

Slant submanifolds of quaternionic space forms

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Abstract. In this paper we establish some inequalities concerning the k -Ricci curvature of a slant submanifold in a quaternionic space form. We also obtain obstructions to the existence of quaternionic slant immersions in quaternionic space forms with unfull first normal bundle.

1. Introduction

According to B.-Y. CHEN [4], one of the most important problems in submanifold theory is “*to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold*”. In [5], B.-Y. CHEN established a sharp inequality between the k -Ricci curvature, one of the main intrinsic invariants, and the squared mean curvature, the main extrinsic invariant, for a submanifold in a real space form with arbitrary codimension. Also, in the same spirit, B.-Y. Chen obtained an optimal inequality between the k -Ricci curvature and the shape operator for submanifolds in real space forms. These inequalities were further extended to many classes of submanifolds in different ambient spaces: complex space forms [25], [26], cosymplectic space forms [23], [24], [38], Sasakian space forms [9], [15], [29], [32], locally conformal Kähler space forms [3], [12], generalized complex space forms [11], [17], [27], locally conformal almost cosymplectic manifolds [16], [37], (κ, μ) -contact space forms [33], Kenmotsu space forms [1], [22], S -space forms [10], [18].

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In quaternionic setting, such inequalities were obtained for quaternionic and totally-real submanifolds [20], [21], [36]. But there are two classes of submanifolds which generalize both quaternionic and totally real submanifolds of quaternionic Kähler manifolds, with no inclusion between them: quaternionic CR-submanifolds (see [2]) and slant submanifolds (see [30]). Some recent results concerning quaternionic CR-submanifolds can be found in [13], [34] and an inequality involving Ricci curvature and squared mean curvature for quaternionic CR-submanifolds in quaternionic space forms was proved in [28]. On the other hand, some optimal inequalities involving scalar curvature, Ricci curvature and squared mean curvature for slant submanifolds in quaternionic space forms were obtained recently in [31], [35]. The main purpose of this paper is to obtain two kinds of inequalities for slant submanifolds in quaternionic space forms: between the k -Ricci curvature and the squared mean curvature and between the k -Ricci curvature and the shape operator. Moreover, we investigate the existence of quaternionic slant immersions in quaternionic space forms with unfull first normal bundle.

2. Preliminaries

Let $\overline{M}$ be a differentiable manifold and assume that there is a rank 3-subbundle σ of $\text{End}(T\overline{M})$ such that a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ satisfying for all $\alpha \in \{1, 2, 3\}$:

$$J_\alpha^2 = -Id, J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \quad (1)$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3. Then the bundle σ is called an almost quaternionic structure on M and $\{J_1, J_2, J_3\}$ is called a canonical local basis of σ . Moreover, $(\overline{M}, \sigma)$ is said to be an almost quaternionic manifold. It is easy to see that any almost quaternionic manifold is of dimension $4m$.

A Riemannian metric $\overline{g}$ on $\overline{M}$ is said to be adapted to the almost quaternionic structure σ if it satisfies:

$$\overline{g}(J_\alpha X, J_\alpha Y) = \overline{g}(X, Y), \quad \forall \alpha \in \{1, 2, 3\} \quad (2)$$

for all vector fields X, Y on $\overline{M}$ and any canonical local basis $\{J_1, J_2, J_3\}$ of σ . Moreover, $(\overline{M}, \sigma, \overline{g})$ is said to be an almost quaternionic Hermitian manifold.

If the bundle σ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$, then $(\overline{M}, \sigma, \overline{g})$ is said to be a quaternionic Kähler manifold. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that we have for all $\alpha \in \{1, 2, 3\}$:

$$\overline{\nabla}_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}, \quad (3)$$

for any vector field X on $\overline{M}$, where the indices are taken from $\{1, 2, 3\}$ modulo 3 (see [14]).

Let $(\overline{M}, \sigma, \overline{g})$ be a quaternionic Kähler manifold and let X be a non-null vector on $\overline{M}$. Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$, denoted by $Q(X)$, is called a quaternionic 4-plane. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say c . It is well-known that a quaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is a quaternionic space form, denoted $\overline{M}(c)$, if and only if its curvature tensor is given by (see [14]):

$$\begin{aligned} \overline{R}(X, Y)Z = \frac{c}{4} \left\{ \overline{g}(Z, Y)X - \overline{g}(X, Z)Y + \sum_{\alpha=1}^3 [\overline{g}(Z, J_\alpha Y)J_\alpha X - \right. \\ \left. - \overline{g}(Z, J_\alpha X)J_\alpha Y + 2\overline{g}(X, J_\alpha Y)J_\alpha Z] \right\} \end{aligned} \quad (4)$$

for all vector fields X, Y, Z on $\overline{M}$ and any local basis $\{J_1, J_2, J_3\}$ of σ .

