

Local distribution of the parts of unequal partitions in arithmetic progressions II

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1. Introduction

This paper contains the main parts of the proofs of the results announced in [6]. We recall below some notations and the main result of this paper but we recommend to the reader to study first [6]. Let $d \in \mathbb{N}^*$, $\mathcal{D}$ a non-empty subset of $\{1, \dots, d\}$ and $\mathcal{D}^c = \{1, \dots, d\} \setminus \mathcal{D}$ its complement. Let $\mathcal{R}_{\mathcal{D}} = \{N_r : r \in \mathcal{D}\}$ be a multiset of $|\mathcal{D}|$ non-negative integers. The main goal of our work is to obtain an asymptotic formula for $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}})$, the number of unequal partitions of n with exactly N_r parts congruent to r modulo d for all $r \in \mathcal{D}$. We adopt the convention $\Pi_d^*(0, \mathcal{R}_{\mathcal{D}}) = 1$ if $\mathcal{R}_{\mathcal{D}} = \{0, \dots, 0\}$ and 0 otherwise.

Recall that if $n \geq 1$ and $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) \geq 1$ then n satisfies

$$n \equiv R_{\mathcal{D}} \pmod{\delta}, \tag{1.1}$$

where $R_{\mathcal{D}} = \sum_{r \in \mathcal{D}} rN_r$ and δ is the *g. c. d.* of the elements of $\mathcal{D}^c \cup \{d\}$. In the introduction of [6], we observed that the N_r , $r \in \mathcal{D}$ may be expected to be close to k_0 with

$$k_0 := \frac{2\sqrt{3} \log 2}{\pi} \frac{\sqrt{n}}{d}. \tag{1.2}$$

More precisely we suppose that for all $r \in \mathcal{D}$ we have

$$|N_r - k_0| \leq \frac{n^{\frac{1}{4}} \sqrt{\log n}}{d^{1/3} |\mathcal{D}|^{2/3} w(n)}, \tag{1.3}$$

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where $w(n)$ is a non-decreasing function such that $w(n) \rightarrow \infty$ if $n \rightarrow \infty$. Let us recall the main result of [6].

Theorem 1.1. *Let $\varepsilon > 0$. The following two propositions hold.*

- (i) *Let $d \leq n^{1/4-\varepsilon}$, $\mathcal{D} = \{1, \dots, d\}$ and $n \equiv R_{\mathcal{D}} \pmod{d}$. Let $\mathcal{R} = \mathcal{R}_{\mathcal{D}} = \{N_1, \dots, N_d\}$ be a multiset of integers satisfying (1.3). Then we have*

$$\begin{aligned} \Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) &= (1 + o(1))q(n) \frac{d}{\sqrt{1 - \frac{12(\log 2)^2}{\pi^2}}} \left(\frac{d}{2\sqrt{3n}} \right)^{d/2} \\ &\times \exp \left\{ - \frac{2\sqrt{3} \log^2 2}{\pi \left(1 - \frac{12(\log 2)^2}{\pi^2}\right) \sqrt{n}} \left(\sum_{r=1}^d (N_r - k_0) \right)^2 - \frac{\pi d}{2\sqrt{3n}} \sum_{r=1}^d (N_r - k_0)^2 \right\}. \end{aligned}$$

- (ii) *We suppose now that $d \leq n^{1/6-\varepsilon}$ and $\mathcal{D} \subset \{1, \dots, d\}$. Then under (1.1) and (1.3) we have*

$$\begin{aligned} \Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) &= q(n) \frac{\delta(1 + o(1))}{\sqrt{1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}} \left(\frac{d}{2\sqrt{3n}} \right)^{|\mathcal{D}|/2} \\ &\times \exp \left(- \frac{2\sqrt{3}(\log 2)^2}{\pi \left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right) \sqrt{n}} \left(\sum_{r \in \mathcal{D}} (N_r - k_0) \right)^2 - \frac{\pi d}{2\sqrt{3n}} \sum_{r \in \mathcal{D}} (N_r - k_0)^2 \right). \end{aligned}$$

First we complete the proof of Theorem 1.1 in the case $\mathcal{D} = \{1, \dots, d\}$, after we will handle the complementary case when $\mathcal{D}^c \neq \emptyset$. The last sections are devoted to the proofs of the different corollaries of [6].

2. The term S_2

We begin to assume that

$$d \leq n^{\frac{1}{2}-\varepsilon} \quad (2.1)$$

with some fixed positive ε and

$$|k - k_0| = o\left(\frac{\sqrt{n}}{d}\right). \quad (2.2)$$

Let

$$x_0 := \frac{\pi}{2\sqrt{3n}}, \quad t := dx_0. \quad (2.3)$$

Then

$$k_0 t = k_0 dx_0 = \log 2. \quad (2.4)$$

We also suppose that

$$|N_r - k_0| = o\left(\frac{\sqrt{n}}{d}\right) \quad (r = 1, \dots, d). \quad (2.5)$$

In Section 4 of [6] we proved that as $n \rightarrow \infty$ then we have

$$\prod_{r=1}^d g_{N_r}(dx_0) = \exp\left(\frac{\pi^2}{12x_0} + \frac{(\log 2)^2}{2x_0} + o(\sqrt{n})\right). \quad (2.6)$$

According to the notations of [6] Sections 3 and 4, we have

$$|S_2| \leq \frac{d}{2\pi} \int_{3\pi x_0 \leq |y| \leq \pi/d} \left\{ \prod_{r=1}^d |g_{N_r}(d(x_0 + iy))| \right\} \exp((n - R - Q)x_0) dy.$$

The main part of this section is the following lemma.

