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Some commutativity theorems for Banach algebras

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A number of theorems in ring theory, mostly due to HERSTEIN, are devoted to showing that certain rings must be commutative as a consequence of conditions which are seemingly too weak to imply commutativity. See [4, Chapter 3]. In [7] we showed that, in the special case of a Banach algebra, some of these results can be sharpened. We continue this program here.

Let R be a ring with center Z. We let $[a, b]$ denote the Lie product $ab - ba$ and $a \cdot b$ the Jordan product $ab + ba$. In a recent paper [2] R. D. GIRI and A. R. DHOBLE showed the following.

Theorem 1. *Suppose that R is a semiprime ring and that n, m are fixed positive integers each larger than one. Suppose that either* (a) $[x^n, y^m] \in Z$ *for every* $x, y \in R$ *or* (b) $x^n \cdot y^m \in Z$ *for every* $x, y \in R$. *Then R is commutative.*

Henceforth A will denote a Banach algebra over the complex field with center Z. For this special case we prove a sharper version of Theorem 1.

Theorem 2. *Suppose that there are non-empty open subsets G_1, G_2 of A such that for each $x \in G_1$ and $y \in G_2$ there are positive integers $n = n(x, y)$, $m = m(x, y)$ depending on x and y, $n > 1$, $m > 1$, such that either $[x^n, y^m] \in Z$ or $x^n \cdot y^m \in Z$. Then A is commutative if A is semiprime.*

Let $p(t) = \sum_{r=0}^{n} b_r t^r$ be a polynomial in the real variable t with coefficients in A where $p(t) \in Z$ for an infinite set of real values t. Then every $b_r \in Z$. For let $f(x)$ be any bounded linear functional on A which vanishes on Z. Then $\sum_{r=0}^{n} f(b_r) t^r = 0$ for an infinite set of reals so that each $f(b_r) = 0$. As Z is a closed linear subspace of A this implies that each $b_r \in Z$.

We begin the PROOF of Theorem 2. Fix $x \in G_1$. For positive integers $n \geq 2,\ m \geq 2$ let $V(n,m)$ be the set of $y \in A$ for which $[x^n, y^m] \notin Z$ and $x^n \cdot y^m \notin Z$. Each $V(n,m)$ is open in A. If every $V(n,m)$ is dense then, by the Baire category theorem, so is the intersection W of all the sets $V(n,m)$. But W being dense would violate the nature of G_1 and G_2. Hence there are integers $r \geq 2$ and $s \geq 2$ so that $V(r,s)$ is not dense. Therefore there is a non-empty open subset G_3 in the complement of $V(r,s)$. For each $y \in G_3$ either $[x^r, y^s] \in Z$ or $x^r \cdot y^s \in Z$. Let $y_0 \in G_3$ and $w \in A$. There is positive real number $a > 0$ such that $y_0 + tw \in G_3$ for all t, $0 \leq t \leq a$. For each such t either

$$[x^r, (y_0 + tw)^s] \in Z \tag{1}$$

or

$$x^r \cdot (y_0 + tw)^s \in Z\,. \tag{2}$$

Therefore at least one of (1) and (2) must be valid for infinitely many real t. Suppose (1) is valid for these t. Now $[x^r, (y_0 + tw)^s]$ can be written as a polynomial in t with coefficients in A. The coefficient of t^s in that polynomial is $[x^r, w^s]$. Therefore $[x^r, w^s] \in Z$. Likewise if (2) is valid for infinitely many values of t then $x^r \cdot w^s \in Z$.

Thus, given $x \in G_1$, there are positive integers $r > 1,\ s > 1$ so that, for each $w \in A$, either $[x^r, w^s] \in Z$ or $x^r \cdot w^s \in Z$. Let $F_1 = \{w \in A : [x^r, w^s] \in Z\}$ and $F_2 = \{w \in A : x^r \cdot w^s \in Z\}$. Now $A = F_1 \cup F_2$ and each F_k is closed. Then, by the Baire category theorem, at least one of F_1 and F_2 must contain a non-empty open subset of A.