For a submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$, we denote by g the metric tensor induced on M . If ∇ is the covariant differentiation induced on M , the Gauss and Weingarten formulas are given by:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM) \quad (5)$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad \forall X \in \Gamma(TM), \quad \forall N \in \Gamma(TM^\perp) \quad (6)$$

where h is the second fundamental form of M , $\nabla^\perp$ is the connection on the normal bundle and A_N is the shape operator of M with respect to N . The shape operator A_N is related to h by:

$$g(A_N X, Y) = \overline{g}(h(X, Y), N), \quad (7)$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$.

If we denote by $\overline{R}$ and R the curvature tensor fields of $\overline{\nabla}$ and ∇ we have the Gauss equation:

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + \overline{g}(h(X, W), h(Y, Z)) - \overline{g}(h(X, Z), h(Y, W)) \quad (8)$$

for all $X, Y, Z, W \in \Gamma(TM)$.

For the second fundamental form h , we define the covariant derivative $\overline{\nabla}h$ of h with respect to the connection on $TM \oplus T^\perp M$ by

$$(\overline{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \quad (9)$$

where D is the linear connection induced on the normal bundle of M in $\overline{M}$. Then the equation of Codazzi is given by

$$(\overline{R}(X, Y)Z)^\perp = (\overline{\nabla}_X h)(Y, Z) - (\overline{\nabla}_Y h)(X, Z). \quad (10)$$

If $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_p M$ and $\{e_{n+1}, \dots, e_{4m}\}$ is an orthonormal basis of $T_p^\perp M$, where $p \in M$, we denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i).$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, \quad r \in \{n+1, \dots, 4m\}$$

and

$$\|h\|^2(p) = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)).$$

A submanifold M of a quaternionic Kähler manifold $\overline{M}$ is called a quaternionic submanifold (resp. totally real submanifold) if each tangent space of M is carried into itself (resp. into the normal space) by each section in σ . Recently, ŞAHİN [30] introduced the slant submanifolds of quaternionic Kähler manifolds, as a natural generalization of both quaternionic and totally real submanifolds. A submanifold M of a quaternionic Kähler manifold $\overline{M}$ is said to be a slant submanifold if for each non-null vector X tangent to M at p , the angle $\theta(X)$ between $J_\alpha(X)$ and $T_p M$, $\alpha \in \{1, 2, 3\}$, is constant, i.e. it does not depend on choice of $p \in M$ and $X \in T_p M$. We can easily see that quaternionic submanifolds are slant submanifolds with $\theta = 0$ and totally-real submanifolds are slant submanifolds with $\theta = \frac{\pi}{2}$. A slant submanifold of a quaternionic Kähler manifold is said to be proper (or θ -slant proper) if it is neither quaternionic nor totally real.

If M is a slant submanifold of a quaternionic Kähler manifold $\overline{M}$, then for any $X \in \Gamma(TM)$ we have the decomposition

$$J_\alpha X = P_\alpha X + F_\alpha X, \quad (11)$$

where $P_\alpha X$ denotes the tangential component of $J_\alpha X$ and $F_\alpha X$ denotes the normal component of $J_\alpha X$.

Similarly for any $U \in \Gamma(TM^\perp)$ we have

$$J_\alpha U = B_\alpha U + C_\alpha U, \quad (12)$$

where $B_\alpha U$ is the tangential component of $J_\alpha U$ and $C_\alpha U$ is the normal component of $J_\alpha U$.

We recall now the following results which we shall need in the sequel.

Theorem 2.1 ([30]). *Let M be a submanifold of a quaternionic Kähler manifold $\overline{M}$. Then M is slant if and only if there exists a constant $\lambda \in [-1, 0]$ such that:*

$$P_\beta P_\alpha X = \lambda X, \quad \forall X \in \Gamma(TM), \alpha, \beta \in \{1, 2, 3\}. \quad (13)$$

Furthermore, in such case, if θ is the slant angle of M , then it satisfies $\lambda = -\cos^2 \theta$.