Lemma 2.1. *Under the notations and hypotheses (1.2), (2.1), (2.2), (2.3), (2.4), and $3\pi x_0 \leq |y| \leq \pi/d$, we have*

$$|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp\left(\frac{-1 + o(1)}{4dx_0}\right).$$

PROOF. This time we start out from the first expression of g_k and develop the logarithms:

$$\begin{aligned} g_k(w) &= \exp\left(\sum_{\nu=1}^k \log \frac{1}{1 - \exp(-\nu w)}\right) = \exp\left(\sum_{\nu=1}^k \sum_{m=1}^{\infty} \frac{1}{m} \exp(-\nu m w)\right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\nu=1}^k \exp(-\nu m w)\right) \\ &= \exp\left(\sum_{\nu=1}^k \exp(-\nu w) + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\nu=1}^k \exp(-\nu m w)\right). \end{aligned} \quad (2.7)$$

We take the moduli

$$\begin{aligned} |g_k(w)| &\leq \exp\left(\left|\sum_{\nu=1}^k \exp(-\nu w)\right| + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\nu=1}^k \exp(-\nu m t)\right) \\ &= g_k(t) \exp\left(\left|\sum_{\nu=1}^k \exp(-\nu w)\right| - \sum_{\nu=1}^k \exp(-\nu t)\right) \end{aligned}$$

$$\begin{aligned}
&= g_k(t) \exp \left(\frac{|1 - \exp(-kw)|}{|\exp(w) - 1|} - \frac{1 - \exp(-kt)}{\exp(t) - 1} \right) \\
&\leq g_k(t) \exp \left(\frac{1 + \exp(-kt)}{|\exp(w) - 1|} - \frac{1 - \exp(-kt)}{\exp(t) - 1} \right).
\end{aligned}$$

When $|\Im w| \leq \pi$,

$$|\exp(w) - 1|^2 = (\exp(t) - 1)^2 + 4e^t(\sin(b/2))^2 \geq 4(\sin(b/2))^2 \geq \frac{4|b|^2}{\pi^2},$$

thus

$$|g_k(w)| \leq g_k(t) \exp \left(\frac{1 + \exp(-kt)}{\frac{2}{\pi}|\Im w|} - \frac{1 - \exp(-kt)}{\exp(t) - 1} \right)$$

if $|\Im w| \leq \pi$. Therefore, $3\pi x_0 \leq |y| \leq \pi/d$ implies that

$$\begin{aligned}
|g_k(d(x_0 + iy))| &\leq g_k(dx_0) \exp \left(\frac{1 + \exp(-kdx_0)}{\frac{2}{\pi}d|y|} - \frac{1 - \exp(-kdx_0)}{\exp(dx_0) - 1} \right) \\
&\leq g_k(dx_0) \exp \left(\frac{1 + \exp(-kdx_0)}{6dx_0} - \frac{1 - \exp(-kdx_0)}{\exp(dx_0) - 1} \right).
\end{aligned}$$

By (1.2), (2.1), (2.2), (2.3), (2.4),

$$\begin{aligned}
|g_k(d(x_0 + iy))| &\leq g_k(dx_0) \exp \left(\frac{\frac{3}{2} + o(1)}{6dx_0} - \frac{\frac{1}{2} + o(1)}{\exp(dx_0) - 1} \right) \\
&= g_k(dx_0) \exp \left(\frac{\frac{3}{2} + o(1)}{6dx_0} - \frac{\frac{1}{2} + o(1)}{dx_0} + O(1) \right) = g_k(dx_0) \exp \left(-\frac{1 + o(1)}{4dx_0} \right).
\end{aligned}$$

This ends the proof of Lemma 2.1. $\square$

By (2.5) we obtain for S_2 ,

$$\begin{aligned}
|S_2| &\leq \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp \left(-\frac{1 + o(1)}{4x_0} \right) \exp((n - R - Q)x_0) \\
&= \exp \left(\frac{\pi\sqrt{n}}{\sqrt{3}} - \frac{\sqrt{3}n}{2\pi} + o(\sqrt{n}) \right), \tag{2.8}
\end{aligned}$$

by (2.6) and according to the estimates of Q and R obtained in Sections 2 and 4 of [6].

3. The term S_1

Next, we will try to give a similar and simple estimation for $n^{-\frac{5}{8}+\varepsilon} \leq |y| \leq 3\pi x_0$.

