Publ. Math. Debrecen
81/1-2 (2012), 127–144
DOI: 10.5486/PMD.2012.5099

Automorphisms on algebras of operator-valued Lipschitz maps

By MARÍA BURGOS (Jerez), A. JIMÉNEZ-VARGAS (Almería)
and MOISÉS VILLEGAS-VALLECILLOS (Puerto Real)

Abstract. Let $\mathrm{Lip}(X,\mathcal{B}(\mathcal{H}))$ and $\mathrm{lip}_\alpha(X,\mathcal{B}(\mathcal{H}))$ $(0<\alpha<1)$ be the big and little Banach *-algebras of $\mathcal{B}(\mathcal{H})$-valued Lipschitz maps on X, respectively, where X is a compact metric space and $\mathcal{B}(\mathcal{H})$ is the $\mathcal{C}^*$-algebra of all bounded linear operators on a complex infinite-dimensional Hilbert space $\mathcal{H}$. We prove that every linear bijective map that preserves zero products in both directions from $\mathrm{Lip}(X,\mathcal{B}(\mathcal{H}))$ or $\mathrm{lip}_\alpha(X,\mathcal{B}(\mathcal{H}))$ onto itself is biseparating. We give a Banach–Stone type description for the *-automorphisms on such Lipschitz *-algebras, and we show that the algebraic reflexivity of the *-automorphism groups of $\mathrm{Lip}(X,\mathcal{B}(\mathcal{H}))$ and $\mathrm{lip}_\alpha(X,\mathcal{B}(\mathcal{H}))$ holds for $\mathcal{H}$ separable.

1. Introduction

Let $\mathcal{A}$ be a Banach *-algebra. A continuous linear map $\Phi:\mathcal{A}\to\mathcal{A}$ is a *local automorphism* if for every $a\in\mathcal{A}$, there exists an automorphism Φ_a of $\mathcal{A}$, possibly depending on a, such that $\Phi(a)=\Phi_a(a)$. Similarly, a continuous linear map $\Phi:\mathcal{A}\to\mathcal{A}$ is an *approximate local automorphism* if for every $a\in\mathcal{A}$, there exists a sequence of automorphisms of $\mathcal{A}$, $\{\Phi_n\}$, that may depend on a, such that $\Phi(a)=\lim_{n\to\infty}\Phi_n(a)$.

Obviously, every automorphism of $\mathcal{A}$ is an (approximate) local automorphism of $\mathcal{A}$, but the converse is not true in general. Precisely, the automorphism

Mathematics Subject Classification: 46L40, 47B48, 47L10.
Key words and phrases: algebraic reflexivity, local automorphism, Lipschitz algebra, $\mathcal{C}^*$-algebra.
Research partially supported by the MCYT projects MTM 2007-65959, MTM 2008-02186 and MTM 2010-17687, and Junta de Andalucía grant FQM-3737. The authors would like to thank the referee for their comments and suggestions.

group of $\mathcal{A}$ is said to be *algebraically reflexive* (*topologically reflexive*) if every local automorphism (respectively, approximate local automorphism) of $\mathcal{A}$ is an automorphism of $\mathcal{A}$. Analogous definitions for *-automorphisms of $\mathcal{A}$ can be given.

The study of local automorphisms of Banach algebras was started by Larson in [19, Some concluding remarks (5), p. 298], and since then they have been a matter of interest, mainly in the theory of operator algebras. Concretely, if $\mathcal{B}(X)$ is the algebra of all bounded linear operators on a complex infinite-dimensional Banach space X, Larson and Sourour [20] proved that every surjective local automorphism of $\mathcal{B}(X)$ is an automorphism. The real case was solved by Brešar and Šemrl in [7]. Moreover, they showed in [8] that the automorphism group of $\mathcal{B}(\mathcal{H})$ (no surjectivity is assumed now) is algebraically reflexive provided that $\mathcal{H}$ is an infinite-dimensional separable Hilbert space. In fact, this group is topologically reflexive as Molnár showed in [21]. Concerning local automorphisms on other operator algebras, we refer to [3], [6], [22].

These results motivated further research on reflexivity in the setting of group algebras and function algebras. Along this line, Molnár and Zalar, [24], studied the algebraic reflexivity of the isometric automorphism group of the convolution algebra $L_p(G)$ of a compact metric group G. Concerning function algebras, Cabello Sánchez and Molnár investigated in [10] the reflexivity of the automorphism group of Banach algebras of holomorphic functions, Fréchet algebras of holomorphic functions, and algebras of continuous functions (see also [9]). In [11], Cabello Sánchez proved that the automorphism group of L_∞ is algebraically reflexive. Recently, Botelho and Jamison [4] have studied the algebraic and topological reflexivity properties of $\ell_p(X)$ spaces.

In this manuscript, we deal with the reflexivity of *-automorphisms on *-algebras of big and little Lipschitz maps taking values in $\mathcal{B}(\mathcal{H})$, the $\mathcal{C}^*$-algebra of all bounded linear operators on a separable complex infinite-dimensional Hilbert space $\mathcal{H}$. Recently, Botelho and Jamison have established in [5], under a different approach, the algebraic reflexivity of the class of *-automorphisms preserving the constant functions on algebras of $\mathcal{B}(\mathcal{H})$-valued big Lipschitz maps.

The study of Banach algebras of complex-valued Lipschitz functions begins with the works by Sherbert [26], [27]. We refer to Weaver's book [28], mainly Chapter 4, for a very comprehensive description of these algebras. The research into spaces of vector-valued Lipschitz functions was initiated by Johnson in [16]. He examined the Banach space properties of scalar-valued and Banach-valued Lipschitz functions. Cao, Zhang and Xu [12] characterized Banach-valued Lipschitz functions (known as *Lipschitz α-operators* in [12]), and studied the Lipschitz extension of such functions.

This paper is organized as follows. In Section 2 we introduce the big and little Banach algebras (*-algebras), $\mathrm{Lip}_\alpha(X,\mathcal{A})$ and $\mathrm{lip}_\alpha(X,\mathcal{A})$, where $0<\alpha\le 1$, of Lipschitz functions on a compact metric space X with values in a Banach algebra (respectively, *-algebra) $\mathcal{A}$. For $\alpha=1$, we write $\mathrm{Lip}(X,\mathcal{A})$ and $\mathrm{lip}(X,\mathcal{A})$.

Assuming that $\mathcal{A}$ is a prime unital Banach algebra, we prove in Section 3 that every linear bijective map that preserves zero products in both directions from $\mathrm{Lip}(X,\mathcal{A})$ or $\mathrm{lip}_\alpha(X,\mathcal{A})$ onto itself is biseparating. The proof uses a technique introduced by Araujo and Jarosz in [2] to state an analogous result in the setting of spaces of operator-valued continuous functions.

Section 4 deals with a Banach–Stone type description of the automorphisms and *-automorphisms on $\mathrm{Lip}(X,\mathcal{B}(\mathcal{H}))$ and $\mathrm{lip}_\alpha(X,\mathcal{B}(\mathcal{H}))$ with $\alpha\in(0,1)$ (no separability is assumed now). Here we apply the results obtained in the preceding section, and some results on biseparating linear maps between spaces of Banach-valued Lipschitz functions established by Araujo and Dubarbie in [1], and by the second and third author of the present manuscript in a joint work with Wang, [15].

In Section 5, taking into account the characterization of the automorphisms of $\mathrm{Lip}(X,\mathbb{C})$ and $\mathrm{lip}_\alpha(X,\mathbb{C})$ with $\alpha\in(0,1)$ by Sherbert [26], we prove that they are algebraically reflexive.

In the last section, we state the algebraic reflexivity of the *-automorphism groups of the Banach *-algebras $\mathrm{Lip}(X,\mathcal{B}(\mathcal{H}))$ and $\mathrm{lip}_\alpha(X,\mathcal{B}(\mathcal{H}))$, with $\alpha\in(0,1)$ and $\mathcal{H}$ being now separable.