Corollary 2.2 ([30]). *Let M be a slant submanifold of a quaternionic Kähler manifold $\overline{M}$, with slant angle θ . Then we have*

$$P_\alpha^2 X = -\cos^2 \theta X, \quad (14)$$

$$B_\alpha F_\alpha X = -\sin^2 \theta X, \quad (15)$$

for any $X \in \Gamma(TM)$ and $\alpha \in \{1, 2, 3\}$.

From the above Theorem we deduce that if M is a θ -slant submanifold of a quaternionic Kähler manifold $\overline{M}$, then we have for any $X, Y \in \Gamma(TM)$:

$$g(P_\alpha X, P_\beta Y) = \cos^2 \theta g(X, Y), \quad \alpha, \beta \in \{1, 2, 3\} \quad (16)$$

and

$$\overline{g}(F_\alpha X, F_\beta Y) = \sin^2 \theta g(X, Y), \quad \alpha, \beta \in \{1, 2, 3\}. \quad (17)$$

Moreover, we can remark that every proper slant submanifold of a quaternionic Kähler manifold is of even dimension $n = 2s$, because we can choose a canonical orthonormal local frame $\{e_1, \sec \theta P_\alpha e_1, \dots, e_s, \sec \theta P_\alpha e_s\}$ of $T_p M$, $p \in M$, called an adapted slant frame, where α is settled in $\{1, 2, 3\}$.

For an n -dimensional Riemannian manifold (M, g) we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_p M$, $p \in M$. If $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space $T_p M$, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K_{ij}, \quad (18)$$

where K_{ij} denotes the sectional curvatures of the 2-plane section spanned by e_i and e_j .

For a k -plane section L of $T_p M$, $p \in M$, and X a unit vector in L , we choose an orthonormal basis $\{e_1, \dots, e_k\}$ of L such that $e_1 = X$. The Ricci curvature of L at X , denoted $\text{Ric}_L(X)$, is defined by

$$\text{Ric}_L(X) = \sum_{j=2}^k K_{1j}. \quad (19)$$

We note that such a curvature is called a k -Ricci curvature. The scalar curvature of a k -plane section L is given by

$$\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}, \quad (20)$$

For an integer k , $2 \leq k \leq n$, B.-Y. Chen introduced a Riemannian invariant Θ_k defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf \{ \text{Ric}_L(X)|L, X \}, \quad p \in M, \quad (21)$$

where L runs over all k -plane sections in $T_p M$ and X runs over all unit vectors in L (see e.g. [8]).

3. k -Ricci curvature and the squared mean curvature

Theorem 3.1. *Let M^n be a θ -slant proper submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then, for any $p \in M$ and any integer k , $2 \leq k \leq n$, one has:*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c}{4} \left(1 + \frac{9}{n-1} \cos^2 \theta \right). \quad (22)$$

PROOF. We choose an adapted slant basis of $T_p M$ at $p \in M$:

$$\{e_1, e_2 = \sec \theta P_\alpha e_1, \dots, e_{2s-1}, e_{2s} = \sec \theta P_\alpha e_{2s-1}\},$$

where $2s = n$, and $\{e_{n+1}, \dots, e_{4m}\}$ an orthonormal basis of $T_p^\perp M$, such that the normal vector e_{n+1} is in the direction of the mean curvature vector $H(p)$ and $\{e_1, \dots, e_n\}$ diagonalize the shape operator A_{n+1} .

Taking now $X = Z = e_i$, $Y = W = e_j$ in the equation of Gauss (8), by summing and using (4), we obtain:

$$n^2 \|H\|^2(p) = 2\tau(p) + \|h\|^2(p) - \frac{n(n-1)c}{4} - \frac{3c}{4} \sum_{\beta=1}^3 \sum_{i,j=1}^n g^2(P_\beta e_i, e_j). \quad (23)$$

On the other hand, because $\{e_1, \dots, e_{2s}\}$ is an adapted slant basis of $T_p M$, using (13) and (16) we can see that we have:

$$g^2(P_\beta e_i, e_{i+1}) = g^2(P_\beta e_{i+1}, e_i) = \cos^2 \theta, \quad \text{for } i = 1, 3, \dots, 2s-1 \quad (24)$$

and

$$g(P_\beta e_i, e_j) = 0, \quad \text{for } (i, j) \notin \{(2l-1, 2l), (2l, 2l-1) | l \in \{1, 2, \dots, s\}\}. \quad (25)$$

