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Two generator p-groups of nilpotency class 2 and their conjugacy classes

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Abstract. We give a classification of 2-generator p-groups of nilpotency class 2. Using this classification, we give a formula for the number of such groups of order p^n in terms of the partitions of n of length 3, and find formulas for the number and size of their conjugacy classes.

1. Introduction

In [1], Bacon and Kappe give a classification of 2-generator p-groups of nilpotency class 2 with the goal of computing their nonabelian tensor squares; that classification was also an attempt to correct errors found in [11]. Later [6], Kappe, Visscher, and Sarmin extended the classification to the case of 2-groups. The idea in these classifications is to start with a generator b of minimal order, and then add a second generator a of minimal order among those that generate the group together with b; then one looks at the intersections $\langle a\rangle \cap [G,G]$ and $\langle b\rangle \cap [G,G]$ and proceeds by considering the different possibilities.

These classifications were used to compute the nonabelian tensor squares of these groups [1], [2], [6], as well as identifying those that are capable [2], [7], [8]. When the first author was developing a formula to count the conjugacy classes of the 2-generator 2-groups of class 2, she discovered that the classification was incomplete; based on her example, we also discovered that the classification for the p-groups, $p > 2$, was likewise incomplete.

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One goal of this paper is to correct these omissions with a complete classification. Our approach to classifying these groups is to exploit the fact they are all central extensions of a cyclic p-group by an abelian p-group of rank 2. This viewpoint simplifies the counting of conjugacy classes, our second goal, and the resulting classification also makes it straightforward to count all the groups of this class of order p^n for any given n. Our count agrees with the one recently obtained by C. Voll using zeta functions [12], providing independent verification of our classification.

As the second and third authors were preparing a separate work concerned with the computation of the nonabelian tensor squares and other functors for these groups, a paper by MIECH [9] was brought to our attention. In this paper, Miech uses an approach very similar to ours to classify the 2-generated p-groups with cyclic commutator subgroup for odd p. His classification is somewhat more complex than ours because of the need for more parameters to account for the groups of class 3 and higher that occur, and seems difficult to extend to the $p = 2$ case. Where Miech's classification overlaps with ours, the two agree.

Since both the classification theorem and our formula for counting the conjugacy classes of these group are self-contained and straightforward we state them now.

Theorem 1.1. *Let p be a prime and $n > 2$ a positive integer. Every 2-generator p-group of order p^n and class 2 corresponds to an ordered 5-tuple of integers, $(\alpha, \beta, \gamma; \rho, \sigma)$, such that:*

(i) $\alpha \geq \beta \geq \gamma \geq 1$,

(ii) $\alpha + \beta + \gamma = n$,

(iii) $0 \leq \rho \leq \gamma$ *and* $0 \leq \sigma \leq \gamma$,

where $(\alpha, \beta, \gamma; \rho, \sigma)$ corresponds to the group presented by

$$G = \left\langle a, b \mid [a,b]^{p^\gamma} = [a,b,a] = [a,b,b] = 1,\ a^{p^\alpha} = [a,b]^{p^\rho},\ b^{p^\beta} = [a,b]^{p^\sigma} \right\rangle.$$

Moreover:

(1) *If $\alpha > \beta$, then G is isomorphic to:*
   (a) $(\alpha, \beta, \gamma; \rho, \gamma)$ *when* $\rho \leq \sigma$.
   (b) $(\alpha, \beta, \gamma; \gamma, \sigma)$ *when* $0 \leq \sigma < \sigma + \alpha - \beta \leq \rho$ *or* $\sigma < \rho = \gamma$.
   (c) $(\alpha, \beta, \gamma; \rho, \sigma)$ *when* $0 \leq \sigma < \rho < \min(\gamma, \sigma + \alpha - \beta)$.

(2) *If $\alpha = \beta > \gamma$, or $\alpha = \beta = \gamma$ and $p > 2$, then G is isomorphic to $(\alpha, \beta, \gamma; \min(\rho, \sigma), \gamma)$.*

(3) *If* $\alpha = \beta = \gamma$ *and* $p = 2$, *then* G *is isomorphic to:*
   (a) $(\alpha, \beta, \gamma; \min(\rho, \sigma), \gamma)$ *when* $0 \leq \min(\rho, \sigma) < \gamma - 1$.
   (b) $(\alpha, \beta, \gamma; \gamma - 1, \gamma - 1)$ *when* $\rho = \sigma = \gamma - 1$.
   (c) $(\alpha, \beta, \gamma; \gamma, \gamma)$ *when* $\min(\rho, \sigma) \geq \gamma - 1$ *and* $\max(\rho, \sigma) = \gamma$.

The groups listed in 1(a)–3(c) *are pairwise nonisomorphic.*

It is family 1(c) that is missing in the classifications from [1], [6]. In addition, we discovered that the families of 2-groups given in [6] were not disjoint, listing the groups $(\gamma, \gamma, \gamma; \gamma, \gamma)$ twice for each $\gamma > 0$. We will discuss this in more detail in the final section.

A direct application of the classification is to identify the orders of the conjugacy classes, the number of conjugacy classes of each order, and total number of conjugacy classes of any 2-generator p-group of class 2. These counts depend only on the order of the group and of its derived subgroup. This information is useful in studying class functions and group representations: for example, it was used to compute the irreducible characters for these groups in [10].

Denote the number of conjugacy classes of G of order p^δ by $\mathcal{C}_\delta(G)$ and denote the number of all conjugacy classes of G by $\mathcal{C}(G)$.

Theorem 1.2. *Let* G *be a* 2*-generator* p*-group of order* p^n *and class* 2 *with derived subgroup of order* p^γ.

(i) *All conjugacy classes of* G *have order* p^δ *for some* $\delta \in \{0, \ldots, \gamma\}$ *and for each* $\delta \in \{0, \ldots, \gamma\}$

$$C_\delta(G) = \begin{cases} p^{n-2\gamma} & \text{if } \delta = 0, \\ p^{n-2\gamma-1}\phi(p^\delta)(p+1) & \text{if } \delta \in \{1, \ldots, \gamma\} \end{cases}$$

where ϕ *is Euler's Totient function, i.e.* $\phi(m)$ *is the number of integers* k *with* $1 \leq k \leq m$ *such that* k *and* m *are coprime.*

(ii) *The group* G *has*

$$\mathcal{C}(G) = p^{n-\gamma}\left(1 + p^{-1} - p^{-(\gamma+1)}\right) \tag{1.3}$$

conjugacy classes.

By a result of P. Hall (see [4] Kapital V, Satz 15.2), any p-group G of order p^n has

$$p^e + (p^2 - 1)\big(m + k(p-1)\big) \tag{1.4}$$

conjugacy classes, where $e \in \{0, 1\}$, $e \equiv n \pmod 2$, and $m = \lfloor n/2 \rfloor$. The nonnegative integer k was called the *abundance of* G in [5], and is denoted by $a(G)$. As

an immediate consequence of Theorem 1.2 we obtain a formula for the abundance of a 2-generator p-groups of class 2 by equating (1.3) and (1.4) and solving for $a(G)$. The formula is given in Corollary 5.12.

The paper is structured as follows. In the next section we fix our notation, establish some preliminary results, and outline our strategy for classifying the 2-generator p-groups of class 2. In Section 3 we prove Theorem 1.1, and in Section 4 we enumerate the number of groups of order p^n in our class for fixed p and n. In Section 5 we prove Theorem 1.2. In the last section, we describe how to translate the descriptions in the previously published classifications into our 5-parameter classification.

2. Preliminaries

We write our groups multiplicatively, with 1 denoting the identity of the group. We say a group is "nilpotent of class 2" to mean the nilpotency class is exactly 2 (that is, $1 \neq [G,G] \subseteq Z(G)$). We let C_m represent the cyclic group of order m. The commutator of x and y is $[x,y] = x^{-1}y^{-1}xy$. If G is nilpotent of class at most 2, then the commutator bracket is alternating bilinear; from this, we obtain the well-known formula $(xy)^k = x^k y^k [y,x]^{\binom{k}{2}}$, where $\binom{k}{2} = \frac{k(k-1)}{2}$ for all integers k, and one easily shows that the order of $[x,y]$ in G divides the orders of $xZ(G)$ and $yZ(G)$ in $G/Z(G)$.

Let G be a 2-generator p-group of order p^n and class 2. Then G' is a central subgroup of G that is isomorphic to C_{p^γ} with $\gamma \geq 1$, and G/G' is isomorphic to $C_{p^\alpha} \times C_{p^\beta}$ with $n = \alpha + \beta + \gamma$. Without loss of generality assume $\alpha \geq \beta$. Let $\{a,b\}$ be a transversal of G/G'. Then a^{p^α} and b^{p^β} are elements of G' and we have

$$[a^{p^\alpha}, b] = 1 = [a,b]^{p^\alpha} \quad \text{and} \quad [a, b^{p^\beta}] = 1 = [a,b]^{p^\beta}.$$

Hence p^γ, the order of G', divides both p^α and p^β. It follows that $1 \leq \gamma \leq \beta \leq \alpha$.