Lemma 3.1. (i) Under the notations and hypotheses (1.2), (2.1), (2.2), (2.3), (2.4), for $n^{-\frac{5}{8}+\varepsilon} \leq |y| \leq 3\pi x_0$ we have

$$|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp\left(-\frac{\sqrt{3}n^{1/4+2\varepsilon}}{27\pi^5 d}\right).$$

(ii) Under the notations and hypotheses (1.2), (2.1), (2.2), (2.3), (2.4), for $n^{-\frac{3}{4}+\frac{\varepsilon}{3}} \leq |y| \leq n^{-\frac{5}{8}+\varepsilon}$ we have

$$|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp\left(-\frac{\sqrt{3}n^{2\varepsilon/3}}{27\pi^5 d}\right).$$

PROOF. First we prove (i). We suppose that $n^{-\frac{5}{8}+\varepsilon} \leq |y| \leq 3\pi x_0$. By (2.7) we have

$$g_k(w) = \exp\left(\sum_{\nu=1}^k \exp(-\nu w) + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\nu=1}^k \exp(-\nu m w)\right).$$

We study again $|g_k(w)|$:

$$\begin{aligned} |g_k(w)| &\leq \exp\left(\Re\left(\sum_{\nu=1}^k \exp(-\nu w)\right) + \sum_{m=2}^{\infty} \frac{1}{m} \exp(-\nu m t)\right) \\ &= g_k(t) \exp\left(\sum_{\nu=1}^k \Re(\exp(-\nu w)) - \sum_{\nu=1}^k \exp(-\nu t)\right) \\ &= g_k(t) \exp\left(\sum_{\nu=1}^k \exp(-\nu t)(\Re \exp(-\nu ib) - 1)\right) \\ &= g_k(t) \exp\left(\sum_{\nu=1}^k \exp(-\nu t)(\cos(\nu b) - 1)\right) \\ &= g_k(t) \exp\left(-2 \sum_{\nu=1}^k \exp(-\nu t) \sin^2\left(\frac{\nu|b|}{2}\right)\right). \end{aligned}$$

Let $K_0 := \lfloor \frac{k_0}{3 \log 2} \rfloor$. If $k = k_0 + o(\frac{\sqrt{n}}{d})$ then $k > K_0$ for n large enough:

$$\begin{aligned} |g_k(w)| &\leq g_k(t) \exp\left(-2 \sum_{\nu=1}^{K_0} \exp(-\nu t) \sin^2\left(\frac{\nu|b|}{2}\right)\right) \\ &\leq g_k(t) \exp\left(-2 \sum_{\nu=1}^{K_0} \exp(-k_0 t) \sin^2\left(\frac{\nu|b|}{2}\right)\right) = g_k(t) \exp\left(-\sum_{\nu=1}^{K_0} \sin^2\left(\frac{\nu|b|}{2}\right)\right). \end{aligned}$$

Writing $w = d(x_0 + iy)$ and using the inequality $|\sin t| \geq \frac{2|t|}{\pi}$ for $|t| \leq \pi/2$, we obtain

$$|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp \left(- \sum_{\nu=1}^{K_0} \left(\frac{\nu d|y|}{\pi} \right)^2 \right)$$

since

$$\frac{\nu d|y|}{2} \leq \frac{K_0 d 3\pi x_0}{2} \leq \frac{k_0 d \pi x_0}{2 \log 2} = \frac{\pi}{2}.$$

Since $\sum_{\nu=1}^{K_0} \nu^2 \geq \frac{K_0^3}{3}$, we have

$$|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp \left(- \left(\frac{d|y|}{\pi} \right)^2 \frac{K_0^3}{3} \right) \leq g_k(dx_0) \exp \left(- \frac{\sqrt{3} n^{1/4+2\varepsilon}}{\pi^5 3^3 d} \right).$$

(ii) can be obtained similarly. $\square$

By (2.5),

$$\begin{aligned} & \left| \frac{d}{2\pi} \int_{n^{-\frac{5}{8}+\varepsilon} \leq |y| \leq 3\pi x_0} \left\{ \prod_{r=1}^d g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy)) dy \right| \\ & \leq \frac{d}{2\pi} \int_{n^{-\frac{5}{8}+\varepsilon} \leq |y| \leq 3\pi x_0} \left| \prod_{r=1}^d g_{N_r}(d(x_0 + iy)) \right| \exp((n - R - Q)x_0) dy \\ & \leq \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp \left(- \frac{\sqrt{3} n^{\frac{1}{4}+2\varepsilon}}{\pi^5 3^3} \right) \exp((n - R - Q)x_0). \end{aligned}$$

We have to stop here since the previously error term $o(\sqrt{n})$ is rough. Otherwise the above proof can be applied, *e.g.*, for $n^{-\frac{3}{4}+\frac{\varepsilon}{3}} \leq |y| \leq n^{-\frac{5}{8}+\varepsilon}$ and results that

$$\begin{aligned} & \left| \frac{d}{2\pi} \int_{n^{-\frac{3}{4}+\frac{\varepsilon}{3}} \leq |y| \leq n^{-\frac{5}{8}+\varepsilon}} \left\{ \prod_{r=1}^d g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy)) dy \right| \\ & \leq \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp \left(- \frac{\sqrt{3} n^{2\varepsilon/3}}{\pi^5 3^3} \right) \exp((n - R - Q)x_0). \end{aligned}$$

Finally we obtain for S_1 :

$$|S_1| \ll \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp \left(- \frac{\sqrt{3} n^{2\varepsilon/3}}{\pi^5 3^3} \right) \exp((n - R - Q)x_0). \quad (3.1)$$

4. The function g_k in the range $|y| < y_1$

Let $|y| \leq y_1 = n^{-\frac{3}{4} + \frac{\varepsilon}{3}}$, $w = t + ib = dx_0 + idy$. Now $\frac{|b|}{t} = O(n^{-\frac{1}{4} + \frac{\varepsilon}{3}})$.