We must point out that our study is motivated by a very nice work by Molnár and Győry, [23], concerning the algebraic reflexivity of the automorphism group of the $\mathcal{C}^*$-algebra $\mathcal{C}_0(X,\mathcal{B}(\mathcal{H}))$ of all continuous functions from X to $\mathcal{B}(\mathcal{H})$ that vanish at infinity, where X is a locally compact Hausdorff space and $\mathcal{H}$ is a separable complex infinite-dimensional Hilbert space.

2. Preliminaries and notation

Let X and Y be metric spaces. We will use the letter d to denote the distance in any metric space. A map $f:X\to Y$ is *Lipschitz* if there exists a constant $k\ge 0$ such that

$$d(f(x),f(y))\le k\ d(x,y),\qquad \forall x,y\in X.$$

The smallest k fulfilling this condition is the *Lipschitz constant* for f, and we denote it by $L(f)$. A map $f:X\to Y$ is a *Lipschitz homeomorphism* if f is bijective, and both f and f^{-1} are Lipschitz.

Let (X, d) be a compact metric space, α a real parameter in $(0, 1]$, and E a complex Banach space. Clearly, the set X with the distance d^α, defined by $d^\alpha(x, y) = d(x, y)^\alpha$, for all $x, y \in X$, is also a compact metric space.

The *big Lipschitz space* $\mathrm{Lip}_\alpha(X, E)$ is the Banach space of all functions $f : X \to E$ such that

$$L_\alpha(f) := \sup\left\{\frac{\|f(x) - f(y)\|}{d(x, y)^\alpha} : x, y \in X,\ x \neq y\right\}$$

is finite, endowed with the norm

$$\|f\|_\alpha := L_\alpha(f) + \|f\|_\infty\,,$$

where

$$\|f\|_\infty := \sup\left\{\|f(x)\| : x \in X\right\}.$$

The *little Lipschitz space* $\mathrm{lip}_\alpha(X, E)$ is the norm-closed linear subspace of $\mathrm{Lip}_\alpha(X, E)$ formed by all functions f satisfying the condition

$$\forall \varepsilon > 0\ \ \exists \delta > 0 : x, y \in X,\ 0 < d(x, y) < \delta \ \Rightarrow\ \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha} < \varepsilon$$

(see [12, Theorem 2.1]). When $\alpha = 1$, we will drop the subscript and write simply $\mathrm{Lip}(X, E)$ and $\mathrm{lip}(X, E)$. For $E = \mathbb{C}$, it is usual to denote $\mathrm{Lip}_\alpha(X)$ and $\mathrm{lip}_\alpha(X)$.

The space $\mathrm{Lip}(X)$ is contained in $\mathrm{lip}_\alpha(X)$ for any $\alpha \in (0, 1)$, contains the constant functions, and separates the points of X [27, Proposition 1.6]. However, there are spaces $\mathrm{lip}(X)$ whose elements are all constant functions (for instance, $\mathrm{lip}[0, 1]$ with the usual metric). Consequently, the spaces $\mathrm{Lip}(X, E)$ and $\mathrm{lip}_\alpha(X, E)$ with $\alpha \in (0, 1)$ contain the constant functions and separate points. Notice that if $g \in \mathrm{Lip}(X)$ ($g \in \mathrm{lip}_\alpha(X)$) and $e \in E$, the map $g \cdot e$ defined by setting $g \cdot e(x) = g(x)e$ for all $x \in X$, belongs to $\mathrm{Lip}(X, E)$ (respectively, $\mathrm{lip}_\alpha(X, E)$) and $\|g \cdot e\|_\alpha = \|g\|_\alpha \|e\|$.

Given a *-algebra $\mathcal{A}$, it is straightforward to verify that $\mathrm{Lip}_\alpha(X, \mathcal{A})$ is a Banach *-algebra with the multiplication and involution defined pointwise, and $\mathrm{lip}_\alpha(X, \mathcal{A})$ is a norm-closed *-subalgebra of $\mathrm{Lip}_\alpha(X, \mathcal{A})$. We denote by $\mathrm{Aut}(\mathcal{A})$ and $\mathrm{Aut}^*(\mathcal{A})$ the group of all automorphisms and the group of all *-automorphisms of $\mathcal{A}$, respectively.

For a metric space X and a Hilbert space $\mathcal{H}$, the unity of $\mathrm{Lip}(X)$ and $\mathrm{lip}(X)$, that is, the function constantly equal to 1 on X, is denoted by 1_X, and the unity of $\mathcal{B}(\mathcal{H})$, that is, the identity operator on $\mathcal{H}$, by $I_{\mathcal{H}}$.

If E is a Banach space, $\mathcal{L}(E)^{-1}$ stands for the set of all linear bijections of E, $\mathcal{B}(E)$ for the space of all bounded linear operators of E equipped with the operator canonical norm, and $\mathrm{Iso}(E)$ for the group of all surjective linear isometries of E.

For any $f : X \to E$, let $c(f) = \{x \in X : f(x) \neq 0\}$ denote its cozero set.

Throughout this paper, we will use the following family of Lipschitz functions on a compact metric space X. For any $x \in X$ and $\delta > 0$, let $h_{x,\delta} : X \to [0,1]$ be defined by

$$h_{x,\delta}(z) = \max\left\{0, 1 - \frac{d(z,x)}{\delta}\right\} \quad (z \in X).$$

Clearly, $h_{x,\delta} \in \mathrm{Lip}(X)$, $h_{x,\delta}(x) = 1$, and $h_{x,\delta}(z) = 0$ if and only if $d(z,x) \geq \delta$.

In order to simplify the notation, from now on we denote by $\mathrm{F}_\alpha(X,E)$ either $\mathrm{Lip}(X,E)$ if $\alpha = 1$ or $\mathrm{lip}_\alpha(X,E)$ if $\alpha \in (0,1)$. Similarly, $\mathrm{F}_\alpha(X)$ stands for $\mathrm{Lip}(X)$ if $\alpha = 1$ or $\mathrm{lip}_\alpha(X)$ if $\alpha \in (0,1)$, and $\mathrm{F}_\alpha(X)^{-1}$ denotes the set of all nowhere vanishing functions in $\mathrm{F}_\alpha(X)$.

3. Zero product preserving maps and separating maps between Lipschitz algebras

Let X be a compact metric space. Given a Banach space E, a linear map $\Phi : \mathrm{F}_\alpha(X,E) \to \mathrm{F}_\alpha(X,E)$ is said to be separating if $c(\Phi(f)) \cap c(\Phi(g)) = \emptyset$ whenever $f, g \in \mathrm{F}_\alpha(X,E)$ satisfy $c(f) \cap c(g) = \emptyset$. Moreover, Φ is called biseparating if it is bijective and both Φ and Φ^{-1} are separating maps.

Given a Banach algebra $\mathcal{A}$, a linear map $\Phi : \mathrm{F}_\alpha(X,\mathcal{A}) \to \mathrm{F}_\alpha(X,\mathcal{A})$ preserves zero products if $fg = 0$ implies $\Phi(f)\Phi(g) = 0$ for all $f, g \in \mathrm{F}_\alpha(X,\mathcal{A})$. It is said that Φ preserves zero products in both directions if it is bijective and both Φ and Φ^{-1} preserve zero products.

Our main goal in this section is to show that every linear bijective map that preserves zero products in both directions from $\mathrm{F}_\alpha(X,\mathcal{B}(\mathcal{H}))$ onto itself is biseparating. This fact will be a key tool to get a Banach–Stone type representation for automorphisms of $\mathrm{F}_\alpha(X,\mathcal{B}(\mathcal{H}))$ in the next section.