From this analysis we may view any 2-generator p-group G of order p^n and class 2 as a central extension of the form

$$1 \longrightarrow C_{p^\gamma} \xrightarrow{\psi} G \xrightarrow{\eta} C_{p^\alpha} \times C_{p^\beta} \longrightarrow 1, \tag{2.1}$$

where $n = \alpha + \beta + \gamma$ and $\alpha \geq \beta \geq \gamma \geq 1$. Therefore to enumerate all 2-generator p-groups of class 2 of order p^n we must consider all positive partitions (α, β, γ) of n of length 3. We denote by $\mathfrak{G}_p(\alpha,\beta,\gamma)$ the set of nonisomorphic central extensions of the form (2.1) of nilpotency class 2. Any group in $\mathfrak{G}_p(\alpha,\beta,\gamma)$ is 2-generated

and has order p^n, where $n = \alpha + \beta + \gamma$. For each positive partition (α, β, γ) of n, the group

$$G = \langle a, b \mid a^{p^\alpha} = b^{p^\beta} = [a, b]^{p^\gamma} = 1,\ [a, b, a] = [a, b, b] = 1 \rangle$$

is in $\mathfrak{G}_p(\alpha, \beta, \gamma)$, so the latter is not empty. Denote by $\mathcal{G}_{p^n}$ the disjoint union of the $\mathfrak{G}_p(\alpha, \beta, \gamma)$ over all positive partitions of length 3 of n so that $\mathcal{G}_{p^n}$ is the set of nonisomorphic 2-generator p-groups of nilpotency class exactly 2 and order p^n.

Let G be a 2-generator p-group of class 2. Then G has a polycyclic series $G > G_1 > G' = 1$ such that $\langle aG_1 \rangle = G/G_1 \cong C_{p^\alpha}$, $\langle bG' \rangle = G_1/G' \cong C_{p^\beta}$, and $\langle [a, b] \rangle = G' \cong C_{p^\gamma}$. Then a, b, $[a, b]$ is a polycyclic generating sequence for G and each element $g \in G$ can be written uniquely as

$$a^i b^j [a, b]^k \tag{2.2}$$

where $0 \le i < p^\alpha$, $0 \le j < p^\beta$ and $0 \le k < p^\gamma$. We use this representation of elements of G to count specific elements such as the central elements.

Proposition 2.3. *Let G be an element $\mathfrak{G}_p(\alpha, \beta, \gamma)$. Then $Z(G)$, the center of G, has order $p^{n-2\gamma}$.*

Proof. It follows from the class of G and the order of G' that $[a^{p^\gamma}, b] = [a, b^{p^\gamma}] = [a, b]^{p^\gamma} = 1$. Hence all elements in the center of G have the form $a^{ip^\gamma} b^{jp^\gamma} [a, b]^k$ where $1 \le i \le p^{\alpha-\gamma}$, $1 \le j \le p^{\beta-\gamma}$, and $1 \le k \le p^\gamma$. Hence there are $p^{\alpha-\gamma} p^{\beta-\gamma} p^\gamma = p^{\alpha+\beta-\gamma} = p^{n-2\gamma}$ different elements in the center. □

3. The groups in $\mathfrak{G}_p(\alpha, \beta, \gamma)$

In this section we determine the nonisomorphic types within each $\mathfrak{G}_p(\alpha, \beta, \gamma)$. We will show that each isomorphism class is determined by a pair of nonnegative integers ρ and σ, with $0 \le \rho \le \gamma$ and $0 \le \sigma \le \gamma$; by placing some conditions on ρ and σ, we can identify each group with a unique ordered 5-tuples of the form $(\alpha, \beta, \gamma; \rho, \sigma)$.

In what follows, we will write "$0 \le \rho, \sigma \le \gamma$" to mean that $0 \le \rho \le \gamma$ and $0 \le \sigma \le \gamma$ both hold.

For any extension

$$1 \longrightarrow N \xrightarrow{\psi} G \xrightarrow{\eta} K \longrightarrow 1,$$

the relations of G are the relations of N (under the injection ψ), the action of K on N via the transversal function $\tau : K \to G$ and the injection ψ, and the

relations determined by the relations of K. If $r = k_1 \cdots k_n$ is a relator of K then $\tau(r)$ need not be the identity in G; it is corrected by an element of the center of N. Hence $\tau(r) \cdot \psi(c) = 1$ for some $c \in Z(N)$ is a relator in G.

From (2.1), we conclude that if $G \in \mathfrak{G}_p(\alpha, \beta, \gamma)$, then the action of $C_{p^\alpha} \times C_{p^\beta}$ on C_{p^γ} is trivial. Let $\langle c' \rangle \cong C_{p^\gamma}$ and let $\psi(\langle c' \rangle) \leq G$ be generated by $\psi(c') = c$. Let $\langle a' \rangle \times \langle b' \rangle \cong C_{p^\alpha} \times C_{p^\beta}$ with relators $[a', b']$, a'^{p^α}, b'^{p^β}. Following our general analysis above and setting $\tau(a') = a$ and $\tau(b') = b$, the relations of any group in $\mathfrak{G}_p(\alpha, \beta, \gamma)$ are $c^{p^\gamma} = 1$, $[a, b] = c^i$, $a^{p^\alpha} = c^j$, $b^{p^\beta} = c^k$, $c^a = c$, and $c^b = c$. Since G' must be cyclic of order p^γ, we will have $\gcd(i, p) = 1$.

Some values i, j, and k give isomorphic groups. Our goal is to select exactly those values of i, j, and k that enumerate all of the nonisomorphic groups in $\mathfrak{G}_p(\alpha, \beta, \gamma)$ with no repetitions. We assume that $0 < i, j, k \leq p^\gamma$.

We begin by making some simplifications. The proposition below shows that the isomorphism type of G depends only on the largest powers of p that divide j and k.

Proposition 3.1. *Fix $\alpha \geq \beta \geq \gamma$, and let i, j, k be positive integers, $0 < i, j, k \leq p^\gamma$, with $\gcd(i, p) = 1$. Write $j = up^\rho$, $k = vp^\sigma$, with $\gcd(uv, p) = 1$, and $0 \leq \rho, \sigma \leq \gamma$. If we let $G, H \in \mathfrak{G}_p(\alpha, \beta, \gamma)$ be the groups*

$$G = \left\langle a, b, c \mid c^a = c,\ c^b = c,\ [a, b] = c^i,\ a^{p^\alpha} = c^j,\ b^{p^\beta} = c^k \right\rangle,$$

$$H = \left\langle x, y, z \mid z^x = z,\ z^y = z,\ [x, y] = z,\ x^{p^\alpha} = z^{p^\rho},\ x^{p^\beta} = z^{p^\sigma} \right\rangle,$$

then G is isomorphic to H.

Proof. If $\rho = \sigma = \gamma$, then $j = k = p^\gamma$. Set $a_1 = a$, $b_1 = b$, and $c_1 = c^i$; then the elements a_1, b_1, c_1 of G satisfy the same relations as $x, y, z \in H$, and hence we have a homomorphism from H onto G that maps $x \mapsto a_1$, $y \mapsto b_1$, and $z \mapsto c_1$. Since the two groups have the same order, this map is an isomorphism.

If $\rho < \sigma = \gamma$, then pick s such that $is \equiv u \pmod{p^{\gamma-\rho}}$, and set $a_1 = a$, $b_1 = b^s$, and $c_1 = c^{is}$. Then $[a_1, b_1] = c^{is} = c_1$, $a_1^{p^\alpha} = c^j = c^{up^\rho} = c^{isp^\rho} = c_1^{p^\rho}$, and $b_1^{p^\beta} = c^{p^\gamma} = c_1^{p^\gamma}$, so again we have a homomorphism from H onto G, which proves the two groups are isomorphic.

If $\sigma < \rho = \gamma$, then pick r such that $ir \equiv v \pmod{p^{\gamma-\sigma}}$, set $a_1 = a^r$, $b_1 = b$, and $c_1 = c^{ir}$; again, we obtain a homomorphism from H onto G, showing that G is isomorphic to H.