In this section we work with a general subset $\mathcal{D} \subset \{1, \dots, d\}$. Instead of (2.2) and (2.5), we suppose that

$$|k - k_0| \leq \frac{n^{\frac{1}{4}} \sqrt{\log n}}{d^{1/3} |\mathcal{D}|^{2/3} w(n)} \quad \text{and} \quad |N_r - k_0| \leq \frac{n^{\frac{1}{4}} \sqrt{\log n}}{d^{1/3} |\mathcal{D}|^{2/3} w(n)} \quad (r \in \mathcal{D}) \quad (4.1)$$

where $w(n)$ is a non-decreasing function such that $w(n) \rightarrow \infty$ if $n \rightarrow \infty$. The aim of this paragraph is to obtain an asymptotic formula for $g_k(x_0 + iy)$ for $|y| \leq y_1$. Instead of (2.1), we suppose that

$$d \leq n^{\frac{1}{4} - 2\varepsilon}. \quad (4.2)$$

Thus (4.1) implies (2.2) and (2.5). We will prove the following Lemma:

Lemma 4.1. *Under (4.1) we have*

$$g_k(w) = f(w) \exp \left\{ -\frac{C_2}{t} + (k - k_0) \log 2 - \frac{(k - k_0)^2 t}{2} + \frac{\log 2}{2} + ib \left(\frac{C_2}{t^2} + \frac{k_0 \log 2}{t} - k_0(k - k_0) \right) + b^2 \left(\frac{C_2}{t^3} + \frac{k_0 \log 2}{t^2} + \frac{k_0^2}{2t} \right) + o \left(\frac{n^{-\varepsilon}}{|\mathcal{D}|} \right) \right\}.$$

We have again by Lemma 4.1 of [6]

$$g_k(w) = f(w) \exp \left\{ -\frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmw) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmw) + O(k^{-1}) \right\}.$$

By (1.2) and (2.2),

$$\frac{1}{k} = O \left(\frac{d}{\sqrt{n}} \right) = O \left(\frac{1}{d} \frac{d^2}{\sqrt{n}} \right).$$

Then $\frac{1}{k} = O(n^{-2\varepsilon}/d)$ and $\frac{|b|}{t} = o(n^{-\varepsilon}d^{-1})$. Since now $|y| \leq y_1$, it is possible to replace w by t in the different $\exp(-kmw)$, in cost of an admissible error term:

Lemma 4.2. (i) *We have*

$$\sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmw) = \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + O \left(\frac{|b|}{t} \right).$$

(ii) *We have*

$$\begin{aligned} \frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmw) &= \frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) - \frac{kib}{w} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) \\ &\quad - \frac{k^2 b^2}{2w} \sum_{m=1}^{\infty} \exp(-kmt) + O \left(\frac{k^3 |b|^3}{|w|} \right) \sum_{m=1}^{\infty} m \exp(-kmt). \end{aligned}$$

PROOF. By standard approximations we have

$$\begin{aligned}
 \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmw) &= \sum_{m=1}^{\infty} \frac{1}{m} (1 - (1 - \exp(-kmb))) \exp(-kmt) \\
 &= \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) O(kmb) \\
 &= \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + O\left(\frac{k|b|}{e^{kt} - 1}\right) = \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + O\left(\frac{|b|}{t}\right).
 \end{aligned}$$

This proves (i). Next we prove (ii). We have

$$\begin{aligned}
 \frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmw) &= \frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) \exp(-ikmb) \\
 &= \frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) \left\{ 1 - ikmb - \frac{(kmb)^2}{2} + O((kmb)^3) \right\}.
 \end{aligned}$$

It remains to develop to end the proof of Lemma 4.2. $\square$

We also have

$$\sum_{m=1}^{\infty} m \exp(-kmt) = \frac{\exp(-kt)}{(1 - \exp(-kt))^2} = \frac{1}{(\exp(kt) - 1)(1 - \exp(-kt))} \leq \frac{1}{(kt)^2},$$

since for $u > 0$,

$$e^u - 1 = u \sum_{n=0}^{\infty} \frac{u^n}{(n+1)!} > u \sum_{n=0}^{\infty} \frac{u^n}{n! 2^n} = ue^{\frac{u}{2}},$$

thus $(1 - e^{-u}) > ue^{-u/2}$ and $(e^u - 1)(1 - e^{-u}) > u^2$.