From (23), (24) and (25) we derive:

$$n^2 \|H\|^2(p) = 2\tau(p) + \|h\|^2(p) - \frac{c}{4} [n(n-1) + 9n \cos^2 \theta]. \quad (26)$$

On the other hand, due to the choosing of the basis of $T_p M$ and $T_p^\perp M$, the shape operators have the following forms:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}, \quad (27)$$

$$A_r = (h_{ij}^r)_{i,j=\overline{1,n}}, \quad \text{trace } A_r = \sum_{i=1}^n h_{ii}^r = 0, \quad \forall r \in \{n+2, \dots, 4m\}. \quad (28)$$

Now, using (27) and (28) in (26) we obtain:

$$n^2 \|H\|^2(p) = 2\tau(p) + \sum_{i=1}^n a_i^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 - \frac{c}{4} [n(n-1) + 9n \cos^2 \theta]. \quad (29)$$

On the other hand, because we have the inequality

$$(n-1) \sum_{i=1}^n a_i^2 \geq 2 \sum_{i < j} a_i a_j,$$

from

$$n^2 \|H\|^2(p) = \left(\sum_{i=1}^n a_i \right)^2 = \sum_{i=1}^n a_i^2 + 2 \sum_{1 \leq i < j \leq n} a_i a_j$$

we derive

$$\sum_{i=1}^n a_i^2 \geq n \|H\|^2(p). \quad (30)$$

Using now (30) in (29) we obtain:

$$n(n-1) \|H\|^2(p) \geq 2\tau(p) - \frac{c}{4} [n(n-1) + 9n \cos^2 \theta]. \quad (31)$$

But, from (18) and (20), it follows that for any k -plane section $L_{i_1 \dots i_k}$ spanned by $\{e_{i_1}, \dots, e_{i_k}\}$, one has:

$$\tau(L_{i_1 \dots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} \text{Ric}_{L_{i_1 \dots i_k}}(e_i) \quad (32)$$

and

$$\tau(p) = \frac{(k-2)!(n-k)!}{(n-2)!} \sum_{1 \leq i_1 < \dots < i_k \leq n} \tau(L_{i_1 \dots i_k}). \quad (33)$$

From (32) and (33) we obtain:

$$\tau(p) \geq \frac{n(n-1)}{2} \cdot \Theta_k(p) \quad (34)$$

and finally, from (31) and (34) one derives (22). $\square$

Applying Theorem 3.1 we may obtain, as a particular case, the corresponding inequality for totally-real submanifolds in quaternionic space forms, established in [21].

Corollary 3.2. *Let M^n be a totally-real submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then, for any $p \in M$ and any integer k , $2 \leq k \leq n$, one has:*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c}{4}. \quad (35)$$

4. k -Ricci curvature and shape operator

Theorem 4.1. *Let $x : M \rightarrow \overline{M}^{4m}(c)$ be an isometric immersion of an n -dimensional θ -slant proper submanifold M into a $4m$ -dimensional quaternionic space form $\overline{M}(c)$. Then, for any $p \in M$ and any integer k , $2 \leq k \leq n$, one has:*

- i. *If $\Theta_k(p) \neq \frac{c}{4}(1 + \frac{9}{n-1} \cos^2 \theta)$, then the shape operator at the mean curvature satisfies*

$$A_H > \frac{n-1}{n} \left[\Theta_k(p) - \frac{c}{4} \left(1 + \frac{9}{n-1} \cos^2 \theta \right) \right] I_n, \quad (36)$$

at p , where I_n denotes the identity map of $T_p M$.

- ii. *If $\Theta_k(p) = \frac{c}{4}(1 + \frac{9}{n-1} \cos^2 \theta)$, then $A_H \geq 0$ at p .*

iii. A unit vector $X \in T_p M$ satisfies

$$A_H X = \frac{n-1}{n} \left[\Theta_k(p) - \frac{c}{4} \left(1 + \frac{9}{n-1} \cos^2 \theta \right) \right] X \quad (37)$$

if and only if $\Theta_k(p) = \frac{c}{4} \left(1 + \frac{9}{n-1} \cos^2 \theta \right)$ and X belongs to the relative null space of M at p :

$$\mathcal{N}_p = \{Z \in T_p M | h(Z, Y) = 0, \forall Y \in T_p M\}.$$

iv. The identity

$$A_H = \frac{n-1}{n} \left[\Theta_k(p) - \frac{c}{4} \left(1 + \frac{9}{n-1} \cos^2 \theta \right) \right] I_n, \quad (38)$$

holds at p if and only if p is a totally geodesic point.