Suppose F_1 contains a ball with center v_0 and radius $r > 0$. Let $z \in A$. For infinitely many t we must have $[x^r, (v_0 + tz)]^s \in Z$. Therefore $[x^r, z^s] \in Z$ for every $z \in A$. Likewise if F_2 has non-void interior then $x^r \cdot z^s \in Z$ for every $z \in A$.

Consequently, given $x \in G_1$, there are positive integers $r > 1,\ s > 1$ so that either $[x^r, z^s] \in Z$ for all $z \in A$ or $x^r \cdot z^s \in Z$ for all $z \in A$.

Now we note that in our set-up with G_1 and G_2 we could replace G_2 by A. Next we reverse the roles of G_1 and G_2 (now replaced by A) in the above arguments. Thus, for each $y \in A$, there are positive integers $r > 1,\ s > 1$ depending on y so that either $[x^r, y^s] \in Z$ for all $x \in A$ or $x^r \cdot y^s \in Z$ for all $x \in A$.

For positive integers $m > 1$ and $n > 1$ let $W(n,m)$ be the set of all $y \in A$ so that either $[x^n, y^m] \in Z$ for all $x \in A$ or $x^n \cdot y^m \in Z$ for all $x \in A$. We check that $W(n,m)$ is closed. For let $\{y_k\}$ be a sequence in $W(n,m)$ and $y_k \to w$. Then either there is an infinite subsequence $\{y_{k_j}\}$ so that $[x^n, y_{k_j}^m] \in Z$ for all $x \in A$ and each k_j or such a subsequence $\{y_{k_j}\}$ where $x^n \cdot y_{k_j}^m \in Z$ for all $x \in A$ and each k_j. Thus $w \in W(n,m)$. Inasmuch as A is the union of all the sets $W(n,m)$ we see by the Baire category theorem that some $W(p,q)$ must contain a non-void open subset G_4 of A. Let $y_0 \in G_4$. For each $v \in A$ there is some real number $b > 0$ so that when $0 \le t \le b$ either $[x^p, (y_0 + tv)^q] \in Z$ for all $x \in A$ or $x^p \cdot (y_0 + tv)^q \in Z$ for all $x \in A$. Now at least one of these alternatives is valid for infinitely many real t. Reasoning already used shows that either $[x^p, v^q] \in Z$ for all $x \in Z,\ v \in A$ or $x^p \cdot v^q \in Z$ for all $x \in A,\ v \in A$. If A is semiprime then A is now seen to be commutative by Theorem 1.

In the proof of Theorem 2 we needed $m > 1$ and $n > 1$ in order to use Theorem 1. If A has an identity we can do with $m \ge 1,\ n \ge 1$, as we do not then cite Theorem 1.

Theorem 3. *Suppose that A has an identity e and that there are non-empty open subsets G_1, G_2 of A where, for each $x \in G_1, y \in G_2$, there are integers $m = m(x,y),\ n = n(x,y),\ m \ge 1,\ n \ge 1$, such that either $[x^n, y^m] \in Z$ or $x^n \cdot y^m \in Z$. If Z is semisimple then A is commutative.*

By the proof of Theorem 2 there exist positive integers p and q, $p \ge 1$, $q \ge 1$, so that either $[x^p, v^p] \in Z$ for all $x, v \in A$ or $x^p \cdot v^q \in Z$ for all $x, v \in A$. In case $[x^p, v^q] \in Z$ for all $x, v \in A$ we may replace v by $e + tv$. Then $[x^p, (e + tv)^q] \in Z$ for all t. The coefficient of t in the polynomial $[x^p, (e + tv)^q]$ is $[x^p, v]$. Then $[x^p, v] \in Z$ for all x and v in A. Now replace x by $e + tx$ and $[(e + tx)^p, v] \in Z$ for all t. Then $[x, v] \in Z$ for all $x, v \in Z$. Likewise if $x^p \cdot v^q \in Z$ for all x and v in A we see that $x \cdot v \in Z$ for all x and v in A.