This gives for $g_k(w)$:

$$\begin{aligned}
 g_k(w) &= f(w) \exp \left\{ -\frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{ikb}{w} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) \right. \\
 &\quad \left. + \frac{k^2 b^2}{2w} \sum_{m=1}^{\infty} \exp(-kmt) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + o\left(\frac{n^{-\varepsilon}}{d}\right) + O\left(k \left(\frac{|b|}{t}\right)^3\right) \right\}.
 \end{aligned}$$

The next step of the proof of Lemma 4.1 consists of “replacing” $\frac{1}{w}$ by $\frac{1}{t}$ and computing the terms arisen by this manipulation. We use the formula

$$\frac{1}{w} = \frac{1}{t(1 - (-i\frac{b}{t}))} = \frac{1}{t} \left(1 - i\frac{b}{t} - \frac{b^2}{t^2} + O\left(\frac{|b|^3}{t^3}\right) \right). \quad (4.3)$$

This gives for $g_k(w)$

$$\begin{aligned}
 g_k(w) = f(w) \exp \Big\{ & -\frac{1}{t} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{ib}{t^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) \\
 & + \frac{b^2}{t^3} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + O\left(\frac{|b|^3}{t^4}\right) + \frac{ikb}{t} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) \\
 & + \frac{kb^2}{t^2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + O\left(\frac{k|b|^3}{t^3} \log\left(1 + \frac{1}{e^{kt} - 1}\right)\right) \\
 & + \frac{k^2 b^2}{2t} \sum_{m=1}^{\infty} \exp(-kmt) + O\left(\frac{k^2 |b|^3}{t^2(e^{kt} - 1)}\right) \\
 & + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + o\left(\frac{n^{-\varepsilon}}{d}\right) + O\left(\frac{k|b|^3}{t^3}\right) \Big\}.
 \end{aligned}$$

We collect the terms with ib , the terms with b^2 :

$$\begin{aligned}
 g_k(w) = f(w) \exp \Big\{ & -\frac{1}{t} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) \\
 & + ib \left(\frac{1}{t^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{k}{t} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) \right) \\
 & + b^2 \left(\frac{1}{t^3} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{k}{t^2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + \frac{k^2}{2t} \sum_{m=1}^{\infty} \exp(-kmt) \right) \\
 & + o\left(\frac{n^{-\varepsilon}}{d}\right) + O\left(\left(k + \frac{1}{t}\right) \frac{|b|^3}{t^3}\right) \Big\}.
 \end{aligned}$$

Now we compute the different summations over m . By (4.1), $\exp(-kmt)$ is close to $\exp(-k_0mt)$ if m is not too large, but we have again some computations to do to control this approximation. For $s = 0, 1, 2$:

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-kmt) = \sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-k_0mt) \exp(-(k - k_0)mt).$$

Since $e^x = \sum_{n=0}^M \frac{x^n}{n!} + \sum_{n=M+1}^{\infty} \frac{x^n}{n!}$, we have

$$\left| e^x - \sum_{n=0}^M \frac{x^n}{n!} \right| \leq |x|^{M+1} \sum_{n=M+1}^{\infty} \frac{|x|^{n-M-1}}{n!} \leq |x|^{M+1} e^{|x|}.$$

Thus we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-kmt) &= \sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-k_0mt) \left\{ 1 - (k - k_0)mt + \frac{1}{2}(k - k_0)^2 m^2 t^2 \right. \\ &\quad \left. - \frac{1}{6}(k - k_0)^3 m^3 t^3 + O(|k - k_0|^4 m^4 t^4 \exp(|k - k_0|mt)) \right\}. \end{aligned}$$

By (2.2), $\exp(|k - k_0|mt) \leq \exp(mk_0t/2)$. Next we use the fact that $k_0t = \log 2$:

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-kmt) &= \sum_{m=1}^{\infty} \frac{1}{m^s 2^m} \left(1 - (k - k_0)mt + \frac{1}{2}(k - k_0)^2 m^2 t^2 \right. \\ &\quad \left. - \frac{1}{6}(k - k_0)^3 m^3 t^3 + O(|k - k_0|^4 m^4 t^4 2^{m/2}) \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{m^s 2^m} - (k - k_0)t \sum_{m=1}^{\infty} \frac{2^{-m}}{m^{s-1}} + \frac{1}{2}(k - k_0)^2 t^2 \sum_{m=1}^{\infty} \frac{2^{-m}}{m^{s-2}} \\ &\quad - \frac{(k - k_0)^3}{6} t^3 \sum_{m=1}^{\infty} \frac{2^{-m}}{m^{s-3}} + O\left(|k - k_0|^4 t^4 \sum_{m=1}^{\infty} \frac{2^{-m/2}}{m^{s-4}}\right). \end{aligned}$$

We obtain for the function g_k

$$\begin{aligned} g_k(w) &= f(w) \exp \left\{ -\frac{1}{t} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} + (k - k_0) \sum_{m=1}^{\infty} \frac{1}{m 2^m} - \frac{(k - k_0)^2 t}{2} \sum_{m=1}^{\infty} 2^{-m} \right. \\ &\quad \left. + O(|k - k_0|^3 t^2) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m 2^m} + O(|k - k_0|t) \right. \\ &\quad \left. + ib \left(\frac{1}{t^2} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} - \frac{(k - k_0)}{t} \sum_{m=1}^{\infty} \frac{1}{m 2^m} + O(|k - k_0|^2) \right) \right. \\ &\quad \left. + \frac{k}{t} \sum_{m=1}^{\infty} \frac{1}{m 2^m} - k(k - k_0) \sum_{m=1}^{\infty} 2^{-m} + O(kt|k - k_0|^2) \right) \\ &\quad + b^2 \left(\frac{1}{t^3} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} + O\left(\frac{|k - k_0|}{t^2}\right) + \frac{k}{t^2} \sum_{m=1}^{\infty} \frac{1}{m 2^m} + O\left(\frac{k|k - k_0|}{t}\right) \right. \\ &\quad \left. + \frac{k^2}{2t} \sum_{m=1}^{\infty} 2^{-m} + O(k^2|k - k_0|) \right) + o(n^{-\varepsilon} d^{-1}) + O\left(\left(k + \frac{1}{t}\right) \frac{|b|^3}{t^3}\right) \Big\}. \end{aligned}$$