PROOF. i. We choose an adapted slant basis of $T_p M$ at $p \in M$:

$$\{e_1, e_2 = \sec \theta P_\alpha e_1, \dots, e_{2s-1}, e_{2s} = \sec \theta P_\alpha e_{2s-1}\},$$

where $2s = n$, and $\{e_{n+1}, \dots, e_{4m}\}$ an orthonormal basis of $T_p^\perp M$, such that the normal vector e_{n+1} is in the direction of the mean curvature vector $H(p)$ and $\{e_1, \dots, e_n\}$ diagonalize the shape operator A_{n+1} . Consequently, the shape operators have the forms (27) and (28).

One can distinguish two cases:

Case I: $H(p) = 0$. In this situation it follows from (22) that $\Theta_k(p) \neq \frac{c}{4} \left(1 + \frac{9}{n-1} \cos^2 \theta \right)$ and the conclusion follows.

Case II: $H(p) \neq 0$. Taking $X = Z = e_i$ and $Y = W = e_j$ in the Gauss equation and using (4), we obtain:

$$a_i a_j = K_{ij} - \frac{c}{4} \left[1 + 3 \sum_{\beta=1}^3 g^2(P_\beta e_i, e_j) \right] - \sum_{r=n+2}^{4m} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \quad (39)$$

From (39) we derive:

$$\begin{aligned} a_1(a_{i_2} + \dots + a_{i_k}) &= \text{Ric}_{L_{1i_2 \dots i_k}}(e_1) - \frac{(k-1)c}{4} - \frac{3c}{4} \sum_{\beta=1}^3 \sum_{j=2}^k g^2(P_\beta e_1, e_{i_j}) \\ &\quad - \sum_{r=n+2}^{4m} \sum_{j=2}^k [h_{11}^r h_{i_j i_j}^r - (h_{1i_j}^r)^2] \end{aligned} \quad (40)$$

which implies

$$\begin{aligned} a_1(a_2 + \cdots + a_n) &= \frac{(k-2)!(n-k)!}{(n-2)!} \sum_{2 \leq i_2 < \cdots < i_k \leq n} \text{Ric}_{L_{i_2 \dots i_k}}(e_1) - \frac{(n-1)c}{4} \\ &\quad - \frac{3c}{4} \sum_{\beta=1}^3 \sum_{j=2}^n g^2(P_\beta e_1, e_j) + \sum_{r=n+2}^{4m} \sum_{j=1}^n (h_{1j}^r)^2. \end{aligned} \quad (41)$$

and taking into account (21), we obtain:

$$a_1(a_2 + \cdots + a_n) \geq (n-1)\theta_k(p) - \frac{(n-1)c}{4} - \frac{3c}{4} \sum_{\beta=1}^3 \sum_{j=2}^n g^2(P_\beta e_1, e_j). \quad (42)$$

Using (24) and (25) in (42) we obtain:

$$a_1(a_2 + \cdots + a_n) \geq (n-1)\Theta_k(p) - \frac{(n-1)c}{4} - \frac{9c}{4} \cos^2 \theta \quad (43)$$

and we find:

$$\begin{aligned} a_1(a_1 + a_2 + \cdots + a_n) &= a_1^2 + a_1(a_2 + \cdots + a_n) \\ &\geq (n-1) \left[\Theta_k(p) - \frac{c}{4} (1 + 9 \cos^2 \theta) \right]. \end{aligned} \quad (44)$$

Similar inequalities hold when the index 1 is replaced by $j \in \{2, \dots, n\}$. Hence, we have

$$a_j(a_1 + a_2 + \cdots + a_n) \geq (n-1) \left[\Theta_k(p) - \frac{c}{4} (1 + 9 \cos^2 \theta) \right], \quad (45)$$

for all $j \in \{1, \dots, n\}$, and because $n\|H\| = a_1 + \cdots + a_n$ we find

$$A_H \geq \frac{n-1}{n} \left[\Theta_k(p) - \frac{c}{4} \left(1 + \frac{9}{n-1} \cos^2 \theta \right) \right] I_n. \quad (46)$$

We remark that the equality does not hold because we are in the case $H(p) \neq 0$.

ii. The statement is clear from i.