In the case that $x \cdot v \in Z$ for all x, v set $v = e$ to see that $2x \in Z$ for all $x \in A$. Then A is commutative. It remains to consider the case where $[x, v] \in Z$ for all x and $v \in A$. By the Kleinecke–Shirokov theorem [1, Prop. 13, p.91] each $w = [x, v]$ is a generalized nilpotent element of A, that is, $\lim \|w^n\|^{1/n} = 0$. Then $[x, v]$ is a generalized nilpotent element in the commutative Banach algebra Z and so is in the radical of Z. As Z is semisimple $[x, v] = 0$ so that A is commutative.

We point out that is easy to show that Z is semisimple if A is semisimple. See, for example, [6, Lemma 2.1].

In the situation of Theorem 3 we next drop the requirement that Z be semisimple. Then A need not be commutative as the following example shows.

First let B be the three-dimensional complex algebra with basis $\{a, b, c\}$ and multiplication given by

$$(\lambda_1 a + \mu_1 b + \nu_1 c)(\lambda_2 a + \mu_2 b + \nu_2 c) = (\lambda_1 \mu_2 - \lambda_2 \mu_1) c$$

where the λ_k, μ_k and ν_k are complex scalars. With the norm, say,

$$\|\lambda a + \mu b + \nu c\| = (|\lambda|^2 + |\mu|^2 + |\nu|^2)^{1/2}.$$

B is a Banach algebra (as the product of any three elements of B is zero, B is associative). Now let A be the Banach algebra obtained by adjoining an identity e to B where $\|\gamma e + x\| = |\gamma| + \|x\|$ for $x \in B$ and γ complex. For x, y in B we have

$$[\gamma_1 e + x,\ \gamma_2 e + y] = [x, y]$$

which is a multiple of c. Therefore, as c is in the center of A, we have $[v, w] \in Z$ for all $v, w \in A$. Hence the requirements of Theorem 3 for G_1 and G_2 hold if $G_1 = G_2 = A$. However A is not commutative.

For the purposes of the next theorem we discuss a point in the theory of non-associative algebras. Let K be a *non-associative* algebra. By the center of K is meant [5, p.14] the set of all $z \in K$ where $zx = xz$ for all $x \in K$ and where

$$(x, y, z) = (z, x, y) = (x, z, y) = 0$$

for all $x, y \in K$. Here (a, b, c) is the associator of the elements a, b and c,

$$(a, b, c) = (ab)c - a(bc).$$

Now we consider A as a non-associative algebra A^J with its multiplication the Jordan multiplication $x \cdot y = xy + yx$. Let Z^J be the center of A^J according to the above definition of center.

For a Lie ideal U of A as in [3, p.5] we set

$$T(U) = \{x \in A : [x, A] \subset U\}.$$

As noted there $T(U)$ is both a subalgebra and a Lie ideal of A and $T(U) \supset U$.

Lemma. *For A we have $Z^J = T(Z)$.*

Proof. A straight-forward calculation shows that

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) = [b, [a, c]]$$

for all a, b and c in A. Then Z^J is the set of all $z \in A$ such that

$$[x, [y, z]] = [z, [x, y]] = [y, [z, x]] = 0$$

for all $x, y \in A$. Thus we see that $Z^J \subset T(Z)$. Conversely suppose that $z \in T(Z)$ so that $[[z, x], y] = 0$ for all $x, y \in A$. Inasmuch as the Jacobi identity gives us

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

for all $x, y, z \in A$, we also get $T(Z) \subset Z^J$. Also, as $Z \subset T(Z)$, we have $Z \subset Z^J$.

Theorem 4. *Let A be a Banach algebra with identity e which satisfies the requirements on G_1 and G_2 of Theorem 3. Then $A = Z^J$.*

PROOF. As shown in the proof of Theorem 3 either A is commutative (so that also $A = Z^J$) or $[x, y] \in Z$ for all $x, y \in A$. Then $A = T(Z) = Z^J$ by the above lemma.

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