Next we compute the different sums on m :

$$g_k(w) = f(w) \exp \left\{ -\frac{C_2}{t} + (k - k_0) \log 2 - \frac{(k - k_0)^2 t}{2} + \frac{1}{2} \log 2 \right.$$

$$\begin{aligned}
& + ib \left(\frac{C_2}{t^2} - \frac{(k-k_0)}{t} \log 2 + \frac{k \log 2}{t} - k(k-k_0) \right) + b^2 \left(\frac{C_2}{t^3} + \frac{k}{t^2} \log 2 + \frac{k^2}{2t} \right) \\
& + O \left(|k-k_0|^3 t^2 + |k-k_0|t + |k-k_0|^2 |b| + \frac{|k-k_0|b^2}{t^2} \right) \\
& + o \left(\frac{n^{-\varepsilon}}{d} \right) + O \left(\left(k + \frac{1}{t} \right) \frac{|b|^3}{t^3} \right) \Big\}.
\end{aligned}$$

Then by (4.1) the above error terms give $o(n^{-\varepsilon}|\mathcal{D}|^{-1})$ and we can replace $-k(k-k_0)$ with $-k_0(k-k_0)$ in the coefficient of ib and analogously in that of b^2 . Finally,

$$\begin{aligned}
g_k(w) = f(w) \exp \Big\{ & -\frac{C_2}{t} + (k-k_0) \log 2 - \frac{(k-k_0)^2 t}{2} + \frac{\log 2}{2} \\
& + ib \left(\frac{C_2}{t^2} + \frac{k_0 \log 2}{t} - k_0(k-k_0) \right) + b^2 \left(\frac{C_2}{t^3} + \frac{k_0 \log 2}{t^2} + \frac{k_0^2}{2t} \right) + o \left(\frac{n^{-\varepsilon}}{|\mathcal{D}|} \right) \Big\},
\end{aligned}$$

as claimed in Lemma 4.1. $\square$

5. The term S_0 , end of the proof of Theorem 1.1 in the case $\mathcal{D} = \{1, \dots, d\}$

As a special case of Lemma 4.1 applied with $\mathcal{D} = \{1, \dots, d\}$, we remark that

$$g_k(dx_0) = f(dx_0) \exp \left\{ -\frac{C_2}{dx_0} + (k-k_0) \log 2 - \frac{(k-k_0)^2 dx_0}{2} + \frac{\log 2}{2} + o \left(\frac{n^{-\varepsilon}}{d} \right) \right\},$$

and

$$\begin{aligned}
\prod_{r=1}^d g_{N_r}(dx_0) = f^d(dx_0) \exp \Big\{ & -\frac{C_2}{x_0} + \sum_{r=1}^d (N_r - k_0) \log 2 \\
& - \frac{dx_0}{2} \sum_{r=1}^d (N_r - k_0)^2 + \frac{d \log 2}{2} + o(n^{-\varepsilon}) \Big\}. \quad (5.1)
\end{aligned}$$

Since $f(w) = \exp \left(\frac{\pi^2}{6w} + \frac{1}{2} \log \frac{w}{2\pi} + O(|w|) \right)$ for $w \rightarrow 0$ in $|\arg w| \leq \kappa < \pi/2$ and $\Re w > 0$, we have for $|y| \leq y_1 \leq n^{-\frac{3}{4} + \frac{\varepsilon}{5}}$:

$$\begin{aligned}
f(d(x_0 + iy)) &= \exp \left(\frac{\pi^2}{6d(x_0 + iy)} + \frac{1}{2} \log \left(\frac{d(x_0 + iy)}{2\pi} \right) + O(dx_0) \right), \\
f^d(d(x_0 + iy)) &= \exp \left(\frac{\pi^2}{6(x_0 + iy)} + \frac{d}{2} \log \left(\frac{d(x_0 + iy)}{2\pi} \right) + O(d^2 x_0) \right)
\end{aligned}$$

$$\begin{aligned}
&= \exp \left(\frac{\pi^2}{6x_0} \left(1 - \frac{iy}{x_0} - \frac{y^2}{x_0^2} + O \left(\frac{|y|^3}{x_0^3} \right) \right) + \frac{d}{2} \log \left(\frac{dx_0}{2\pi} \right) \right. \\
&\quad \left. + \frac{d}{2} \log \left(1 + \frac{iy}{x_0} \right) + O(d^2 x_0) \right) \\
&= \exp \left(\frac{\pi^2}{6x_0} - iy \frac{\pi^2}{6x_0^2} - \frac{\pi^2 y^2}{6x_0^3} + \frac{d}{2} \log \left(\frac{dx_0}{2\pi} \right) + o(n^{-\varepsilon}) \right).
\end{aligned}$$