Finally, assume that $\rho, \sigma < \gamma$. Pick r such that $ir \equiv v \pmod{p^{\gamma-\sigma}}$, s such that $is \equiv u \pmod{p^{\gamma-\rho}}$, and set $t \equiv irs \pmod{p^\gamma}$. Let $a_1 = a^r$, $b_1 = b^s$, and $c_1 = c^t$. The nontrivial relations to check are:

$$[a_1, b_1] = [a, b]^{rs} = c^{irs} = c^t = c_1 = c_1^{a_1} = c_1^{b_1},$$

$$a_1^{p^\alpha} = a^{rp^\alpha} = c^{rup^\rho} = c^{risp^\rho} = c^{tp^\rho} = c_1^{p^\rho},$$
$$b_1^{p^\beta} = b^{sp^\beta} = c^{svp^\sigma} = c^{srip^\sigma} = c^{tp^\sigma} = c_1^{p^\sigma}.$$

Therefore, there is a homomorphism from H onto G, and hence G is isomorphic to H, as claimed. □

Thus we see that the isomorphism type of a 2-generator p-group of class 2 depends on five parameters: α and β, which determine the isomorphism type of G^{ab}; γ, which gives the isomorphism type of $[G,G]$; and on parameters ρ and σ that determine the relations of a^{p^α} and b^{p^β} with $[a,b]$. We establish the following notation:

Definition 3.2. Let $\alpha \geq \beta \geq \gamma \geq 1$ be positive integers, and let ρ, σ be integers, $0 \leq \rho, \sigma \leq \gamma$. We will use the ordered 5-tuple $(\alpha, \beta, \gamma; \rho, \sigma)$ to denote the group $G \in \mathfrak{G}_p(\alpha, \beta, \gamma)$ presented by

$$G = \langle a, b, c \mid [a,b] = c,\ c^a = c,\ c^b = c,\ a^{p^\alpha} = c^{p^\rho},\ b^{p^\beta} = c^{p^\sigma} \rangle.$$

Proposition 3.1 guarantees that every group in $\mathfrak{G}_p(\alpha, \beta, \gamma)$ is isomorphic to at least one group of the form $(\alpha, \beta, \gamma; \rho, \sigma)$. We still need to determine which choices of ρ and σ may lead to isomorphic groups. The goal of the next few results is to help discover when $(\alpha, \beta, \gamma; \rho, \sigma)$ is isomorphic to $(\alpha, \beta, \gamma; \bar\rho, \bar\sigma)$.

The following result may be deduced from the main result of [3], or obtained directly:

Lemma 3.3. *Let p be a prime, and let $\alpha \geq \beta > 0$ be integers. Let $A = C_{p^\alpha} \times C_{p^\beta}$, with a and b generating the respective cyclic factors. Then every automorphism of A is of the form*

$$a \mapsto a^k b^\ell, \qquad b \mapsto a^{sp^{\alpha-\beta}} b^t,$$

where k is determined modulo p^α; ℓ, s, and t are determined modulo p^β; and $kt - \ell s p^{\alpha-\beta}$ is relatively prime to p.

Theorem 3.4. *Let p be a prime, and fix $\alpha \geq \beta \geq \gamma \geq 1$. Let ρ, σ, $\bar\rho$, $\bar\sigma$ be integers, $0 \leq \rho, \sigma, \bar\rho, \bar\sigma \leq \gamma$, and let G be the group $(\alpha, \beta, \gamma; \rho, \sigma)$. Then G is isomorphic to the group $(\alpha, \beta, \gamma; \bar\rho, \bar\sigma)$ if and only if there exist integers k, ℓ, s, t, v, w such that $\gcd(p, kt - \ell s p^{\alpha-\beta}) = 1$ and $\gcd(p, vw) = 1$, with $(a^k b^\ell)^{p^\alpha} = c^{vp^r}$ and $(a^{sp^{\alpha-\beta}} b^t)^{p^\beta} = c^{wp^s}$.*

PROOF. Let H be the group $(\alpha,\beta,\gamma;\bar\rho,\bar\sigma)$, and suppose that there is an isomorphism $f: H\to G$. To avoid possible confusion, denote the generators of H by a_H, b_H, and c_H. The isomorphism f induces an isomorphism $\overline{f}: H^{\mathrm{ab}}\to G^{\mathrm{ab}}$, so we know there exist integers k, ℓ, s, t, x, y with $\gcd(p, kt-\ell sp^{\alpha-\beta})=1$ such that $f(a_H)=a^kb^\ell c^x$ and $f(b_H)=a^{sp^{\alpha-\beta}}b^tc^y$. We also know that f restricts to an isomorphism from $[H,H]$ to $[G,G]$, so $f(c_H)=c^v$ for some integer v that is relatively prime to p. Since c is central and $\alpha\geq\beta\geq\gamma$, we have:

$$(a^kb^\ell)^{p^\alpha}=(a^kb^\ell c^x)^{p^\alpha}=f(a_H^{p^\alpha})=f(c_H^{p^{\bar\rho}})=c^{vp^{\bar\rho}},$$

$$\text{and}\quad (a^{sp^{\alpha-\beta}}b^t)^{p^\beta}=(a^{sp^{\alpha-\beta}}b^tc^y)^{p^\beta}=f(b_H^{p^\beta})=f(c_H^{p^{\bar\sigma}})=c^{vp^{\bar\sigma}}.$$

Setting $w=v$ proves the necessity.

Conversely, suppose we have integers k, ℓ, s, t, v, and w with the given properties. Set $a_1=a^kb^\ell$, $b_1=a^{sp^{\alpha-\beta}}b^t$, and $c_1=c$. Since $\gcd(p,kt-\ell sp^{\alpha-\beta})=1$, the images of a_1 and b_1 generate G^{ab}. By Proposition 3.1, there is an isomorphism between $H=(\alpha,\beta,\gamma;\bar\rho,\bar\sigma)$ and $\langle a_1,b_1,c_1\rangle$; since the latter generate G, we obtain the desired isomorphism. □

Thus, to determine whether the group $(\alpha,\beta,\gamma;\rho,\sigma)$ is isomorphic to the group $(\alpha,\beta,\gamma;\bar\rho,\bar\sigma)$, it suffices to check if there exist integers k, ℓ, s, t, v, w, with $\gcd(p,kt-\ell sp^{\alpha-\beta})=\gcd(p,vw)=1$ such that $(a^kb^\ell)^{p^\alpha}=c^{wp^{\bar\rho}}$ and $(a^{sp^{\alpha-\beta}}b^t)^{p^\beta}=c^{vp^{\bar\sigma}}$. Conversely, if we find all possible exponents $wp^{\bar\rho}$ and $vp^{\bar\sigma}$ for suitable choices of k, ℓ, s, t (and by Proposition 3.1 it suffices to determine the highest powers of p that divide those exponents), we will determine all such isomorphisms.

Suppose G is the group $(\alpha,\beta,\gamma;\rho,\sigma)$ with $0\leq\rho,\sigma\leq\gamma$. For future reference, we have the following computations:

$$\left(a^kb^\ell\right)^{p^\alpha}=a^{kp^\alpha}b^{\ell p^\alpha}[b,a]^{k\ell\binom{p^\alpha}{2}}=c^{kp^\rho+\ell p^{\sigma+\alpha-\beta}-k\ell\binom{p^\alpha}{2}},\tag{3.5}$$

and

$$\left(a^{sp^{\alpha-\beta}}b^t\right)^{p^\beta}=a^{sp^\alpha}b^{tp^\beta}[b,a]^{stp^{\alpha-\beta}\binom{p^\beta}{2}}=c^{sp^\rho+tp^\sigma-stp^{\alpha-\beta}\binom{p^\beta}{2}}.\tag{3.6}$$

We only need to determine the largest powers of p that divide the exponents of c in the above expressions. The binomial coefficients and the factor $p^{\alpha-\beta}$ lead us to consider separate cases: when $\alpha>\beta$; when $\alpha=\beta$ and either $\beta>\gamma$ or $p>2$; and when $\alpha=\beta=\gamma$ and $p=2$. We treat each of these cases in turn.

Theorem 3.7. *Let p be a prime, and fix $\alpha>\beta\geq\gamma\geq 1$. The groups $(\alpha,\beta,\gamma;\rho,\sigma)$ and $(\alpha,\beta,\gamma;\bar\rho,\bar\sigma)$, where $0\leq\rho,\sigma,\bar\rho,\bar\sigma\leq\gamma$, are isomorphic if and only if:*

(i) $\rho = \bar{\rho}$ *and* $\sigma = \bar{\sigma}$*; or*

(ii) $\rho = \bar{\rho}$, $\sigma \geq \rho$, *and* $\bar{\sigma} \geq \bar{\rho}$*; or*

(iii) $\sigma = \bar{\sigma}$, $\rho \geq \sigma + (\alpha - \beta)$, *and* $\bar{\rho} \geq \bar{\sigma} + (\alpha - \beta)$.

In particular, the group $(\alpha, \beta, \gamma; \rho, \sigma)$ *is isomorphic to one and only one of the groups listed in* 1(a)–1(c) *of Theorem 1.1, according to the conditions listed there.*

PROOF. Let $\alpha > \beta \geq \gamma \geq 1$, and let G be the group $(\alpha, \beta, \gamma; \rho, \sigma)$. By Theorem 3.4, any isomorphism is determined by a pair of elements $a^k b^\ell$ and $a^{sp^{\alpha-\beta}} b^t$, with $\gcd(kt, p) = 1$. The exponent of c in (3.5) simplifies to $kp^\rho + \ell p^{\sigma+\alpha-\beta}$, while the exponent in (3.6) simplifies to $sp^\rho + tp^\sigma$. If $\rho \leq \sigma$, then the highest power of p that divides the former is exactly p^ρ (since $\gcd(p, k) = 1$), and the highest power of p that divides the latter is at least p^ρ, possibly larger depending on the value of $s + tp^{\sigma-\rho}$; this is condition (ii). If we have $\rho \geq \sigma + \alpha - \beta$, then the highest power of p that divides the exponent of c in (3.5) is at least $p^{\sigma+\alpha-\beta}$, possibly larger depending on the value of $kp^{\rho-\sigma-\alpha+\beta} + \ell$; whereas the highest power of p that divides the exponent of c in (3.6) is exactly p^σ since $\gcd(p, t) = 1$. Thus, we are in the case contemplated in condition (iii). Finally, if $\sigma < \rho < \sigma + \alpha - \beta$, then the highest power of p that divides the exponent of c in (3.5) is exactly p^ρ since $\gcd(p, k) = 1$, and the largest power in (3.6) is exactly p^σ (again, since $\gcd(p, t) = 1$), and we are in the case contemplated in (i). Thus, the given conditions are necessary for an isomorphism.