Let X be a compact metric space and $\mathcal{A}$ be a unital Banach algebra with unity $\mathbb{1}_{\mathcal{A}}$, and $f \in \mathrm{F}_\alpha(X,\mathcal{A})$. We fix some additional notation according to [2]:

$$\begin{aligned}
L(f) &= \{g \in \mathrm{F}_\alpha(X,\mathcal{A}) : gf = 0\}, \\
R(f) &= \{g \in \mathrm{F}_\alpha(X,\mathcal{A}) : fg = 0\}, \\
\mathcal{AI} &= \{g \in \mathrm{F}_\alpha(X,\mathcal{A}) : L(g) \subset R(g)\}, \\
C(f) &= \{g \in \mathrm{F}_\alpha(X,\mathcal{A}) : R(f) \cap \mathcal{AI} \subset R(g)\}.
\end{aligned}$$

For every $f, g \in \mathrm{F}_\alpha(X,\mathcal{A})$, it is clear that $fg = 0$ whenever $c(f) \cap c(g) = \emptyset$. If $\mathcal{A}$ is prime (that is, $a\mathcal{A}b = \{0\}$ implies $a = 0$ or $b = 0$) and g lies in $\mathcal{AI}$, the converse also holds. Indeed, assume $fg = 0$. If $x \in c(f) \cap c(g)$, then $f(x)$ and $g(x)$ are nonzero elements in $\mathcal{A}$. Since $\mathcal{A}$ is prime, there is $a \in \mathcal{A} \setminus \{0\}$ such that $g(x)af(x) \neq 0$. Let $h \in \mathrm{F}_\alpha(X, \mathcal{A})$ be given by $h = (1_X \cdot a)f$. It is clear that $hg = 0$ and $gh(x) = g(x)af(x) \neq 0$. Hence $g \notin \mathcal{AI}$.

Lemma 3.1. *Let X be a compact metric space and $\mathcal{A}$ be a unital Banach algebra. For every $f \in \mathrm{F}_\alpha(X, \mathcal{A})$, we have*

$$C(f) \subset \left\{ g \in \mathrm{F}_\alpha(X, \mathcal{A}) : c(g) \subset \mathrm{int}(\overline{c(f)}) \right\},$$

and the equality holds whenever A is prime.

Proof. Let $f, g \in \mathrm{F}_\alpha(X, \mathcal{A})$ be such that $c(g)$ is not included in $\mathrm{int}(\overline{c(f)})$. Let us show that $g \notin C(f)$. We can choose $x \in c(g)$, $\varepsilon > 0$ with $B(x, \varepsilon) \subset c(g)$, $y \in B(x, \varepsilon)$ and $\delta > 0$ such that $B(y, \delta) \cap c(f) = \emptyset$. As usual, $B(x, \varepsilon) = \{z \in X : d(z, x) < \varepsilon\}$. Let h be the map $h_{y,\delta} \cdot \mathbb{1}_{\mathcal{A}}$ defined on X. It is clear that $h \in \mathcal{AI}$ and $c(h) = B(y, \delta)$. Hence $c(h) \cap c(f) = \emptyset$ and, in particular, $fh = 0$. Nevertheless as $gh(y) = g(y) \neq 0$, $gh \neq 0$, which proves that $g \notin C(f)$.

Notice that by the comments above, if A is prime, we have

$$C(f) = \{g \in \mathrm{F}_\alpha(X, \mathcal{A}) : \text{ for all } h \in \mathcal{AI}[c(f) \cap c(h) = \emptyset \Rightarrow c(g) \cap c(h) = \emptyset]\}.$$

Let $g \in \mathrm{F}_\alpha(X, \mathcal{A})$ be for which $c(g) \subset \mathrm{int}(\overline{c(f)})$. Take $h \in \mathcal{AI}$ with $c(f) \cap c(h) = \emptyset$. Then $\overline{c(f)} \cap c(h) = \emptyset$ and thus $\mathrm{int}(\overline{c(f)}) \cap c(h) = \emptyset$. It follows that $c(g) \cap c(h) = \emptyset$, that is, $g \in C(f)$. □

Lemma 3.2. *Let X be a compact metric space and $\mathcal{A}$ be a unital Banach algebra. If $c(f_1) \cap c(f_2) = \emptyset$, then $C(f_1) \cap C(f_2) = \{0\}$ for every $f_1, f_2 \in \mathrm{F}_\alpha(X, \mathcal{A})$. The converse holds if $\mathcal{A}$ is prime.*

Proof. Let $f_1, f_2 \in \mathrm{F}_\alpha(X, \mathcal{A})$. By Lemma 3.1, if $g \in C(f_1) \cap C(f_2)$ and $g \neq 0$, then $\emptyset \neq c(g) \subset \mathrm{int}(\overline{c(f_1)}) \cap \mathrm{int}(\overline{c(f_2)})$. It follows easily that $c(f_1) \cap c(f_2) \neq \emptyset$. Conversely, if $\mathcal{A}$ is prime and $c(f_1) \cap c(f_2) \neq \emptyset$, let $x \in X$ and $\delta > 0$ be so that $B(x, \delta) \subset c(f_1) \cap c(f_2)$. For $g = h_{x,\delta} \cdot \mathbb{1}_{\mathcal{A}}$, it is clear that $c(g) = B(x, \delta) \subset \mathrm{int}(\overline{c(f_1)}) \cap \mathrm{int}(\overline{c(f_2)})$ and hence, by Lemma 3.1, $g \in C(f_1) \cap C(f_2)$. □

The following result is inspired in [2, Theorem 2].

Theorem 3.3. *Let X be a compact metric space and let $\mathcal{A}$ be a prime unital Banach algebra. Let $\Phi : \mathrm{F}_\alpha(X, \mathcal{A}) \to \mathrm{F}_\alpha(X, \mathcal{A})$ be a bijective linear map preserving zero products in both directions. Then Φ is biseparating.*

Proof. As Φ is bijective and preserves zero products in both directions, an easy verification shows that $\mathcal{A}\mathcal{I} = \Phi(\mathcal{A}\mathcal{I})$ and $C(\Phi(f)) = \Phi(C(f))$ for every $f \in \mathrm{F}_\alpha(X, \mathcal{A})$.

Let $f_1, f_2 \in \mathrm{F}_\alpha(X, \mathcal{A})$ be such that $c(f_1) \cap c(f_2) = \emptyset$. By Lemma 3.2, $C(f_1) \cap C(f_2) = \{0\}$, that is, $C(\Phi(f_1)) \cap C(\Phi(f_2)) = \{0\}$. As A is prime, we have also $c(\Phi(f_1)) \cap c(\Phi(f_2)) = \emptyset$. Hence Φ is separating. The same argument applied to Φ^{-1} shows that Φ is biseparating. $\square$

As a direct consequence we obtain the above announced result.

Corollary 3.4. *Let X be a compact metric space and let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space. Then every bijective linear map from $\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$ onto itself that preserves zero products in both directions is biseparating.*

4. Banach–Stone type representation of automorphisms of $\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$

We describe the general form of the automorphisms and *-automorphisms on Lipschitz *-algebras $\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$. For the little Lipschitz spaces we require the next result on Lipschitz functional calculus.

Lemma 4.1. *Let X be a compact metric space, E be a Banach space, and $\alpha \in (0,1)$. If $h \in \mathrm{lip}_\alpha(X, E)$ and $\varphi : X \to X$ is Lipschitz, then $h \circ \varphi \in \mathrm{lip}_\alpha(X, E)$.*

Proof. First observe that $h \circ \varphi \in \mathrm{Lip}_\alpha(X, E)$ since

$$\|h(\varphi(x)) - h(\varphi(y))\| \leq L_\alpha(h) d(\varphi(x), \varphi(y))^\alpha \leq L_\alpha(h) L(\varphi)^\alpha d(x,y)^\alpha$$

for every $x, y \in X$. Now, let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$x, y \in X,\ 0 < d(x,y) < \delta \ \Rightarrow\ \frac{\|h(x) - h(y)\|}{d(x,y)^\alpha} < \frac{\varepsilon}{1 + L(\varphi)^\alpha}.$$

Let $x, y \in X$ with $0 < d(x,y) < \delta/(1 + L(\varphi))$. If $\varphi(x) \neq \varphi(y)$ (otherwise, the result is trivial) then $0 < d(\varphi(x), \varphi(y)) < \delta$ and

$$\begin{aligned} \frac{\|h(\varphi(x)) - h(\varphi(y))\|}{d(x,y)^\alpha} &= \frac{\|h(\varphi(x)) - h(\varphi(y))\|}{d(\varphi(x), \varphi(y))^\alpha} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} \\ &< \frac{\varepsilon}{1 + L(\varphi)^\alpha} L(\varphi)^\alpha < \varepsilon. \end{aligned}$$

This shows that $h \circ \varphi \in \mathrm{lip}_\alpha(X, E)$. $\square$

Let us recall that every *-automorphism of a $\mathcal{C}^*$-algebra $\mathcal{A}$ is an isometry, and every automorphism of $\mathcal{A}$ is continuous in the norm topology in $\mathcal{A}$ and its norm equals the norm of its inverse (see, for example, [25, Corollary 1.2.6, Lemma 4.1.12 and Proposition 4.1.13]). Therefore $\mathrm{Aut}^*(\mathcal{A}) \subset \mathrm{Aut}(\mathcal{A}) \subset \mathcal{B}(\mathcal{A})$. We next consider these sets as metric spaces with the metric induced by the operator canonical norm.