iii. If $X \in T_p M$ is a unit vector such that (37) holds, then we have equalities both in (42) and (44). Consequently, we obtain $a_1 = 0$ and $h_{1j}^r = 0$, for all $j \in \{1, \dots, n\}$ and $r \in \{n+2, \dots, 4m\}$, which implies $\Theta_k(p) = \frac{c}{4} (1 + \frac{9}{n-1} \cos^2 \theta)$ and $X \in \mathcal{N}_p$. The converse part is clear.

iv. The equality (41) holds for any $X \in T_p M$ if and only if $\mathcal{N}_p = T_p M$, i.e. p is a totally geodesic point. This completes the proof of the theorem. $\square$

Corollary 4.2. *Let $x : M \rightarrow \overline{M}^{4m}(c)$ be an isometric immersion of an n -dimensional totally-real submanifold M into a $4m$ -dimensional quaternionic space form $\overline{M}(c)$. Then, for any $p \in M$ and any integer k , $2 \leq k \leq n$, one has:*

- i. *If $\Theta_k(p) \neq \frac{c}{4}$, then the shape operator at the mean curvature satisfies*

$$A_H > \frac{n-1}{n} \left[\Theta_k(p) - \frac{c}{4} \right] I_n, \quad (47)$$

at p , where I_n denotes the identity map of $T_p M$.

- ii. *If $\Theta_k(p) = \frac{c}{4}$, then $A_H \geq 0$ at p .*

- iii. *A unit vector $X \in T_p M$ satisfies*

$$A_H X = \frac{n-1}{n} \left[\Theta_k(p) - \frac{c}{4} \right] X \quad (48)$$

if and only if $\Theta_k(p) = \frac{c}{4}$ and $X \in \mathcal{N}_p$.

- iv. *The identity*

$$A_H = \frac{n-1}{n} \left[\Theta_k(p) - \frac{c}{4} \right] I_n, \quad (49)$$

holds at p if and only if p is a totally geodesic point.

5. Quaternionic slant submanifolds with unfull first normal bundle

Let M be a submanifold isometrically immersed in a Riemannian manifold $(\overline{M}, \overline{g})$. If p is a point of M , then the first normal space at p is defined to be $\text{Im } h_p$, the image space of the second fundamental form h at p . Moreover, $\text{Im } h$ is called the first normal bundle of M in $\overline{M}$. The submanifold is said to have full first normal bundle if $\text{Im } h_p = T_p M^\perp$, for any $p \in M$ (see [6], [7]).

The existence of Kählerian slant submanifolds of smallest possible codimension in complex space forms, having unfull first normal bundle, has been investigated in [19]. Next we'll study this problem in the context of slant submanifolds in quaternionic space forms. The quaternionic version of Kählerian slant submanifolds has been introduced in [30], under the name of quaternionic slant submanifolds. Therefore a proper slant submanifold M of a quaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is said to be quaternionic slant submanifold if it satisfies the condition

$$\overline{\nabla}_X P_\alpha = \omega_{\alpha+2}(X) P_{\alpha+1} - \omega_{\alpha+1}(X) P_{\alpha+2}, \quad (50)$$

for any vector field X on $\overline{M}$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

We have the following characterization of quaternionic slant submanifolds.

Theorem 5.1 ([30]). *Let M be a proper slant submanifold of a quaternionic Kähler manifold $\overline{M}$. Then M is quaternionic slant submanifold if and only if*

$$A_{F_\alpha Y}Z = A_{F_\alpha Z}Y \quad (51)$$

for all $Y, Z \in \Gamma(TM)$ and $\alpha \in \{1, 2, 3\}$.

Lemma 5.2. *Let M be a slant submanifold of a quaternionic Kähler manifold $\overline{M}$. Then we have*

$$P_\alpha^2 = -Id - B_\alpha F_\alpha, \quad (52)$$

$$C_\alpha F_\alpha + F_\alpha P_\alpha = 0, \quad (53)$$

$$C_\alpha^2 = -Id - F_\alpha B_\alpha, \quad (54)$$

$$P_\alpha B_\alpha + B_\alpha C_\alpha = 0 \quad (55)$$

for $\alpha \in \{1, 2, 3\}$.

PROOF. For any $X \in \Gamma(TM)$, taking into account (1) and (11), we derive

$$-X = J_\alpha^2 X = P_\alpha^2 X + F_\alpha P_\alpha X + B_\alpha F_\alpha X + C_\alpha F_\alpha X.$$

Equating the tangent and normal parts of both the sides we obtain (52) and (53).