Conversely, condition (i) is trivially sufficient. Assume next that $\rho = \bar{\rho}$, $\rho \leq \sigma$ and $\bar{\rho} \leq \bar{\sigma}$, and we want to prove that $(\alpha, \beta, \gamma; \rho, \sigma)$ is isomorphic to $(\alpha, \beta, \gamma; \bar{\rho}, \bar{\sigma})$. Setting $k = 1$, $\ell = 0$, $s = p^{\bar{\sigma}-\bar{\rho}} - p^{\sigma-\rho}$, and $t = 1$, the exponent of c in (3.5) is of course $p^\rho = p^{\bar{\rho}}$, while the exponent of c in (3.6) is $(p^{\bar{\sigma}-\rho} - p^{\sigma-\rho})p^\rho + p^\sigma = p^{\bar{\sigma}} - p^\sigma + p^\sigma = p^{\bar{\sigma}}$, proving that $(\alpha, \beta, \gamma; \rho, \sigma)$ is isomorphic to $(\alpha, \beta, \gamma; \bar{\rho}, \bar{\sigma})$. Thus, (ii) is sufficient. Finally, suppose that $\sigma = \bar{\sigma}$, $\rho \geq \sigma + \alpha - \beta$, and $\bar{\rho} \geq \bar{\sigma} + \alpha - \beta$. Then set $k = 1$, $\ell = p^{\bar{\rho}-(\sigma+\alpha-\beta)} - p^{\rho-(\sigma+\alpha-\beta)}$, $s = 0$, and $t = 1$. The exponent in (3.5) is $p^\rho + (p^{\bar{\rho}-(\sigma+\alpha-\beta)} - p^{\rho-(\sigma+\alpha-\beta)})p^{\sigma+\alpha-\beta} = p^{\bar{\rho}}$, while the exponent in (3.6) is $p^\sigma = p^{\bar{\sigma}}$, proving that $(\alpha, \beta, \gamma; \rho, \sigma)$ is indeed isomorphic to $(\alpha, \beta, \gamma; \bar{\rho}, \bar{\sigma})$ as claimed. □

Theorem 3.8. *Let* p *be a prime, and fix* $\alpha = \beta \geq \gamma \geq 1$. *If* $p > 2$ *or* $\beta > \gamma$, *then the groups* $(\alpha, \beta, \gamma; \rho, \sigma)$ *and* $(\alpha, \beta, \gamma; \bar{\rho}, \bar{\sigma})$ *are isomorphic if and only if* $\min(\rho, \sigma) = \min(\bar{\rho}, \bar{\sigma})$. *In particular, the group* $(\alpha, \beta, \gamma; \rho, \sigma)$ *is isomorphic to* $(\alpha, \beta, \gamma; \min(\rho, \sigma), \gamma)$, *as in* (2) *of Theorem 1.1.*

PROOF. For the necessity of the condition, note that selecting $k = 0$, $\ell = 1$, $s = 1$, and $t = 0$ as in Theorem 3.4 shows we may assume without loss of generality

that $\rho \leq \sigma$. Then the exponents of c in (3.5) and (3.6) simplify to $p^\rho(k + \ell p^{\sigma-\rho})$ and $p^\rho(s + tp^{\sigma-\rho})$. If $\sigma > \rho$, since at most one of k and s are multiples of p we obtain that at least one of these two expressions will be divisible by exactly p^ρ and no higher power, so $\min(\bar{\rho}, \bar{\sigma})$ is equal to ρ. On the other hand, if $\sigma = \rho$ and $k + \ell$ is divisible by p, then $kt - \ell s \equiv kt + ks \equiv k(t + s) \pmod{p}$ and the fact that $kt - \ell s$ is prime to p yields that $s + t$ is prime to p; symmetrically, if $s + t$ is divisible by p then $k + \ell$ is prime to p, so once again we have $\min(\bar{\rho}, \bar{\sigma}) = \rho$. This proves the necessity.

For sufficiency, we may assume without loss of generality that $\rho = \bar{\rho}$, $\rho \leq \sigma$, and $\bar{\rho} \leq \bar{\sigma}$. Then set $k = 1$, $\ell = 0$, $s = p^{\bar{\sigma}-\rho} - p^{\sigma-\rho}$, and $t = 1$; the exponent of c in (3.5) is $p^\rho = p^{\bar{\rho}}$, and the exponent in (3.6) simplifies to $p^{\bar{\sigma}}$, proving that $(\alpha, \beta, \gamma; \rho, \sigma)$ is isomorphic to $(\alpha, \beta, \gamma; \bar{\rho}, \bar{\sigma})$, as claimed. □

Theorem 3.9. *Let $p = 2$ and fix $\alpha = \beta = \gamma \geq 1$. The groups $(\alpha, \beta, \gamma; \rho, \sigma)$ and $(\alpha, \beta, \gamma; \bar{\rho}, \bar{\sigma})$, where $0 \leq \rho, \sigma, \bar{\rho}, \bar{\sigma} \leq \gamma$, are isomorphic if and only if:*

(i) $\min(\rho, \sigma) = \min(\bar{\rho}, \bar{\sigma})$ *and* $\max(\rho, \sigma) = \max(\bar{\rho}, \bar{\sigma})$*; or*

(ii) *exactly one of $\rho, \sigma, \bar{\rho}, \bar{\sigma}$ is equal to $\gamma - 1$ and the remaining three are equal to γ; or*

(iii) $\min(\rho, \sigma) = \min(\bar{\rho}, \bar{\sigma}) < \gamma - 1$.

In particular, the group $(\alpha, \beta, \gamma; \rho, \sigma)$ is isomorphic to exactly one of the groups in 3(a)–3(c) *of Theorem 1.1 according to the conditions listed there.*

Proof. Let G be the group $(\alpha, \beta, \gamma; \rho, \sigma)$; without loss of generality we may assume that $\rho \leq \sigma$, since all conditions are symmetric and picking $k = 0$, $\ell = 1$, $s = 1$, and $t = 0$ will yield an isomorphism between $(\alpha, \beta, \gamma; \rho, \sigma)$ and $(\alpha, \beta, \gamma; \sigma, \rho)$.

Since $p = 2$ and $\alpha = \beta = \gamma$, the binomial coefficients in (3.5) and (3.6) are congruent to $2^{\gamma-1}$ modulo 2^γ. Thus, the exponent of c in (3.5) simplifies to $k2^\rho + \ell 2^\sigma + k\ell 2^{\gamma-1}$, while the exponent of c in (3.6) becomes $s2^\rho + t2^\sigma + st2^{\gamma-1}$.

To prove necessity of the conditions, assume first that $\rho = \sigma = \gamma - 1$; since $k + \ell + k\ell$ and $s + t + st$ are both odd, the highest power of 2 that divides the exponents of c in both (3.5) and (3.6) is exactly $2^{\gamma-1}$, so we are in case (i). If $\rho = \gamma - 1$ and $\sigma = \gamma$, then the exponents of c simplify to $k2^{\gamma-1}(1 + \ell)$ and $s2^{\gamma-1}(1+t)$. At most one of ℓ and t are even, so at least one of the two exponents is divisible by 2^γ, and the other is divisible by at least $2^{\gamma-1}$, yielding either case (i) or (ii). If $\rho = \sigma = \gamma$, then the exponents simplify to $k\ell 2^{\gamma-1}$ and $st2^{\gamma-1}$. We cannot have all of k, s, t, and ℓ odd, so at least one of the two exponents is divisible by 2^γ and the other by at least $2^{\gamma-1}$, again yielding cases (i) or (ii). Finally, consider the case where $\rho < \gamma - 1$; then the exponent in (3.5) simplifies to

$2^\rho(k+\ell 2^{\sigma-\rho}+k\ell 2^{\gamma-1-\rho})$, and the one in (3.6) simplifies to $2^\rho(s+t2^{\sigma-\rho}+st2^{\gamma-1-\rho})$. If $\sigma=\rho$, since at most one of $k+\ell$ and $s+t$ is even (as $kt-\ell s$ is odd), then at least one of the two exponents is divisible by 2^ρ and no higher power of 2, yielding case (iii). And if $\sigma>\rho$, since at most one of k and s is even, we again have that at most one of the two exponents is divisible by a power of 2 higher than 2^ρ, again yielding case (iii). Thus, the three conditions are necessary.