Theorem 4.2. *Let X be a compact metric space, $\mathcal{H}$ be a complex infinite-dimensional Hilbert space, and $\alpha \in (0,1)$. A map Φ of $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ into $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ is an automorphism if and only if there exist a unique Lipschitz map τ from (X, d^α) into $\mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ and a unique Lipschitz homeomorphism $\varphi : X \to X$ such that*

$$\Phi(f)(x) = \tau(x)(f(\varphi(x))) \quad (f \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H})),\ x \in X). \tag{1}$$

Moreover, if $\Phi : \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \to \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ is an automorphism, and $\tau : X \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ is the map given above, then Φ is a *-automorphism if and only if $\tau(x)$ is a *-automorphism for every $x \in X$.

Proof. Let $\Phi : \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \to \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ be a map of the form (1) with τ, φ being as in the statement above. It is straightforward to check that Φ is linear, injective and multiplicative. Observe that $\tau \in \mathrm{Lip}_\alpha(X, \mathcal{B}(\mathcal{B}(\mathcal{H})))$ by hypothesis. We prove that $\tau \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{B}(\mathcal{H})))$. Indeed, by (1)

$$\tau(x)(a) = \Phi(1_X \cdot a)(x) \quad (x \in X,\ a \in \mathcal{B}(\mathcal{H})),$$

and since $\Phi(1_X \cdot a) \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$, for every $a \in \mathcal{B}(\mathcal{H})$, the map $\tau(\cdot)(a)$ belongs to $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$. From this we show that $\tau \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{B}(\mathcal{H})))$. Suppose to the contrary that there exist $\varepsilon > 0$ and, for each $n \in \mathbb{N}$, $x_n, y_n \in X$ with $x_n \neq y_n$ such that $d(x_n, y_n) \to 0$ as $n \to \infty$, but

$$\frac{\|\tau(x_n) - \tau(y_n)\|}{d(x_n, y_n)^\alpha} \geq \varepsilon$$

for all n. Then we can find some $a \in \mathcal{B}(\mathcal{H})$ with $\|a\| = 1$ such that

$$\frac{\|(\tau(x_n) - \tau(y_n))\,(a)\|}{d(x_n, y_n)^\alpha} \geq \frac{\varepsilon}{2}$$

for all n, and this says us that $\tau(\cdot)(a)$ is not in $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$, which is impossible.

It remains to show that Φ is surjective. To this end, pick $h \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ and let $f : X \to \mathcal{B}(\mathcal{H})$ be defined by

$$f(x) = \tau(\varphi^{-1}(x))^{-1}(h(\varphi^{-1}(x))) \quad (x \in X).$$

We only need to prove that $f \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$, since $\Phi(f) = h$. Notice that for any $x, y \in X$ and $a \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned}\left\|\left(\tau(x)^{-1} - \tau(y)^{-1}\right)(a)\right\| &= \left\|\left(\tau(x)^{-1}\tau(y)\tau(y)^{-1} - \tau(x)^{-1}\tau(x)\tau(y)^{-1}\right)(a)\right\| \\ &\le \|\tau(x)\| \left\|\left(\tau(x) - \tau(y)\right)\left(\tau(y)^{-1}(a)\right)\right\|.\end{aligned}$$

From this inequality, it follows that for every $x, y \in X$

$$\begin{aligned}\|f(x) - f(y)\| &= \|\tau(\varphi^{-1}(x))^{-1}(h(\varphi^{-1}(x)) - h(\varphi^{-1}(y))) \\ &\quad + (\tau(\varphi^{-1}(x))^{-1} - \tau(\varphi^{-1}(y))^{-1})(h(\varphi^{-1}(y)))\| \\ &\le \|\tau(\varphi^{-1}(x))\| \|h(\varphi^{-1}(x)) - h(\varphi^{-1}(y))\| \\ &\quad + \|\tau(\varphi^{-1}(x))\| \|(\tau(\varphi^{-1}(x)) - \tau(\varphi^{-1}(y)))(\tau(\varphi^{-1}(y))^{-1})(h(\varphi^{-1}(y)))\| \\ &\le \|\tau\|_\infty \|h(\varphi^{-1}(x)) - h(\varphi^{-1}(y))\| \\ &\quad + \|\tau\|_\infty^2 \|\tau(\varphi^{-1}(x)) - \tau(\varphi^{-1}(y))\| \|h \circ \varphi^{-1}\|_\infty \\ &\le \|\tau\|_\infty L_\alpha(h \circ \varphi^{-1}) d(x, y)^\alpha + \|\tau\|_\infty^2 \|h \circ \varphi^{-1}\|_\infty L_\alpha(\tau) d(\varphi^{-1}(x), \varphi^{-1}(y))^\alpha \\ &\le \|\tau\|_\infty L_\alpha(h \circ \varphi^{-1}) d(x, y)^\alpha + \|\tau\|_\infty^2 \|h \circ \varphi^{-1}\|_\infty L_\alpha(\tau) L(\varphi^{-1})^\alpha d(x, y)^\alpha.\end{aligned}$$

Hence $f \in \mathrm{Lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$. Moreover, $h \circ \varphi^{-1}$ belongs to $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ and $\tau \circ \varphi^{-1}$ lies in $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{B}(\mathcal{H}))$ by Lemma 4.1, so the second inequality yields $f \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$, as desired.

To prove the converse implication, we need some results from [15] on biseparating linear maps between spaces $\mathrm{lip}_\alpha(X, E)$, with $0 < \alpha < 1$. Let Φ be an automorphism of $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$. It is clear that Φ preserves zero products in both directions, and according to Corollary 3.4, Φ is a biseparating linear map. Then, by [15, Theorem 4.1], there exist a map $\tau : X \to \mathcal{L}(\mathcal{B}(\mathcal{H}))^{-1}$ and a homeomorphism $\varphi : X \to X$ such that

$$\Phi(f)(x) = \tau(x)(f(\varphi(x))), \quad \forall f \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H})),\ \forall x \in X.$$

Since Φ is a homomorphism, it follows easily that $\tau(x)$ is multiplicative for every $x \in X$. Hence $\tau(x) \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ for all $x \in X$, and Φ is continuous by [15, Theorem 4.2]. Now, according to [15, Theorem 4.3], φ is a Lipschitz homeomorphism. Moreover, as

$$\tau(x)(a) = \Phi(1_X \cdot a)(x) \quad (x \in X,\ a \in \mathcal{B}(\mathcal{H})),$$

for every $x, y \in X$ and $a \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{gathered}\|(\tau(x) - \tau(y))(a)\| = \|\Phi(1_X \cdot a)(x) - \Phi(1_X \cdot a)(y)\| \\ \le L_\alpha(\Phi(1_X \cdot a)) d(x, y)^\alpha \le \|\Phi(1_X \cdot a)\|_\alpha d(x, y)^\alpha \le \|\Phi\| \|a\| d(x, y)^\alpha.\end{gathered}$$

Hencefore $\|\tau(x)-\tau(y)\| \leq \|\Phi\| d(x,y)^\alpha$ for all $x, y \in X$, and thus τ is a Lipschitz map from (X, d^α) into $\mathrm{Aut}(\mathcal{B}(\mathcal{H}))$.

