As previously, with the help of the expression (17), we have the convergence

$$\sum_{k=1}^{n} q^{-k} \sum_{\partial(p)=k} (f_n(p) - 1) - it\alpha(n) = \int_{\mathbb{R}} \frac{e^{itu} - 1 - itu^*}{u^{*2}} d\Psi_0(u) + o(1) = \\ := \kappa(t) + o(1)$$

uniformly in $|t| \le T$ for each $T > 0$. Further, for $\ell \ge 1$,

$$\frac{\ell}{n^\ell} \sum_{k=1}^{n} q^{-k} k^\ell \sum_{\partial(p)=k} f_n(p) = \mathcal{T}_\ell(t) + o(1) \tag{19}$$

where $\mathcal{T}_\ell(t)$ is the characteristic function of the distribution $V_\ell(u)$. As in [13] we apply the Main Lemma. The formulas (18) and (19) yield

$$\sum_{k=1}^{n} (\mathcal{F}(k) - \mathcal{P}(k)) e^{-zk/n} = it\alpha(n) + \kappa(t) + \sum_{\ell=1}^{\infty} \frac{(-1)^\ell (\mathcal{T}_\ell(t) - 1) z^\ell}{\ell!\ell} + o(1) \\ := it\alpha(n) + q(z,t) + o(1)$$

uniformly in $|t| \le T$ and $|\operatorname{Im} z| \le K$. The factor $H_n(1)/H_0(1)$ tends to 1 as $n \to \infty$ with the same uniformity. Hence

$$\exp\{-it\alpha(n)\} q^{-n} A^{-1} \sum_{\partial(a)=n} \exp\{ith(a)/\beta(n)\} = \\ = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^z}{z} \exp\{q(z,t)\} dz + o(1)$$

where the convergence of the integral is also uniform in $|t| \le T$. This ends the proof.

In order to *prove* the Corollary we need only to observe that summation by parts yields

$$\sum_{k=1}^{N} k^{-1}\left(\exp\left\{it\left(\frac{k}{n}\right)^{\rho}\right\}-1\right)\exp\{-zk/n\}=\int_0^1 \frac{e^{itu^{\rho}}-1}{u}e^{-zu}du+o(1)$$

uniformly in $|t|\le T$ for each $T>0$.

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