The sufficiency of (i) follows since $\alpha=\beta$, as noted above. For (ii), we simply note that $(\gamma,\gamma,\gamma;\gamma-1,\gamma)$ is isomorphic to $(\gamma,\gamma,\gamma;\gamma,\gamma)$ by setting $k=\ell=t=1$ and $s=0$. Finally, if ρ is chosen with $\rho<\gamma-1$, and σ and $\bar\sigma$ are both greater than or equal to ρ and less than or equal to γ, then we want to show that $(\gamma,\gamma,\gamma;\rho,\sigma)$ is isomorphic to $(\gamma,\gamma,\gamma,\rho,\bar\sigma)$; this can be seen by setting $k=t=1$, $\ell=0$, and $s=2^{s-\rho}-2^{\sigma-\rho}-2^{\gamma-1-\rho}(2^{\bar\sigma-\rho}-2^{\sigma-\rho})$. □

Putting the previous three theorems together yields Theorem 1.1 in Section 1.

4. The number of nonisomorphic groups in $\mathfrak{G}_p(\alpha,\beta,\gamma)$

In this section we use our classification to give a formula for the number of groups in $\mathfrak{G}_p(\alpha,\beta,\gamma)$, the set of nonisomorphic groups of class two that have abelianization isomorphic to $C_{p^\alpha}\times C_{p^\beta}$ and commutator subgroup isomorphic to C_{p^γ}. We seek a formula for the cardinality of $\mathfrak{G}_p(\alpha,\beta,\gamma)$ in terms of α, β, and γ (the analysis below will show the number of elements does not depend on p).

Consider first the case where $\alpha=\beta$. If $p>2$ or $\beta>\gamma$, Theorem 1.1 says that each group in the class we are interested in is isomorphic to one and only one of $(\alpha,\beta,\gamma;\rho,\gamma)$ with $0\le\rho\le\gamma$, giving $\gamma+1$ nonisomorphic groups. If $p=2$ and $\beta=\gamma$, then all of these are pairwise nonisomorphic, with the exception of $(\alpha,\beta,\gamma;\gamma-1,\gamma)$ and $(\alpha,\beta,\gamma;\gamma,\gamma)$ which are isomorphic, giving only γ nonisomorphic groups. However, in this case there is a further group, namely $(\alpha,\beta,\gamma;\gamma-1,\gamma-1)$, which is not isomorphic to any of the γ groups already counted, so we again obtain $\gamma+1$ nonisomorphic groups. Thus, when $\alpha=\beta$, the set $\mathfrak{G}_p(\alpha,\beta,\gamma)$ has exactly $\gamma+1$ elements.

Next, we consider the case where $\alpha-\beta>\gamma$. In addition to the $\gamma+1$ groups given by $(\alpha,\beta,\gamma;\rho,\gamma)$ with $0\le\rho\le\gamma$, we also have one group for each choice of a pair (ρ,σ) satisfying $0\le\sigma<\rho\le\gamma$; this gives $\binom{\gamma+1}{2}$ further groups. Adding the two totals, we obtain $(\gamma+1)+\frac{1}{2}\gamma(\gamma+1)$ elements in $\mathfrak{G}_p(\alpha,\beta,\gamma)$ (we will see below the reason for expressing the count in this manner).

Finally, we come to the case where $0<\alpha-\beta\le\gamma$. There are $\gamma+1$ nonisomorphic groups corresponding to the 5-tuple $(\alpha,\beta,\gamma;\rho,\gamma)$ with $0\le\rho\le\gamma$. In

addition, we also have $\alpha-\beta$ groups of the form $(\alpha,\beta,\gamma;\rho,\sigma)$ with $\sigma<\rho\leq\sigma+\alpha-\beta$ for each choice of σ that satisfies $0\leq\sigma\leq\gamma-(\alpha-\beta)$. Finally, for $\sigma=\gamma-(\alpha-\beta)+k$ with $0<k<\alpha-\beta$, we will have exactly $(\alpha-\beta)-k$ choices of ρ that satisfy $\rho\leq\gamma$ and $\sigma<\rho\leq\sigma+\alpha-\beta$, and each such choice of ρ yields a further nonisomorphic group. The total number is then q, where:

$$\begin{aligned} q &= (\gamma+1)+(\alpha-\beta)(\gamma-(\alpha-\beta)+1)+((\alpha-\beta-1)+\cdots+1)\\ &= (\gamma+1)+(\alpha-\beta)(\gamma-(\alpha-\beta-1))+\frac{1}{2}(\alpha-\beta-1)(\alpha-\beta)\\ &= (\gamma+1)+\frac{1}{2}(\alpha-\beta)(2\gamma+1-(\alpha-\beta)). \end{aligned}$$

Consider now the expression $(\gamma+1)+\frac{1}{2}\kappa(2\gamma+1-\kappa)$. If we set $\kappa=0$, we obtain the number of nonisomorphic groups when $\alpha=\beta$. If $\kappa=\gamma$, we obtain the number of nonisomorphic groups when $\gamma<\alpha-\beta$. And if $\kappa=\alpha-\beta$, we obtain the number of nonisomorphic groups when $0<\alpha-\beta<\gamma$. Therefore, we obtain the following result:

Theorem 4.1. *Let p be a prime, and let $\alpha\geq\beta\geq\gamma\geq 1$ be integers. The cardinality of $\mathfrak{G}_p(\alpha,\beta,\gamma)$ is:*

$$|\mathfrak{G}_p(\alpha,\beta,\gamma)| = (\gamma+1)+\frac{1}{2}\min(\gamma,\alpha-\beta)(2\gamma+1-\min(\gamma,\alpha-\beta)).$$

Note that if $\gamma=0$, the expression in Theorem 4.1 evaluates to 1, which corresponds to the unique group with trivial commutator subgroup and abelianization isomorphic to $C_{p^\alpha}\times C_{p^\beta}$. Thus, if instead of considering only positive values of α, β, γ with $\alpha+\beta+\gamma=n$ we consider nonnegative values, we obtain the number of nonisomorphic 2-generator p-groups of order p^n and class *at most* two (recall that a group is a k generator group if it can be generated by k elements, although it may be generated by fewer). Thus, we have:

Theorem 4.2. *Let p be a prime and n be a positive integer. The number of nonisomorphic 2-generator p-groups of order p^n and class 2 is given by*

$$\sum_{\substack{\alpha+\beta+\gamma=n\\ \alpha\geq\beta\geq\gamma>0}}\left((\gamma+1)+\frac{1}{2}\min(\gamma,\alpha-\beta)\,(2\gamma+1-\min(\gamma,\alpha-\beta))\right).$$

The number of nonisomorphic 2-generator p-groups of order p^n and class at most 2 is given by:

$$\sum_{\substack{\alpha+\beta+\gamma=n\\ \alpha\geq\beta\geq\gamma\geq0}}\left((\gamma+1)+\frac{1}{2}\min(\gamma,\alpha-\beta)\,(2\gamma+1-\min(\gamma,\alpha-\beta))\right).$$

n	1	2	3	4	5	6	7	8	9	10
class two	0	0	2	3	5	9	13	18	26	34
class at most two	1	2	4	6	8	13	17	23	31	40

n	11	12	13	14	15	16	17	18	19	20
class two	44	58	72	89	111	134	160	193	227	266
class at most two	50	65	79	97	119	143	169	203	237	277

n	21	22	23	24	25	26	27	28	29	30
class two	312	361	415	479	545	619	703	792	888	998
class at most two	323	373	427	492	558	633	717	807	903	1014

Table 1. Number of nonisomorphic 2-generator p-groups of order p^n and class at most 2.

In Table 1 we give the number of nonisomorphic 2-generator p-groups of order p^n and class 2, and of class at most 2, for $1 \leq n \leq 30$.

We note that Voll has computed the Dirichlet generating function for the sequence given in the second line of Table 1; see [12]. The counts obtained this way agree with the ones there.

5. Conjugacy classes

The goal of this section is to prove Theorem 1.2. We start by counting the number of conjugates for an arbitrary element of G in $\mathfrak{G}_p(\alpha, \beta, \gamma)$ and computing $\mathcal{C}_\delta(G)$ (Propositions 5.1 and 5.4). Our formula of $\mathcal{C}_\delta(G)$, however, depends on α, β, and γ whereas Theorem 1.2 depends only on n and γ. Corollary 5.10 remedies this situation. We conclude with two corollaries to Theorem 1.2: Corollary 5.11 shows the distinct values of $\mathcal{C}(G)$ are bounded by n for the groups in $\mathcal{G}_{p^n}$ and Corollary 5.12 provides a formula for the abundance of G.

Proposition 5.1. *Let G be an element of $\mathfrak{G}_p(\alpha, \beta, \gamma)$ and let $g = a^i b^j [a, b]^k$ be an arbitrary element of G where $0 \leq i < p^\alpha$, $0 \leq j < p^\beta$, and $0 \leq k < p^\gamma$. Then g has $p^{\gamma-\zeta}$ conjugates in G where p^ζ is the largest p-power factor of i, j and p^γ.*

The following lemma will facilitate our proof of Proposition 5.1.