To prove the uniqueness, assume that there are a Lipschitz map τ' from (X, d^α) into $\mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ and a Lipschitz homeomorphism $\varphi' : X \to X$ such that $\Phi(f)(x) = \tau'(x)(f(\varphi'(x)))$ for all $x \in X$ and $f \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$. For any $x \in X$ and $a \in \mathcal{B}(\mathcal{H})$, it is clear that $\tau'(x)(a) = \Phi(1_X \cdot a)(x) = \tau(x)(a)$ and thus $\tau' = \tau$. Therefore, given any $x \in X$, we have $\tau(x)(f(\varphi'(x))) = \tau(x)(f(\varphi(x)))$ for all $f \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$, which yields $f(\varphi'(x)) = f(\varphi(x))$ for all $f \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$. Since $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ separates the points of X, we infer that $\varphi'(x) = \varphi(x)$. This holds for every $x \in X$, and so we conclude that $\varphi' = \varphi$.

We finish the proof by characterizing the *-automorphisms of $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$. Let $\Phi : \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \to \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ be an automorphism and let τ, φ be the maps that permit us to express Φ in the form (1). Suppose first that Φ preserves the involution in $\mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$. Then, given $x \in X$, we have

$$\begin{aligned}\tau(x)(a^*) &= \Phi(1_X \cdot a^*)(x) = \Phi((1_X \cdot a)^*)(x) \\ &= (\Phi(1_X \cdot a))^*(x) = (\Phi(1_X \cdot a)(x))^* = (\tau(x)(a))^*\end{aligned}$$

for all $a \in \mathcal{B}(\mathcal{H})$, and therefore $\tau(x)$ is a *-automorphism. Conversely, assume that $\tau(x)$ is a *-automorphism for every $x \in X$. Given $f \in \mathrm{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$, we have

$$\begin{aligned}\Phi(f^*)(x) &= \tau(x)(f^*(\varphi(x))) = \tau(x)(((f(\varphi(x))^*) \\ &= (\tau(x)(f(\varphi(x))))^* = (\Phi(f)(x))^* = (\Phi(f))^*(x)\end{aligned}$$

for every $x \in X$, and so Φ preserves the involution. □

The following result may be proved in the same way as Theorem 4.2. We only need some facts from [1] on biseparating linear maps between spaces $\mathrm{Lip}(X, E)$.

Theorem 4.3. *Let X be a compact metric space, and let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space. A map $\Phi : \mathrm{Lip}(X, \mathcal{B}(\mathcal{H})) \to \mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$ is an automorphism if and only if there exist a unique Lipschitz map $\tau : X \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ and a unique Lipschitz homeomorphism $\varphi : X \to X$ such that Φ is of the form*

$$\Phi(f)(x) = \tau(x)(f(\varphi(x))) \quad (f \in \mathrm{Lip}(X, \mathcal{B}(\mathcal{H})),\ x \in X).$$

Moreover, if $\Phi : \mathrm{Lip}(X, \mathcal{B}(\mathcal{H})) \to \mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$ is an automorphism and $\tau : X \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ is the map given above, then Φ is *-preserving if and only if $\tau(x)$ is *-preserving for every $x \in X$.

PROOF. Just the "only if" part deserves some comment. Let Φ be an automorphism of $\mathrm{Lip}(X, \mathcal{B}(\mathcal{H}))$. By Corollary 3.4, Φ is biseparating. From [1, Theorem 3.1], there are a map $\tau : X \to \mathcal{L}(\mathcal{B}(\mathcal{H}))^{-1}$ and a Lipschitz homeomorphism $\varphi : X \to X$ so that

$$\Phi(f)(x) = \tau(x)(f(\varphi(x))), \quad \forall f \in \mathrm{Lip}(X, \mathcal{B}(\mathcal{H})),\ \forall x \in X.$$

Since Φ is a homomorphism, $\tau(x)$ is multiplicative and thus continuous, for all $x \in X$. Equivalently, the set $Y_d := \{x \in X : \tau(x) \text{ is discontinuous}\}$ is empty and therefore Φ is continuous by [1, Theorem 3.4]. A glance at the comments preceding [1, Proposition 3.2] reveals that

$$\|\tau(x) - \tau(y)\| \leq \|\Phi\|\, d(x, y), \quad \forall x, y \in X,$$

and thus $\tau : X \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ is Lipschitz. The uniqueness of τ and φ is proved similarly as in Theorem 4.2. $\square$

For the proof of our results we also need the following well-known facts on the general form of the automorphisms of $\mathrm{Lip}(X)$ and $\mathrm{lip}_\alpha(X)$ with $0 < \alpha < 1$.

Theorem 4.4. *Let X be a compact metric space.*

(1) [26, Corollary 5.2] *A map $\Phi : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ is an automorphism if and only if there exists a Lipschitz homeomorphism $\varphi : X \to X$ such that $\Phi(f) = f \circ \varphi$ for every $f \in \mathrm{Lip}(X)$.*

(2) [15, Corollary 5.3] *Given $\alpha \in (0, 1)$, a map $\Phi : \mathrm{lip}_\alpha(X) \to \mathrm{lip}_\alpha(X)$ is an automorphism if and only if Φ is of the form $\Phi(f) = f \circ \varphi$ for all $f \in \mathrm{lip}_\alpha(X)$, where φ is a Lipschitz homeomorphism of X.*

5. Algebraic reflexivity of the automorphism group of $\mathrm{F}_\alpha(X)$

In this section we prove that $\mathrm{Aut}(\mathrm{F}_\alpha(X))$ is algebraically reflexive. In view of Theorem 4.4, note that $\mathrm{Aut}(\mathrm{F}_\alpha(X)) = \mathrm{Aut}^*(\mathrm{F}_\alpha(X))$.

Let us recall that for a compact metric space X, Sherbert proved in [26, Theorem 5.1] that a map $\Phi : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ is a unital homomorphism if and only if there exists a Lipschitz map $\varphi : X \to X$ such that $\Phi(f) = f \circ \varphi$ for every $f \in \mathrm{Lip}(X)$. By using the same idea of the proof of this statement, we can see that an analogous result holds for unital endomorphisms of $\mathrm{lip}_\alpha(X)$, with $0 < \alpha < 1$.

Theorem 5.1. *Let X be a compact metric space. Then the automorphism group of $\mathrm{F}_\alpha(X)$ is algebraically reflexive.*

Proof. Let Φ be a local automorphism of $\mathrm{F}_\alpha(X)$. Then, for each $f \in \mathrm{F}_\alpha(X)$, there exists an automorphism Φ_f of $\mathrm{F}_\alpha(X)$ so that $\Phi(f) = \Phi_f(f)$. This implies that Φ is injective. As $\Phi_f(1_X) = 1_X$, for every $f \in \mathrm{F}_\alpha(X)$, it follows that $\Phi(1_X) = \Phi_{1_X}(1_X) = 1_X$. By Theorem 4.4, there exists a Lipschitz homeomorphism $\varphi_f : X \to X$ such that

$$\Phi_f(g)(z) = g(\varphi_f(z)) \quad (g \in \mathrm{F}_\alpha(X),\ z \in X).$$

In particular,

$$\Phi(f)(z) = \Phi_f(f)(z) = f(\varphi_f(z)) \quad (z \in X).$$

Fix $x \in X$, and define the unital linear functional $\Phi_x : \mathrm{F}_\alpha(X) \to \mathbb{C}$ by

$$\Phi_x(f) = \Phi(f)(x), \quad \forall f \in \mathrm{F}_\alpha(X).$$

Let $f \in \mathrm{F}_\alpha(X)^{-1}$. Since $\Phi_x(f) = \Phi(f)(x) = f(\varphi_f(x))$, we have $\Phi_x(f) \neq 0$. By the Gleason–Kahane–Żelazko theorem [13], [17], we infer that Φ_x is multiplicative. Hence Φ is a homomorphism, that is, there exists a Lipschitz map $\varphi : X \to X$ such that

$$\Phi(f)(z) = f(\varphi(z)) \quad (f \in \mathrm{F}_\alpha(X),\ z \in X). \tag{2}$$

We claim that φ is onto. Suppose, to the contrary, that there exists $x \in X \backslash \varphi(X)$. Then $d(x, \varphi(X)) > 0$ since $\varphi(X)$ is closed. For $\delta = d(x, \varphi(X))$, the Lipschitz map $h_{x,\delta} \in \mathrm{Lip}(X) \subset \mathrm{F}_\alpha(X)$ satisfies $h_{x,\delta}(\varphi(X)) = \{0\}$. By (2), $\Phi(h_{x,\delta}) = 0$, but $h_{x,\delta}(x) = 1$, which contradicts the fact that Φ is linear and injective.