We obtain for the integrand

$$\begin{aligned}
P &:= \left\{ \prod_{r=1}^d g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy)) \\
&= \exp \left\{ \frac{\pi^2}{6x_0} - iy \frac{\pi^2}{6x_0^2} - \frac{\pi^2 y^2}{6x_0^3} + \frac{d}{2} \log \left(\frac{dx_0}{2\pi} \right) + o(n^{-\varepsilon}) \right. \\
&\quad + (n - R - Q)(x_0 + iy) - \frac{C_2}{x_0} + \sum_{r=1}^d (N_r - k_0) \log 2 - \frac{1}{2} \sum_{r=1}^d (N_r - k_0)^2 dx_0 \\
&\quad + \frac{d}{2} \log 2 + idy \left(\frac{C_2}{dx_0^2} + \frac{k_0 \log 2}{x_0} - k_0 \sum_{r=1}^d (N_r - k_0) \right) \\
&\quad \left. + d^2 y^2 \left(\frac{C_2}{d^2 x_0^3} + \frac{k_0 \log 2}{dx_0^2} + \frac{k_0^2}{2x_0} \right) + o(n^{-\varepsilon}) \right\}.
\end{aligned}$$

We collect terms in iy , y^2 :

$$\begin{aligned}
P &= \exp \left\{ \frac{\pi^2}{6x_0} + \frac{d}{2} \log \left(\frac{dx_0}{2\pi} \right) + (n - R - Q)x_0 \right. \\
&\quad - \frac{C_2}{x_0} + \sum_{r=1}^d (N_r - k_0) \log 2 - \frac{1}{2} \sum_{r=1}^d (N_r - k_0)^2 dx_0 + \frac{d \log 2}{2} \\
&\quad + iy \left(n - R - Q - \frac{\pi^2}{6x_0^2} + \frac{C_2}{x_0^2} + \frac{dk_0 \log 2}{x_0} - dk_0 \sum_{r=1}^d (N_r - k_0) \right) \\
&\quad \left. + y^2 \left(-\frac{\pi^2}{6x_0^3} + \frac{C_2}{x_0^3} + \frac{dk_0 \log 2}{x_0^2} + \frac{d^2 k_0^2}{2x_0} \right) + o(n^{-\varepsilon}) \right\}.
\end{aligned}$$

As a special case, we obtained that

$$\begin{aligned}
&\left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0) = \exp \left\{ \frac{\pi^2}{6x_0} + \frac{d}{2} \log \left(\frac{dx_0}{2\pi} \right) + (n - R - Q)x_0 \right. \\
&\quad \left. - \frac{C_2}{x_0} + \sum_{r=1}^d (N_r - k_0) \log 2 - \frac{1}{2} \sum_{r=1}^d (N_r - k_0)^2 dx_0 + \frac{d \log 2}{2} + o(n^{-\varepsilon}) \right\}. \quad (5.2)
\end{aligned}$$

Consequently,

$$\begin{aligned}
 & \left\{ \prod_{r=1}^d g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy)) \\
 &= \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0) \\
 & \quad \times \exp \left\{ iy \left(n - R - Q - \frac{\pi^2}{6x_0^2} + \frac{1}{x_0^2} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} + \frac{dk_0 \log 2}{x_0} - dk_0 \sum_{r=1}^d (N_r - k_0) \right) \right. \\
 & \quad \left. + y^2 \left(-\frac{\pi^2}{6x_0^3} + \frac{1}{x_0^3} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} + \frac{dk_0 \log 2}{x_0^2} + \frac{d^2 k_0^2}{2x_0} \right) + o(n^{-\varepsilon}) \right\}. \quad (5.3)
 \end{aligned}$$

The coefficient of y^2 in (5.3) is

$$\frac{1}{x_0^3} \left(-\frac{\pi^2}{6} + \frac{\pi^2}{12} - \frac{\log^2 2}{2} + dk_0 x_0 \log 2 + \frac{d^2 k_0^2 x_0^2}{2} \right) = -\frac{2\sqrt{3}}{\pi} n^{\frac{3}{2}} \left(1 - \frac{12 \log^2 2}{\pi^2} \right),$$

where $\frac{12 \log^2 2}{\pi^2} < \frac{12 \cdot 0.49}{\pi^2} < \frac{6}{\pi^2} < 1$.

The coefficient of iy in (5.3) is

$$\begin{aligned}
 & n - R - Q - \frac{1}{x_0^2} \left(\frac{\pi^2}{6} - \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} - dk_0 x_0 \log 2 \right) - dk_0 \sum_{r=1}^d (N_r - k_0) \\
 &= -R - Q + \frac{\log^2 2}{2x_0^2} - dk_0 \sum_{r=1}^d (N_r - k_0) = -2dk_0 \sum_{r=1}^d (N_r - k_0) + O(n^{\frac{3}{4}-2\varepsilon}).
 \end{aligned}$$