Lemma 5.2. *Let $\bar{\imath}$, $\bar{\jmath}$, s, and t be integers, with $0 \leq s < t$, and let $\gcd(\bar{\imath}, \bar{\jmath}) = d$. If $\gcd(d, t) = 1$ then there exist integers i' and j' such that $0 \leq i', j' < t$ and $(\bar{\imath}j' - i'\bar{\jmath}) \equiv s \pmod{t}$.*

Proof. Let x and y be integers such that $\bar{\imath}x - \bar{\jmath}y = d$, and let w and z be integers such that $wd + zt = 1$. Then

$$s = s(wd + zt) \equiv swd \equiv sw(\bar{\imath}x - \bar{\jmath}y) \equiv \bar{\imath}(swx) - \bar{\jmath}(swy) \pmod{t}.$$

So we may take i' and j' to be the integers, $0 \leq i', j' < t$, such that $i' \equiv swy \pmod{t}$ and $j' \equiv swx \pmod{t}$. □

Proof of Proposition 5.1. Let $g = a^i b^j [a, b]^k$ and $h = a^{i'} b^{j'} [a, b]^{k'}$ be arbitrary elements of G following (2.2). Set p^ζ to be the largest common p-power factor of i and j. If $\zeta \geq \gamma$ then p^γ is a common factor of both i and j so $g \in Z(G)$ (see Proposition 2.3). Therefore without loss of generality $0 \leq \zeta \leq \gamma$. Conjugating g with h we obtain

$$g^h = a^i b^j [a, b]^{ij' - i'j} [a, b]^k = a^i b^j [a, b]^{p^\zeta(\bar{\imath}j' - i'\bar{\jmath}) + k} \tag{5.3}$$

where $i = \bar{\imath}p^\zeta$ and $j = \bar{\jmath}p^\zeta$. Hence the conjugates of g are parameterized by i' and j' where $0 \leq i' < p^\alpha$ and $0 \leq j' < p^\beta$.

If $\zeta = \gamma$ then g is in the center of G and has $1 = p^{\gamma - \zeta}$ element, namely itself, conjugate to it.

Suppose that $\zeta < \gamma$. Set $d = \gcd(\bar{\imath}, \bar{\jmath})$. Since $\zeta < \gamma$ then $\gcd(d, p^\gamma) = 1$. Applying Lemma 5.2 with $t = p^\gamma$, for each value s, $0 \leq s < p^\gamma$ there exist i' and j' such that $\bar{\imath}j' - i'\bar{\jmath} \equiv s \pmod{p^\gamma}$. Thus the expression $p^\zeta(\bar{\imath}j' - i'\bar{\jmath}) + k$ has $p^{\gamma - \zeta}$ distinct values modulo p^γ. Hence for any element $g = a^{\bar{\imath}p^\zeta} b^{\bar{\jmath}p^\zeta} [a, b]^k$ of G with $0 \leq \zeta \leq \gamma$, the number of elements conjugate to g is $p^{\gamma - \zeta}$. □

Proposition 5.4. *Let G be any group in $\mathfrak{G}_p(\alpha, \beta, \gamma)$ and set*

$$\Phi_\alpha^\delta = \phi\left(p^{\alpha - (\gamma - \delta)}\right) \quad \text{and} \quad \Phi_\beta^\delta = \phi\left(p^{\beta - (\gamma - \delta)}\right),$$

where ϕ is Euler's Totient function and $1 \leq \delta \leq \gamma$. Then for each $\delta = 1, 2, \ldots, \gamma$

$$\mathcal{C}_\delta(G) = p^{\gamma - \delta}\left(\Phi_\alpha^\delta + \Phi_\beta^\delta + \Phi_\alpha^\delta \Phi_\beta^\delta + \Phi_\alpha^\delta \sum_{i=\gamma-\delta+1}^{\beta-1} \phi(p^{\beta - i}) + \Phi_\beta^\delta \sum_{i=\gamma-\delta+1}^{\alpha-1} \phi(p^{\alpha - i})\right). \tag{5.5}$$

PROOF. Let $G \in \mathfrak{G}_p(\alpha, \beta, \gamma)$ and $g = a^i b^j [a,b]^k$ be an arbitrary element of G. By (5.3), the conjugates of g have the form $a^i b^j [a,b]^{k'}$ where $k' = p^\zeta(\bar{\imath}\bar{\jmath}' - \bar{\imath}'\bar{\jmath}) + k$. Moreover, each $k' \equiv k \pmod{p^\zeta}$ defines an element conjugate to g. Hence for any given i, j pair, all elements of the form $a^i b^j [a,b]^k$ have conjugacy classes of the same size, and each of the p^ζ distinct values of k define elements in distinct conjugacy classes.

To count the number of conjugacy classes of order p^δ, we set $\zeta = \gamma - \delta$. Then for each i, j pair with greatest common p-power divisor equal to p^ζ, there are $p^\zeta = p^{\gamma-\delta}$ distinct values of k that form distinct conjugacy classes. This observation reduces the problem to determining all i, j pairs that define elements with conjugacy classes of order p^δ. We break this analysis into five pairwise mutually exclusive cases.

Case 1. If $i = \bar{\imath}p^\zeta$ and $j = 0$. There are $p^{\alpha-\zeta}$ multiples of p^ζ between 1 and p^α; $\bar{\imath}$ needs to be a number between 1 and $p^{\gamma-\zeta}$ that is relatively prime to p; hence, there are $\phi(p^{\alpha-\zeta}) = \phi(p^{\alpha-(\gamma-\delta)}) = \Phi_\alpha^\delta$ possible values for $\bar{\imath}$.

Case 2. If $i = 0$ and $j = \bar{\jmath}p^\zeta$. The analysis follows as in Case 1, so there are Φ_β^δ possible values for $\bar{\jmath}$.

Case 3. If $i = \bar{\imath}p^\zeta$, $j = \bar{\jmath}p^\zeta$, where $p \nmid \bar{\imath}$, and $p \nmid \bar{\jmath}$. Since the largest p-power divisor of both i and j is p^ζ, the number of possible $\bar{\imath}$ and $\bar{\jmath}$ pairs is $\phi(p^{\alpha-(\gamma-\delta)})\phi(p^{\beta-(\gamma-\delta)}) = \Phi_\alpha^\delta \Phi_\beta^\delta$, by our analysis from Case 1 and 2.

Case 4. If $i = \bar{\imath}p^\zeta$, $j = \bar{\jmath}p^\zeta$, where $p \nmid \bar{\imath}$, and $p \mid \bar{\jmath}$. In this case p divides $\bar{\jmath}$ and we have to account for all p-powers from $\zeta+1$ to $\beta-1$. Once this largest p-power that divides j is fixed, the argument for counting the number of conjugacy classes of size p^δ follows from Case 3. Hence we sum over the possible p-powers to obtain

$$\sum_{i=\zeta+1}^{\beta-1} \phi(p^{\alpha-\zeta})\phi(p^{\beta-i}) = \phi(p^{\alpha-\zeta}) \sum_{i=\zeta+1}^{\beta-1} \phi(p^{\beta-i}) = \Phi_\alpha^\delta \sum_{i=\gamma-\delta+1}^{\beta-1} \phi(p^{\beta-i}).$$

Case 5. If $i = \bar{\imath}p^\zeta$, $j = \bar{\jmath}p^\zeta$, where $p \mid \bar{\imath}$, and $p \nmid \bar{\jmath}$, the analysis follows as in Case 4 to yield

$$\sum_{i=\zeta+1}^{\alpha-i} \phi(p^{\alpha-i})\phi(p^{\beta-\zeta}) = \phi(p^{\beta-\zeta}) \sum_{i=\zeta+1}^{\alpha-1} \phi(p^{\alpha-i}) = \Phi_\beta^\delta \sum_{i=\gamma-\delta+1}^{\alpha-1} \phi(p^{\alpha-i}).$$

The five cases now correspond to the five summands on the right hand side of the expression for $\mathcal{C}_\delta(G)$ in the statement of Proposition 5.4. □

The following lemma and corollary are used to simplify (5.5).