To show that φ is injective, let $x, y \in X$ be such that $\varphi(x) = \varphi(y)$. Define $h : X \to \mathbb{R}$ by

$$h(z) = d(z, \varphi(x)), \quad \forall z \in X.$$

Clearly, h belongs to $\mathrm{Lip}(X)$, and $h(z) = 0$ if and only if $z = \varphi(x)$. Since Φ is a local automorphism of $\mathrm{F}_\alpha(X)$,

$$\Phi(h)(z) = h(\varphi_h(z)) \quad (z \in X), \tag{3}$$

where φ_h is a Lipschitz homeomorphism of X. From (2) and (3), it follows

$$h(\varphi_h(x)) = \Phi(h)(x) = h(\varphi(x)) = 0, \qquad h(\varphi_h(y)) = \Phi(h)(y) = h(\varphi(y)) = 0.$$

This implies that $\varphi_h(x) = \varphi_h(y) = \varphi(x)$, and as φ_h is injective, we get $x = y$.

By taking into account Theorem 4.4, it remains to show that $\varphi^{-1} : X \to X$ is Lipschitz to ensure that Φ is an automorphism of $\mathrm{F}_\alpha(X)$. In order to prove this, we follow the argument used by Botelho and Jamison in [4, Theorem 2.1]. Assume that φ^{-1} is not Lipschitz. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, with $x_n \neq y_n$ for all n, such that

$$\lim_{n\to\infty} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} = 0.$$

Let $\widetilde{X} = \{(x, y) \in X^2 : x \neq y\}$, and let $F : \widetilde{X} \to \mathbb{R}$ be defined by

$$F(x, y) = \frac{d(\varphi(x), \varphi(y))}{d(x, y)}.$$

Denote by $\beta\widetilde{X}$ the Stone-Čech compactification of $\widetilde{X}$, and by βF the unique continuous extension of F to $\beta\widetilde{X}$. By the compactness of $\beta\widetilde{X}$, there exists a subnet $\{(x_i, y_i)\}$ converging to $\xi \in \beta\widetilde{X}$. By using the continuity of βF, we have $\beta F(\xi) = 0$. Moreover, $\xi \notin \widetilde{X}$ since $F(x, y) \neq 0$ for all $(x, y) \in \widetilde{X}$. As X is compact, taking subnets if necessary, we may assume that $\{x_i\}$ converges to some $x \in X$ and $y_i \neq x$ for all i. Define $k : X \to \mathbb{R}$ by

$$k(z) = d(z, \varphi(x)), \quad \forall z \in X.$$

Since Φ is a local automorphism of $\mathrm{F}_\alpha(X)$, we get

$$\Phi(k)(z) = k(\varphi_k(z)) \quad (z \in X), \tag{4}$$

for some Lipschitz homeomorphism $\varphi_k : X \to X$. By applying (2) and (4), we have

$$d(\varphi(z), \varphi(x)) = d(\varphi_k(z), \varphi(x)) \quad (z \in X).$$

In particular, $\varphi_k(x) = \varphi(x)$ and thus

$$d(\varphi(z), \varphi(x)) = d(\varphi_k(z), \varphi_k(x)) \quad (z \in X).$$

Therefore

$$\frac{d(\varphi(y_i), \varphi(x))}{d(y_i, x)} = \frac{d(\varphi_k(y_i), \varphi_k(x))}{d(y_i, x)} \geq \frac{1}{L(\varphi_k^{-1})} > 0$$

for all i. If we use that

$$|\beta F(y_i, x) - \beta F(\xi)| \leq |\beta F(y_i, x) - \beta F(y_i, x_i)| + |\beta F(y_i, x_i) - \beta F(\xi)|$$

for all i and the uniform continuity of βF, it follows that $\{\beta F(y_i, x)\}$ converges to $\beta F(\xi)$. Hence $\beta F(\xi) \geq 1/L(\varphi_k^{-1})$, a contradiction. This proves that φ^{-1} is Lipschitz, as desired. $\square$

6. Algebraic reflexivity of the *-automorphism group of $\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$

Our aim is to prove that the *-automorphism group of $\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$ is algebraically reflexive whenever $\mathcal{H}$ is a separable complex infinite-dimensional Hilbert space. We will use the following three lemmas. The first two appear essentially in the manuscript by Győry and Molnár [23].

We begin by showing that the set of all scalar multiples of *-automorphisms on $\mathcal{B}(\mathcal{H})$ is algebraically reflexive.

Lemma 6.1. *Let $\mathcal{H}$ be a separable complex infinite-dimensional Hilbert space. Let $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a continuous linear map with the property that for each $a \in \mathcal{B}(\mathcal{H})$, there exist $\lambda_a \in \mathbb{C}$ and $\tau_a \in \mathrm{Aut}^*(\mathcal{B}(\mathcal{H}))$ such that $\Psi(a) = \lambda_a \tau_a(a)$. Then there exist $\lambda \in \mathbb{C}$ and $\tau \in \mathrm{Aut}^*(\mathcal{B}(\mathcal{H}))$ such that $\Psi(a) = \lambda\tau(a)$ for every $a \in \mathcal{B}(\mathcal{H})$.*

Proof. Since the *-automorphisms of $\mathcal{B}(\mathcal{H})$ are both automorphisms and surjective linear isometries, [23, Lemmas 2.3 and 2.4] ensure that there are $\lambda_1, \lambda_2 \in \mathbb{C}$, $\tau_1 \in \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ and $\tau_2 \in \mathrm{Iso}(\mathcal{B}(\mathcal{H}))$ such that $\Psi = \lambda_1\tau_1$ and $\Psi = \lambda_2\tau_2$. If $\lambda_1 = 0$, then $\Psi = 0 = 0I_{\mathcal{H}}$. So we can assume that $\lambda_1 \neq 0$. Therefore $\tau_1 = (\lambda_2/\lambda_1)\tau_2$. Since τ_1 is unital, it follows that $|\lambda_2/\lambda_1| = 1$, and so $\tau_1 \in \mathrm{Iso}(\mathcal{B}(\mathcal{H}))$. According to [18, Lemma 8], τ_1 is a *-automorphism of $\mathcal{B}(\mathcal{H})$, which proves the lemma. □

Lemma 6.2. [23, Lemma 2.2]. *Let $\mathcal{H}$ be a separable complex infinite-dimensional Hilbert space. Let τ, τ_1, τ_2 be in $\mathrm{Aut}^*(\mathcal{B}(\mathcal{H}))$, and let λ and $0 \neq \lambda_1, \lambda_2$ be in $\mathbb{C}$ satisfying that $\lambda\tau(a) = \lambda_1\tau_1(a) + \lambda_2\tau_2(a)$ for every $a \in \mathcal{B}(\mathcal{H})$. Then $\tau_1 = \tau_2$.*

Lemma 6.3. *Let X be a compact metric space and let E be a Banach space. Then $\mathrm{F}_\alpha(X, E)$ is the uniformly closed linear span of the set of functions $\{g \cdot e : g \in \mathrm{F}_\alpha(X),\ e \in E\}$.*

Proof. Let $f \in \mathrm{F}_\alpha(X, E)$ and $\epsilon > 0$. For every $x \in X$ the set

$$U_x = \left\{ y \in X : \|f(y) - f(x)\| < \frac{\epsilon}{2} \right\}$$

is open in X. Since $X = \bigcup_{x\in X} U_x$ and X is compact, there exist $x_1, \dots, x_n \in X$ such that $X = \bigcup_{k=1}^n U_{x_k}$. Let $\{g_1, \dots, g_n\} \subset \mathrm{F}_\alpha(X)$ be a partition of unity on X subordinate to the open covering $\{U_{x_1}, \dots, U_{x_n}\}$ (see, for example, [14, Lemma 2.2]). Thus, $g_1, \dots, g_n$ are functions in $\mathrm{F}_\alpha(X)$ from X into $[0, 1]$ such that $\sum_{k=1}^n g_k = 1_X$ and $\mathrm{supp}(g_k) \subset U_{x_k}$ for every $k = 1, \dots, n$. Here $\mathrm{supp}(g_k)$ denotes the closure of the cozero set of g_k.