On the other hand, taking into account (1) and (12) we conclude that for any $U \in \Gamma(TM^\perp)$ we have

$$-U = J_\alpha^2 U = P_\alpha B_\alpha U + F_\alpha B_\alpha U + B_\alpha C_\alpha U + C_\alpha^2 U.$$

Equating now the tangent and normal parts of both the sides we obtain (54) and (55). $\square$

Lemma 5.3. *Let M be a θ -slant proper submanifold of a quaternionic Kähler manifold $\overline{M}$. Then for any vectors $U, V \in T_p M^\perp$, $p \in M$, we have*

$$\overline{g}(C_\alpha U, C_\alpha V) = \cos^2 \theta \overline{g}(U, V), \quad \alpha = 1, 2, 3. \quad (56)$$

PROOF. Because M is a θ -slant proper submanifold of $\overline{M}$, it follows that there exist $X_\alpha, Y_\alpha \in T_p M$ such that $U = F_\alpha X_\alpha$, $V = F_\alpha Y_\alpha$. Then, by using (16), (17) and (53), we derive

$$\begin{aligned} \overline{g}(C_\alpha U, C_\alpha V) &= \overline{g}(C_\alpha F_\alpha X_\alpha, C_\alpha F_\alpha Y_\alpha) = \overline{g}(F_\alpha P_\alpha X_\alpha, F_\alpha P_\alpha Y_\alpha) \\ &= \sin^2 \theta \overline{g}(P_\alpha X_\alpha, P_\alpha Y_\alpha) = \sin^2 \theta \cos^2 \theta \overline{g}(X_\alpha, Y_\alpha) \\ &= \cos^2 \theta \overline{g}(F_\alpha X_\alpha, F_\alpha Y_\alpha) = \cos^2 \theta \overline{g}(U, V). \end{aligned} \quad \square$$

From Theorem 5.1 and Lemma 5.3, using the same techniques as in [19], we can state now the following result.

Lemma 5.4. *Let M be a quaternionic slant submanifold of a quaternionic Kähler manifold $\overline{M}$. Then*

$$B_\alpha(\operatorname{Im} h_p)^\perp = \mathcal{N}_p, \quad \alpha = 1, 2, 3,$$

where $(\operatorname{Im} h_p)^\perp$ denotes the orthogonal complementary subspace of $\operatorname{Im} h_p$ in $T_p M^\perp$ and $\mathcal{N}_p$ is the relative null space of M at p .

PROOF. For $Z \in B_\alpha(\operatorname{Im} h_p)^\perp$ it follows that there exists $U \in (\operatorname{Im} h_p)^\perp$ such that $Z = B_\alpha U$. Then, by using (7), (51), (54) and (56), we obtain for all vector $X, Y \in T_p M$ and $\alpha = 1, 2, 3$:

$$\begin{aligned} \overline{g}(h(X, Z), F_\alpha Y) &= g(A_{F_\alpha Y} Z, X) = g(A_{F_\alpha Z} Y, X) = \overline{g}(h(X, Y), F_\alpha Z) \\ &= \overline{g}(h(X, Y), F_\alpha B_\alpha U) = \sin^2 \theta \overline{g}(h(X, Y), U) = 0. \end{aligned}$$

Therefore it follows that $h(X, Z) = 0$, for any $X \in T_p M$ and thus we obtain $Z \in \mathcal{N}_p$.

If we take now $Z \in \mathcal{N}_p$, it is clear that for any $X, Y \in T_p M$ and $\alpha = 1, 2, 3$ we have

$$\overline{g}(h(X, Y), F_\alpha Z) = \overline{g}(h(Z, X), F_\alpha Y) = 0.$$

Thus it follows $F_\alpha Z \in (\operatorname{Im} h_p)^\perp$ and therefore we derive

$$B_\alpha F_\alpha Z \in B_\alpha(\operatorname{Im} h_p)^\perp. \quad (57)$$

From (15) and (57) we conclude that $Z \in B_\alpha(\operatorname{Im} h_p)^\perp$ and the proof is now complete. $\square$

Theorem 5.5. *Let $x : M \rightarrow \overline{M}(c)$ be an isometric immersion of a quaternionic slant submanifold M of minimal codimension into a quaternionic space form $\overline{M}(c)$. If the first normal bundle is not full, then $c = 0$.*

PROOF. First of all we remark that if the dimension of $\overline{M}(c)$ is $4m$, then the minimal codimension of a proper slant submanifold M of $\overline{M}(c)$ is $2m$; in this case we can choose an adapted slant basis of $T_p M$ at $p \in M$:

$$\{e_1, e_2 = \sec \theta P_\alpha e_1, \dots, e_{2m-1}, e_{2m} = \sec \theta P_\alpha e_{2m-1}\},$$

and an orthonormal basis of $T_p^\perp M$:

$$\{e_{2m+1} = \operatorname{cosec} \theta F_\alpha e_1, e_{2m+2} = \operatorname{cosec} \theta F_\alpha e_2, \dots, e_{4m} = \operatorname{cosec} \theta F_\alpha e_{2m}\},$$

where α is settled in $\{1, 2, 3\}$.