Lemma 5.6. *Let $k > \ell > j > 0$ be integers and set $m = k - \ell$. Then:*

$$\sum_{i=j}^{k-1} \phi(p^{k-i}) = p^m \left(\sum_{i=j}^{\ell-1} \phi(p^{\ell-i}) + p^{-m} \sum_{i=1}^{m} \phi(p^i) \right), \tag{5.7}$$

$$\sum_{i=j}^{\ell-1} \phi(p^{\ell-i}) = p^{-m} \left(\sum_{i=j}^{k-1} \phi(p^{k-i}) - \sum_{i=1}^{m} \phi(p^i) \right), \tag{5.8}$$

$$\sum_{i=1}^{m} \phi(p^i) = p^m - 1. \tag{5.9}$$

Proof. These follow immediately using $\phi(p^m) = (p-1)p^{m-1}$ for all $m \geq 1$ and applying the formula for a finite geometric series. □

Corollary 5.10. *Let G and H be 2-generator p-groups of order p^n and class 2 whose derived subgroups have order p^γ. Then G and H have the same number of conjugacy classes.*

Proof. Fix n and let (α, β, γ) and $(\alpha', \beta', \gamma)$ be two positive partitions of n. If $\alpha = \alpha'$, then $\beta = \beta'$, and then both G and H are elements of $\mathfrak{G}_p(\alpha, \beta, \gamma)$, hence they have the same number of conjugacy classes by Proposition 5.4. Suppose then that $\alpha \neq \alpha'$, and without loss generality take $\alpha > \alpha'$. Then $\alpha - \alpha' = \beta' - \beta = \mu$. Since $\alpha + \beta = \alpha' + \beta'$, the centers of G and H have the same orders by Proposition 2.3. Hence it suffices to show that $\mathcal{C}_\delta(G) = \mathcal{C}_\delta(H)$ for $\delta = 1, \ldots, \gamma$. Since $\alpha = \alpha' + \mu$ and $\beta = \beta' - \mu$, we set

$$\Phi_\alpha^\delta = \phi(p^{\alpha-(\gamma-\delta)}) = (p-1)p^{\alpha'+\mu-(\gamma-\delta)-1} = (p-1)p^{\alpha'-(\gamma-\delta)-1}p^\mu = \Phi_{\alpha'}^\delta p^\mu$$

and similarly we obtain $\Phi_\beta^\delta = \Phi_{\beta'}^\delta p^{-\mu}$. Thus,

$$\Phi_\alpha^\delta \Phi_\beta^\delta = \Phi_{\alpha'}^\delta p^\mu \Phi_{\beta'}^\delta p^{-\mu} = \Phi_{\alpha'}^\delta \Phi_{\beta'}^\delta.$$

We complete the proof by showing that the sum of the remaining four terms of (5.5) for $\mathcal{C}_\delta(G)$ and $\mathcal{C}_\delta(H)$ are equal. We express the sum

$$\Phi_\alpha^\delta + \Phi_\beta^\delta + \Phi_\alpha^\delta \sum_{i=\gamma-\delta+1}^{\beta-1} \phi(p^{\beta-i}) + \Phi_\beta^\delta \sum_{i=\gamma-\delta+1}^{\alpha-1} \phi(p^{\alpha-i})$$

in terms of α' and β' using (5.7) and (5.8) to obtain

$$\Phi^\delta_{\alpha'}p^\mu + \Phi^\delta_{\beta'}p^{-\mu} + \Phi^\delta_{\alpha'}p^\mu\bigg(p^{-\mu}\bigg(\sum_{i=\gamma-\delta+1}^{\beta'-1}\phi(p^{\beta'-i}) - \sum_{i=1}^{\mu}\phi(p^i)\bigg)\bigg) + \Phi^\delta_{\beta'}p^{-\mu}\bigg(p^\mu\bigg(\sum_{i=\gamma-\delta+1}^{\alpha'-1}\phi(p^{\alpha'-i}) + p^{-\mu}\sum_{i=1}^{\mu}\phi(p^i)\bigg)\bigg).$$

Simplifying, we get

$$\begin{aligned}
&\Phi^\delta_{\alpha'}p^\mu + \Phi^\delta_{\beta'}p^{-\mu} + \Phi^\delta_{\alpha'}\sum_{i=\gamma-\delta+1}^{\beta'-1}\phi(p^{\beta'-i}) - \Phi^\delta_{\alpha'}\sum_{i=1}^{\mu}\phi(p^i)\\
&\quad + \Phi^\delta_{\beta'}\sum_{i=\gamma-\delta+1}^{\alpha'-1}\phi(p^{\alpha'-i}) + \Phi^\delta_{\beta'}p^{-\mu}\sum_{i=1}^{\mu}\phi(p^i)\\
&= \quad \Phi^\delta_{\alpha'}\sum_{i=\gamma-\delta+1}^{\beta'-1}\phi(p^{\beta'-i}) + \Phi^\delta_{\beta'}\sum_{i=\gamma-\delta+1}^{\alpha'-1}\phi^{\alpha'-i}\\
&\quad + \Phi^\delta_{\alpha'}\bigg(p^\mu - \sum_{i=1}^{\mu}\phi(p^i)\bigg) + \Phi^\delta_{\beta'}p^{-\mu}\bigg(1 + \sum_{i=1}^{\mu}\phi(p^i)\bigg)\\
&= \Phi^\delta_{\alpha'} + \Phi^\delta_{\beta'} + \Phi^\delta_{\alpha'}\sum_{i=\gamma-\delta+1}^{\beta'-1}\phi(p^{\beta'-i}) + \Phi^\delta_{\beta'}\sum_{i=\gamma-\delta+1}^{\alpha'-1}\phi(p^{\alpha'-i}),
\end{aligned}$$

where the last equality is obtained using (5.9). Hence $\mathcal{C}_\delta(G) = \mathcal{C}_\delta(H)$ as desired. □

Proof of Theorem 1.2. Let G be a 2-generator p-group of order p^n and class 2 and suppose that G' has order p^γ. To count the number of conjugacy classes we may, without loss of generality, assume that $G \in \mathfrak{G}_p(n-2\gamma,\gamma,\gamma)$ by Corollary 5.10.

(i) For $\delta = 0$, by Proposition 2.3, the order of the center is

$$p^{n-2\gamma+\gamma-\gamma} = p^{n-2\gamma}.$$

For $\delta > 0$, we have $\mu = n - 3\gamma$. Then $\Phi^\delta_\alpha = \phi(p^{\mu+\delta})$, $\Phi^\delta_\beta = \phi(p^\delta)$,

$$\sum_{i=\gamma-\delta+1}^{\alpha-1}\phi(p^{\alpha-i}) = \sum_{i=1}^{\mu+\delta-1}\phi(p^i), \quad \text{and} \quad \sum_{i=\gamma-\delta+1}^{\beta-1}\phi(p^{\beta-i}) = \sum_{i=1}^{\delta-1}\phi(p^i).$$

Using (5.9) to simplify the equations above we obtain:

$$\mathcal{C}_\delta(G) = p^{\gamma-\delta}\Big(\phi(p^{\mu+\delta}) + \phi(p^\delta) + \phi(p^{\mu+\delta})\phi(p^\delta)\Big)$$

$$\begin{aligned}&+ p^{\gamma-\delta}\Big(\phi(p^{\mu+\delta})(p^{\delta-1}-1)+\phi(p^{\delta})(p^{\mu+\delta-1}-1)\Big)\\ &= p^{\gamma-\delta}\Big(\phi(p^{\mu+\delta})\phi(p^{\delta})+\phi(p^{\mu+\delta})p^{\delta-1}+\phi(p^{\delta})p^{\mu+\delta-1}\Big)\\ &= p^{\gamma-\delta}\phi(p^{\delta})p^{\mu+\delta-1}(p+1) = p^{n-2\gamma-1}\phi(p^{\delta})(p+1).\end{aligned}$$

ii) Summing over $\delta \in \{0,\dots,\gamma\}$, we obtain the formula for the total number of conjugacy classes of G again with the help of (5.9):

$$\begin{aligned}\sum_{\delta=0}^{\gamma}\mathcal{C}_{\delta}(G) &= p^{n-2\gamma}+p^{n-2\gamma-1}(p+1)\sum_{\delta=1}^{\gamma}\phi(p^{\delta})\\ &= p^{n-2\gamma}+p^{n-2\gamma-1}(p+1)(p^{\gamma}-1) = p^{n-\gamma}\left(1+p^{-1}-p^{-(\gamma+1)}\right),\end{aligned}$$

as claimed. □

We conclude this section with direct consequences of Corollary 5.10 and Theorem 1.2, respectively.

Corollary 5.11. *Let (α,β,γ) and (α',β',γ') be two partitions of n of length 3, and let $G \in \mathfrak{G}_p(\alpha,\beta,\gamma)$ and $H \in \mathfrak{G}_p(\alpha',\beta',\gamma')$. Then G and H have the same number of conjugacy classes if and only if $\gamma = \gamma'$. Moreover, the set*

$$\big\{k \mid \text{there exists } G \in \mathcal{G}_{p^n} \text{such that } G \text{ has exactly } k \text{ conjugacy classes}\big\}$$

has exactly $\lfloor n/3 \rfloor$ elements.

Proof. If $\gamma = \gamma'$, then the number of conjugacy classes are equal by Corollary 5.10.

Conversely, if the number of conjugacy classes of G and H are equal, then

$$p^{n-\gamma}+p^{n-\gamma-1}-p^{n-1} = p^{n-\gamma'}+p^{n-\gamma'-1}-p^{n-1}$$

by Theorem 1.2 (ii). Since $0 < \gamma < n$, the largest power of p that divides the left hand side is $p^{n-\gamma-1}$, and the highest power of p that divides the right hand side is $p^{n-\gamma'-1}$, so $\gamma = \gamma'$. Since $0 < \gamma \leq \lfloor n/3 \rfloor$ must hold, there are exactly $\lfloor n/3 \rfloor$ possible values of γ. □

Corollary 5.12. *Let G be a 2-generator p-group of order p^n and class 2. If G has derived subgroup of order p^{γ}, then $a(G)$, the abundance of G, is*

$$a(G) = \frac{p^{n-\gamma}\left(1+p^{-1}-p^{-(\gamma+1)}\right)-p^{e}-m(p^2-1)}{(p^2-1)(p-1)},$$

where $m = \lfloor n/2 \rfloor$ and $e \in \{0,1\}$, $e \equiv n \pmod 2$.