Define $f_n = \sum_{k=1}^{n} g_k \cdot e_k$ on X, where $e_k = f(x_k)$ for every $k = 1, \ldots, n$. Clearly, $f_n \in \mathrm{F}_\alpha(X, E)$. Given $x \in X$ and $k \in \{1, \ldots, n\}$, we have either $\|f(x) - e_k\| < \epsilon/2$ or $g_k(x) = 0$. It follows that

$$\|f(x) - f_n(x)\| = \left\| \sum_{k=1}^{n} g_k(x)\,(f(x) - e_k) \right\| \leq \sum_{k=1}^{n} g_k(x)\,\|f(x) - e_k\| < \frac{\epsilon}{2},$$

and thus $\|f - f_n\|_\infty < \epsilon$, as desired. □

We now are ready to state the last result of this paper.

Theorem 6.4. *Let X be a compact metric space, and let $\mathcal{H}$ be a separable complex infinite-dimensional Hilbert space. Then the *-automorphism group of $\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$ is algebraically reflexive.*

Proof. Let Φ be a local *-automorphism of $\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$, that is, Φ is a continuous linear map satisfying that for every $f \in \mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$, there is $\Phi_f \in \mathrm{Aut}^*(\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H})))$ such that $\Phi(f) = \Phi_f(f)$. In light of Theorems 4.2 and 4.3, for every $f \in \mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$ there are a Lipschitz map τ_f from (X, d^α) into $\mathrm{Aut}^*(\mathcal{B}(\mathcal{H}))$ and a Lipschitz homeomorphism $\varphi_f : X \to X$ such that

$$\Phi(f)(x) = \tau_f(x)(f(\varphi_f(x))) \quad (x \in X). \tag{5}$$

Since $\tau_f(x)$ is a linear isometry for every $x \in X$, we have

$$\|\Phi(f)(x)\| = \|\tau_f(x)(f(\varphi_f(x)))\| = \|f(\varphi_f(x))\|$$

for all $x \in X$, and hence $\|\Phi(f)\|_\infty = \|f\|_\infty$. Consequently, Φ preserves the supremum norm.

Moreover, for every $g \in \mathrm{F}_\alpha(X)$ there are a unique Lipschitz map $\tau_{g \cdot I_\mathcal{H}}$ from (X, d^α) into $\mathrm{Aut}^*(\mathcal{B}(\mathcal{H}))$ and a unique Lipschitz homeomorphism $\varphi_{g \cdot I_\mathcal{H}} : X \to X$ such that

$$\begin{aligned} \Phi(g \cdot I_\mathcal{H})(x) &= \tau_{g \cdot I_\mathcal{H}}(x)(g \cdot I_\mathcal{H}(\varphi_{g \cdot I_\mathcal{H}}(x))) \\ &= g(\varphi_{g \cdot I_\mathcal{H}}(x))\tau_{g \cdot I_\mathcal{H}}(x)(I_\mathcal{H}) = g(\varphi_{g \cdot I_\mathcal{H}}(x))I_\mathcal{H} \qquad (x \in X). \end{aligned} \tag{6}$$

Let $\Psi : \mathrm{F}_\alpha(X) \to \mathrm{F}_\alpha(X)$ be the map given by $\Psi(g) = g \circ \varphi_{g \cdot I_\mathcal{H}}$ for all $g \in \mathrm{F}_\alpha(X)$. By the uniqueness of $\varphi_{g \cdot I_\mathcal{H}}$ and (6), Ψ is well-defined and, clearly, it is linear and continuous. Notice that Ψ is a local *-automorphism of $\mathrm{F}_\alpha(X)$, and since $\mathrm{Aut}^*(\mathrm{F}_\alpha(X))$ is algebraically reflexive by Theorem 5.1, we deduce that there is a Lipschitz homeomorphism $\varphi : X \to X$ such that $\Psi(g) = g \circ \varphi$ for all $g \in \mathrm{F}_\alpha(X)$. Then we can rewrite (6) as

$$\Phi(g \cdot I_{\mathcal{H}})(x) = g(\varphi(x))I_{\mathcal{H}} \quad (g \in \mathrm{F}_\alpha(X),\ x \in X). \tag{7}$$

Fix a function $g \in \mathrm{F}_\alpha(X)^{-1}$ and a point $x \in X$, and consider $\Phi_{g,x} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\Phi_{g,x}(a) = \Phi(g \cdot a)(x) \quad (a \in \mathcal{B}(\mathcal{H})). \tag{8}$$

Clearly, $\Phi_{g,x}$ is linear and continuous. Since Φ is a local *-automorphism of $\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$, from Theorems 4.2 and 4.3, for each $a \in \mathcal{B}(\mathcal{H})$ there exist a Lipschitz homeomorphism φ_a of X, a complex number $g(\varphi_a(x))$ and a *-automorphism $\tau_a(x)$ of $\mathcal{B}(\mathcal{H})$ such that

$$\Phi_{g,x}(a) = \Phi(g \cdot a)(x) = \tau_a(x)(g \cdot a(\varphi_a(x))) = g(\varphi_a(x))\tau_a(x)(a).$$

Then, by Lemma 6.1, there are $\lambda_{g,x} \in \mathbb{C}$ and $\tau_{g,x} \in \mathrm{Aut}^*(\mathcal{B}(\mathcal{H}))$ for which

$$\Phi_{g,x}(a) = \lambda_{g,x}\tau_{g,x}(a) \quad (a \in \mathcal{B}(\mathcal{H})). \tag{9}$$

By using (7) and taking $a = I_{\mathcal{H}}$ in (8) and (9), we deduce that

$$g(\varphi(x))I_{\mathcal{H}} = \Phi(g \cdot I_{\mathcal{H}})(x) = \lambda_{g,x}\tau_{g,x}(I_{\mathcal{H}}) = \lambda_{g,x}I_{\mathcal{H}}, \tag{10}$$

and thus $g(\varphi(x)) = \lambda_{g,x}$. Now from (9), we obtain

$$\Phi(g \cdot a)(x) = g(\varphi(x))\tau_{g,x}(a) \quad (a \in \mathcal{B}(\mathcal{H})).$$

Since g and x are arbitrary, we have proved that

$$\Phi(g \cdot a)(x) = g(\varphi(x))\tau_{g,x}(a) \quad (g \in \mathrm{F}_\alpha(X)^{-1},\ x \in X,\ a \in \mathcal{B}(\mathcal{H})). \tag{11}$$