Moreover, if the first normal bundle is not full, then it follows that there exists a unit normal vector $U \in T_p M^\perp$ at a point $p \in M$ such that $\bar{g}(h(X, Y), U) = 0$, for any vector $X, Y \in T_p M$ and without loss of generality we can suppose $e_{4m} = U$. Applying Lemma 5.4 it follows $B_\alpha e_{4m} \in \mathcal{N}_p$, $\alpha = 1, 2, 3$, and from (15) we conclude $e_{2m} \in \mathcal{N}_p$. Thus we have

$$h(e_i, e_{2m}) = 0, \quad i = 1, \dots, 2m - 1. \quad (58)$$

By using now (9) and (58) in (10) we obtain for $i = 1, \dots, 2m - 1$:

$$(\bar{R}(e_i, e_{2m})e_{2m})^\perp = h(e_i, \nabla_{e_{2m}} e_{2m})$$

and taking into account (58) and the definition of the Christoffel symbols Γ_{ij}^k :

$$\nabla_{e_i} e_j = \sum_{k=1}^{2m} \Gamma_{ij}^k e_k$$

we obtain

$$(\bar{R}(e_i, e_{2m})e_{2m})^\perp = \sum_{1 \leq k, l < 2m} \Gamma_{2m2m}^k h_{ik}^{2m+l} e_{2m+l}. \quad (59)$$

On the other hand, from (4) we obtain

$$\bar{R}(e_i, e_{2m})e_{2m} = \frac{c}{4} \left[e_i + 3 \sum_{\beta=1}^3 \bar{g}(e_i, J_\beta e_{2m}) J_\beta e_{2m} \right]$$

and therefore

$$(\bar{R}(e_i, e_{2m})e_{2m})^\perp = \frac{3c}{4} \sum_{\beta=1}^3 g(e_i, P_\beta e_{2m}) F_\beta e_{2m}. \quad (60)$$

But, since M is a slant submanifold, we can easily remark that

$$P_1 X = P_2 X = P_3 X, \quad X \in T_p M. \quad (61)$$

On the other hand, using (17) we obtain for all $\beta \in \{1, 2, 3\}$ and $k \in \{1, \dots, 2m\}$:

$$\bar{g}(F_\beta e_{2m}, e_{2m+k}) = \operatorname{cosec} \theta \bar{g}(F_\beta e_{2m}, F_\alpha e_k) = \operatorname{cosec} \theta \sin^2 \theta g(e_{2m}, e_k) = \sin \theta \delta_{2mk},$$

where δ_{ij} denotes the Kronecker delta. Thus we derive

$$F_1 e_{2m} = F_2 e_{2m} = F_3 e_{2m} = \sin \theta e_{4m}. \quad (62)$$

From (60), (61) and (62) we derive

$$(\bar{R}(e_i, e_{2m})e_{2m})^\perp = \frac{9c}{4}g(e_i, P_\alpha e_{2m})\sin \theta e_{4m}$$

and considering the decomposition of $P_\alpha e_{2m}$ with respect to the adapted slant basis of $T_p M$:

$$P_\alpha e_{2m} = \sum_{j=1}^{2m-1} \lambda_j e_j$$

we obtain

$$(\bar{R}(e_i, e_{2m})e_{2m})^\perp = \frac{9c}{4}\lambda_i \sin \theta e_{4m}. \quad (63)$$

Comparing now (59) and (63) we derive

$$9c\lambda_i \sin \theta = 0, \quad i = 1, \dots, 2m-1,$$

and since M is a proper slant submanifold of $\bar{M}$ and $\sum_{i=1}^{2m-1} \lambda_i^2 \neq 0$, we conclude that $c = 0$. $\square$

Corollary 5.6. *There do not exist quaternionic slant immersions of minimal codimension in $P^m(\mathbb{H})$ with unfull first normal bundle.*

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