6. Connections to previously published descriptions

As mentioned in the introduction, the attempts to classify the 2-generated p-groups of class 2 that appeared in [6], [1] were incomplete, and in the case of $p = 2$, two families that were claimed to be disjoint are not. There is also overlap between our Theorem 1.1 and the results in [9]. In these cases the two results agree. Theorem 1.1 and the results in [9], where the two agree.

In this final section, we connect our description with those given in the works mentioned above; this is particularly important for the classifications in [6], [1], since the lists given there have been used in subsequent work by these authors and others, e.g., [2], [7], [8].

We begin with the work of Miech, since it is closest to our description. Miech considers 2-generated nonabelian p-groups with cyclic commutator subgroup. Miech uses x, y, z where we use a, b, c, and uses a, b, c where we use α, β, γ, but otherwise his approach is essentially the same as ours, with the added complications necessitated by not assuming the groups are of class two. The three parameters, a, b, c describe the same quantities as our α, β, γ: $p^a \geq p^b$ are the abelian invariants of the abelianization of the group, and p^c is the order of the commutator subgroup. Because of the more general situation considered in [9], only the inequalities $a \geq b$, $a \geq c$ may be taken a priori.

Once these three quantities are fixed, Miech parameterizes the groups with 4-tuples, $[Rp^r, Sp^s, p^m, p^n]$, that describe the groups

$$\left\langle x, y \;\middle|\; \begin{array}{ll} [x,y]^{p^c} = [x,y,x]^{p^m} = [x,y,y]^{p^n} = 1, & \\ x^{p^a} = [x,y]^{Rp^r}, & y^{p^b} = [x,y]^{Sp^s} \end{array} \right\rangle,$$

with the parameters satisfying certain inequalities. Because the groups considered include groups of higher class, one cannot restrict attention to r and s in general, as we did, hence the need to keep track of the parameters R and S. The groups then break down into 21 families that are described in eight theorems, depending on the relative values of b and c, and of $a-b$ and c; these families, like ours, specify inequalities between the parameters R, S, r, s, m, and n. Only those families in which $m = n = c$ is possible correspond to groups of class two; in those, the inequalities always force $R = S = 1$, as we expect from our classification. The parameters r and s then correspond to our ρ and σ, respectively. The only other difference is that rather than use p^c as the parameter when $x^{p^a} = 1$ or $y^{p^b} = 1$, Miech sometimes sets $Rp^r = 0$ or $Sp^s = 0$.

Our four families for odd p (Families 1(a)–1(c) and 2 from Theorem 1.1) fall into seven of the families described by Miech; setting $m = n = c$ and simplifying

the ancillary inequalities and conditions on the parameters to account for this, they are:

(I) $[p^r, 0, 0, 0]$ when $b \geq c$ and $a - b > c$ [9, Theorem 2(a)], which are included in our Family 1(a).

(II) $[p^r, p^s, 0, 0]$ when $b \geq c$, $a - b > c$, and $0 \leq s < r \leq c$ [9, Theorem 2(c)], which fall either in our Family 1(b) or in 1(c).

(III) $[0, p^s, 0, 0]$ when $a > b \geq c$, $a - b \leq c$, and $s < c - (a - b)$ [9, Theorem 3(b)], which are included in our Family 1(b).

(IV) $[p^r, 0, 0, 0]$ when $a > b \geq c$, $a - b \leq c$, and $r \leq c$ [9, Theorem 3(c)]; these are included in our Family 1(a).

(V) $[p^r, p^s, 0, 0]$ when $a > b \geq c$, $a - b \leq c$, $s < r < s + a - b + 1$, and $s < c$ [9, Theorem 3(f)], which are in our Family 1(c).

(VI) $[0, p^s, 0, 0]$ when $a > b \geq c$, $a - b \leq c$, $s \leq c - (a - b)$ [9, Theorem 3(h)], which are in our Family 1(b).

(VII) $[0, p^s, 0, 0]$ when $a = b$ and $0 \leq s \leq c$ [9, Theorem 5(b)]; these correspond to our Family 2, with the roles of a and b reversed.

None of the other families or possible values of the parameters given by Miech correspond to groups of class 2.

Moving now to the descriptions found in [1], [6], the groups are described in terms of four families, one of which can only occur in the case $p = 2$. The notation in these papers is very hard to reconcile with our own, since they use both a, b, c and α, β, γ but for purposes very different from ours. We replace these variables in the descriptions that follow with x, y, z for the elements, and τ, υ, ϑ, ω for the parameters at play. The descriptions below for $p > 2$ appear in [1]*Theorem 2.4, and in [6]*Theorem 2.5 for $p = 2$.

(i) $\big(\langle z\rangle \times \langle x\rangle\big) \rtimes \langle y\rangle$, with $[x, y] = z$, $|z| = p^\tau$, $|y| = p^\upsilon$, and $\tau \geq \upsilon \geq \vartheta \geq 1$. By setting $a = x$, $b = y$, and $c = z$, we see that these groups are of type $(\tau, \upsilon, \vartheta; \vartheta, \vartheta)$.

(ii) $\langle x\rangle \rtimes \langle y\rangle$, with $[x, y] = x^{p^{\tau-\vartheta}}$, $|x| = p^\tau$, $|y| = p^\upsilon$, $|[x, y]| = p^\vartheta$, $\tau \geq \upsilon$, $\tau \geq 2\vartheta$, $\upsilon \geq \vartheta \geq 1$; when $p = 2$, we also place the restriction $\tau + \upsilon > 3$. If $\tau - \vartheta \geq \upsilon$, then the groups are of type $(\tau - \vartheta, \upsilon, \vartheta; \vartheta, 0)$, which can be seen by setting $a = x$, $b = y$, and $c = x^{p^{\tau-\vartheta}}$; if $\tau - \vartheta < \upsilon$, then we get $(\upsilon, \tau - \vartheta, \vartheta; 0, \vartheta)$ by setting $a = y$, $b = x$, and $c = x^{p^{\tau-\vartheta}}$.

(iii) $\big(\langle z\rangle \times \langle x\rangle\big) \rtimes \langle y\rangle$, with $[x, y] = x^{p^{\tau-\vartheta}} z$, $[z, y] = x^{-p^{2(\tau-\vartheta)}} z^{-p^{\tau-\vartheta}}$, $|x| = p^\tau$, $|y| = p^\upsilon$, $|z| = p^\omega$, $|[x, y]| = p^\vartheta$, $\vartheta > \omega \geq 1$, $\tau + \omega \geq 2\vartheta$, $\upsilon \geq \vartheta$; if p is odd we also require $\tau \geq \upsilon$. If $\tau + \omega - \vartheta \geq \upsilon$, then we let $a = x$, $b = y$, and

$c = x^{p^{\tau-\vartheta}}z$ and we obtain that the group is of type $(\tau+\omega-\vartheta, \upsilon, \vartheta; \omega, \vartheta)$. If $\tau+\omega-\vartheta < \upsilon$, we reverse the choice of a and b and get that the group is of type $(\upsilon, \tau+\omega-\vartheta, \vartheta; \vartheta, \omega)$.

(iv) $\big(\langle z\rangle \times \langle x\rangle\big)\langle y\rangle$, with $|x| = |y| = 2^{\vartheta+1}$, $|z| = 2^{\vartheta-1}$, $[x,y] = x^2z$, $[z,y] = x^{-4}y^{-2}$, $|[x,y]| = 2^{\vartheta}$, $x^{2^{\vartheta}} = y^{2^{\vartheta}}$, $\vartheta > 0$. These groups have no counterparts for odd prime, and they correspond to our family 3(b), groups of type $(\vartheta, \vartheta, \vartheta; \vartheta-1, \vartheta-1)$, with $a = x$, $b = y$, and $c = x^2z$.

As is clear, these families miss all the groups in family 1(c). The smallest group that does not occur in these families is group $(4, 2, 2; 1, 0)$, of order p^8; it was the realization by the first author that this group (with $p = 2$) was not included in any of the families (i)–(iv) above that led to our Theorem 1.1.

In addition to this omission, when $p = 2$ the four families are not disjoint. If we let $\upsilon = \vartheta = \tau - 1 = \omega + 1$ in family (iii) above, the values lead to the group $(\vartheta, \vartheta, \vartheta; \vartheta-1, \vartheta)$, which is isomorphic to the group $(\vartheta, \vartheta, \vartheta; \vartheta, \vartheta)$ that occurs in family (i). The condition $\tau + \upsilon > 3$ in family (ii) prevents this group from occurring for a third time when $\vartheta = 1$. This overlap shows up inadvertently in [8, Theorem 8.1(d)].

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