Now let $x \in X$ and $g_1, g_2 \in \mathrm{F}_\alpha(X)^{-1}$. By (11) and (5), we get that

$$\begin{aligned} g_1(\varphi(x))\tau_{g_1,x}(a) + g_2(\varphi(x))\tau_{g_2,x}(a) &= \Phi(g_1 \cdot a)(x) + \Phi(g_2 \cdot a)(x) \\ &= \Phi((g_1 + g_2) \cdot a)(x) = (g_1 + g_2)(\varphi_{(g_1+g_2)\cdot a}(x))\tau_{(g_1+g_2)\cdot a}(x)(a) \end{aligned}$$

for every $a \in \mathcal{B}(\mathcal{H})$. By Lemma 6.2, it follows that $\tau_{g_1,x} = \tau_{g_2,x}$. Therefore $\tau : X \to \mathrm{Aut}^*(\mathcal{B}(\mathcal{H}))$ given by $\tau(x) = \tau_{g,x}$ for some $g \in \mathrm{F}_\alpha(X)^{-1}$ is well-defined. From (11) we infer

$$\Phi(g \cdot a)(x) = \tau(x)(g \cdot a(\varphi(x))) \quad (g \in \mathrm{F}_\alpha(X)^{-1},\ x \in X,\ a \in \mathcal{B}(\mathcal{H})). \tag{12}$$

To see that τ is a Lipschitz map from (X, d^α) into $\mathrm{Aut}^*(\mathcal{B}(\mathcal{H}))$, let $x, y \in X$. If we set $g = 1_X$ in (12), we have

$$\begin{gathered}\|(\tau(x) - \tau(y))(a)\| = \|\Phi(1_X \cdot a)(x) - \Phi(1_X \cdot a)(y)\| \\ \leq L_\alpha(\Phi(1_X \cdot a))d(x,y)^\alpha \leq \|\Phi(1_X \cdot a)\|_\alpha d(x,y)^\alpha \leq \|\Phi\|\|a\|d(x,y)^\alpha\end{gathered}$$

for all $a \in \mathcal{B}(\mathcal{H})$, and thus $\|\tau(x) - \tau(y)\| \leq \|\Phi\| d(x,y)^\alpha$.

Since every function in $\mathrm{F}_\alpha(X)$ can be expressed as a linear combination of functions in $\mathrm{F}_\alpha(X)^{-1}$, from (12) we deduce

$$\Phi(g \cdot a)(x) = \tau(x)(g \cdot a(\varphi(x))) \quad (g \in \mathrm{F}_\alpha(X),\ x \in X,\ a \in \mathcal{B}(\mathcal{H})). \tag{13}$$

As Φ is linear and continuous for the supremum norm, Lemma 6.3 together with (13) yield

$$\Phi(f)(x) = \tau(x)(f(\varphi(x))) \quad (f \in \mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H})),\ x \in X).$$

In view of Theorems 4.2 and 4.3, Φ is a *-automorphism of $\mathrm{F}_\alpha(X, \mathcal{B}(\mathcal{H}))$, and the proof is complete. □

References

[1] J. Araujo and L. Dubarbie, Biseparating maps between Lipschitz function spaces, *J. Math. Anal. Appl.* **357** (2009), 191–200.

[2] J. Araujo and K. Jarosz, Biseparating maps between operator algebras, *J. Math. Anal. Appl.* **282** (2003), 48–55.

[3] C. J. K. Batty and L. Molnár, On topological reflexivity of the groups of *-automorphisms and surjective isometries of $B(H)$, *Arch. Math.* **67** (1996), 415–421.

[4] F. Botelho and J. E. Jamison, Algebraic and topological reflexivity properties of $\ell_p(X)$ spaces, *J. Math. Anal. Appl.* **346** (2008), 141–144.

[5] F. Botelho and J. E. Jamison, Homomorphisms on algebras of Lipschitz functions, *Studia Math.* **199** (2010), 95–106.

[6] F. Botelho and J. E. Jamison, Algebraic reflexivity of sets of bounded operators on vector valued Lipschitz functions, *Linear Algebra Appl.* **432** (2010), 3337–3342.

[7] M. Brešar and P. Šemrl, Mappings which preserve idempotents, local automorphisms, and local derivations, *Canad. J. Math.* **45** (1993), 483–496.

[8] M. Brešar and P. Šemrl, On local automorphisms and mappings that preserve idempotents, *Studia Math.* **113** (1995), 101–108.

[9] F. Cabello Sánchez, Local isometries on spaces of continuous functions, *Math. Z.* **251** (2005), 735–749.

[10] F. Cabello Sánchez and L. Molnár, Reflexivity of the isometry group of some classical spaces, *Rev. Mat. Iberoamericana* **18** (2002), 409–430.

[11] F. Cabello Sánchez, The group of automorphisms of L_∞ is algebraically reflexive, *Studia Math.* **161** (2004), 19–32.

[12] H. X. Cao, J. H. Zhang and Z. B. Xu, Characterizations and extensions of Lipschitz-α operators, *Acta Math. Sin. (English Series)* **22** (2006), 671–678.

[13] A. M. Gleason, A characterization of maximal ideals, *J. Analyse Math.* **19** (1967), 171–172.

[14] A. Jiménez-Vargas and Moisés Villegas-Vallecillos, Order isomorphisms of little Lipschitz algebras, *Houston J. Math.* **34** (2008), 1185–1195.

[15] A. Jiménez-Vargas, Moisés Villegas-Vallecillos and Ya-Shu Wang, Banach-Stone theorems for vector-valued little Lipschitz functions, *Publ. Math. Debrecen* **74** (2009), 81–100.

[16] J. A. Johnson, Banach spaces of Lipschitz functions and vector-valued Lipschitz functions, *Trans. Amer. Math. Soc.* **148** (1970), 147–169.

[17] J. P. Kahane and W. Żelazko, A characterization of maximal ideals in commutative Banach algebras, *Studia Math.* **29** (1968), 339–343.

[18] R. V. Kadison, Isometries of operator algebras, *Ann. of Math.* **54** (1951), 325–338.

[19] D. R. Larson, Reflexivity, algebraic reflexivity and linear interpolation, *Amer. J. Math.* **110** (1988), 283–299.

[20] D. R. Larson and A. R. Sourour, Local derivations and local automorphisms of $B(X)$, *Proc. Symp. Pure Math.* **51, Part. 2** (1990), 187–194.

[21] L. Molnár, The set of automorphisms of $B(H)$ is topologically reflexive in $B(B(H))$, *Studia Math.* **122** (1997), 183–193.

[22] L. Molnár, Reflexivity of the automorphism and isometry groups of $\mathcal{C}^*$-algebras in BDF theory, *Arch. Math.* **74** (2000), 120–128.

[23] L. Molnár and M. Győry, Reflexivity of the automorphism and isometry groups of the suspension of $B(H)$, *J. Funct. Anal.* **159** (1998), 568–586.

[24] L. Molnár and B. Zalar, On local automorphisms of group algebras of compact groups, *Proc. Amer. Math. Soc.* **128** (2000), 93–99.

[25] S. Sakai, $\mathcal{C}^*$-algebras and W^*-algebras, *Springer-Verlag*, *New York/Berlin*, 1971.

[26] D. Sherbert, Banach algebras of Lipschitz functions, *Pacific J. Math.* **13** (1963), 1387–1399.

[27] D. Sherbert, The structure of ideals and point derivations in Banach algebras of Lipschitz functions, *Trans. Amer. Math. Soc.* **111** (1964), 240–272.

[28] N. Weaver, Lipschitz algebras, *World Scientific*, *Singapore*, 1999.

MARÍA BURGOS
CAMPUS DE JEREZ
FACULTAD DE CIENCIAS SOCIALES
Y DE LA COMUNICACIÓN
AV. DE LA UNIVERSIDAD S/N
11405 JEREZ, CÁDIZ
SPAIN

E-mail: maria.burgos@uca.es

A. JIMÉNEZ-VARGAS
DEPARTAMENTO DE ÁLGEBRA
Y ANÁLISIS MATEMÁTICO
UNIVERSIDAD DE ALMERÍA
04120 ALMERÍA
SPAIN

E-mail: ajimenez@ual.es

MOISÉS VILLEGAS-VALLECILLOS
CAMPUS UNIVERSITARIO DE PUERTO REAL
FACULTAD DE CIENCIAS
11510 PUERTO REAL, CÁDIZ
SPAIN

E-mail: moises.villegas@uca.es

(Received January 19, 2011; revised December 2, 2011)