Since $|y|O(n^{\frac{3}{4}-2\varepsilon}) = o(n^{-\varepsilon})$ we infer from (5.3) that

$$\begin{aligned}
 & \left\{ \prod_{r=1}^d g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy)) \\
 &= \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0) \\
 & \quad \times \exp \left\{ -iy 2dk_0 \sum_{r=1}^d (N_r - k_0) - y^2 \frac{2\sqrt{3}n^{\frac{3}{2}}}{\pi} \left(1 - \frac{12 \log^2 2}{\pi^2} \right) + o(n^{-\varepsilon}) \right\}. \quad (5.4)
 \end{aligned}$$

Let $A = \frac{2\sqrt{3}n^{\frac{3}{2}}}{\pi} \left(1 - \frac{12 \log^2 2}{\pi^2} \right)$, ($A > 0$) and $B = 2dk_0 \sum_{r=1}^d (N_r - k_0)$. Then, from (5.4)

$$S_0 = \frac{d}{2\pi} \int_{-y_1}^{y_1} \left\{ \prod_{r=1}^d g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy)) dy$$

$$= \frac{d}{2\pi} \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0) \int_{-y_1}^{y_1} \exp(-iyB - Ay^2 + o(n^{-\varepsilon})) dy.$$

Lemma 5.1. *We have:*

$$\begin{aligned} \int_{-y_1}^{y_1} \exp(-iyB - Ay^2 + o(n^{-\varepsilon})) dy \\ = \sqrt{\frac{\pi}{A}} \exp\left(-\frac{B^2}{4A}\right) \left\{ 1 + o(n^{-\varepsilon}) \exp\left(\frac{B^2}{4A}\right) \right\}. \end{aligned} \quad (5.5)$$

PROOF. These are standard manipulations on Gaussian integrals thus we won't write all the details. Let $I_{AB}(y_1)$ be the integral of the left hand side of (5.5). Since for $|y| \leq y_1$, $\exp(-iyB - Ay^2 + o(n^{-\varepsilon})) = (1 + o(n^{-\varepsilon})) \exp(-iyB - Ay^2)$, we have:

$$\begin{aligned} I_{AB}(y_1) &= \int_{-\infty}^{+\infty} \exp(-iyB - Ay^2) dy \\ &\quad + O\left(\int_{y_1}^{+\infty} \exp(-Ay^2) dy\right) + o(n^{-\varepsilon}) \int_{-\infty}^{+\infty} \exp(-Ay^2) dy. \end{aligned}$$

The main term is a Gaussian integral:

$$\int_{-\infty}^{+\infty} \exp(-iyB - Ay^2) dy = \sqrt{\frac{\pi}{A}} \exp\left(-\frac{B^2}{4A}\right).$$

For the error terms we have

$$\int_{y_1}^{+\infty} \exp(-Ay^2) dy \ll \frac{1}{Ay_1} \exp(-Ay_1^2) \quad \text{and} \quad \int_{-\infty}^{+\infty} \exp(-Ay^2) dy \ll \frac{1}{\sqrt{A}},$$

the Lemma follows.

Furthermore

$$\frac{B^2}{4A} = O\left(n^{-\frac{3}{2}} \left(\sqrt{n} d \frac{n^{\frac{1}{4}} \sqrt{\log n}}{dw(n)}\right)^2\right) = o(\log n).$$

Thus the error term in Lemma 5.1 is:

$$o(n^{-\varepsilon}) \exp\left(\frac{B^2}{4A}\right) = o(1).$$

Therefore

$$S_0 = (1 + o(1)) \sqrt{\frac{\pi}{A}} \exp\left(-\frac{B^2}{4A}\right) \frac{d}{2\pi} \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0).$$

Adding the estimates for the trivial parts (see (2.8), (3.1)) we obtain that for $n \equiv R \pmod{d}$,

$$\begin{aligned} \Pi_d^*(n, \mathcal{R}) &= \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0) \\ &\times \left\{ (1 + o(1)) \sqrt{\frac{\pi}{A}} \exp\left(-\frac{B^2}{4A}\right) \frac{d}{2\pi} + O\left(\exp\left(-\frac{1 + o(1)}{4x_0}\right)\right) \right. \\ &\quad \left. + O\left(\exp\left(-\frac{\sqrt{3}}{\pi^5 3^3} n^{\frac{1}{4} + 2\varepsilon}\right)\right) + O\left(\exp\left(-\frac{\sqrt{3}}{\pi^5 3^3} n^{2\varepsilon/3}\right)\right) \right\} \\ &= (1 + o(1)) \sqrt{\frac{\pi}{A}} \exp\left(-\frac{B^2}{4A}\right) \frac{d}{2\pi} \left\{ \prod_{r=1}^d g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0). \end{aligned} \quad (5.6)$$

To end the proof, it remains to insert the classical formula

$$q(n) = (1 + o(1)) \frac{1}{4 \cdot 3^{1/4} n^{3/4}} \exp\left(\frac{\pi \sqrt{n}}{\sqrt{3}}\right), \quad (5.7)$$

and our previous results on the $g_{N_r}(dx_0)$ (see (5.2)), and to do the convenient computations. We obtain

$$\begin{aligned} \Pi_d^*(n, \mathcal{R}) &= (1 + o(1)) 4 \cdot 3^{1/4} q(n) \frac{1}{\sqrt{1 - \frac{12(\log 2)^2}{\pi^2}}} \frac{d}{2\sqrt{2} 3^{1/4}} \left(\frac{d}{2\sqrt{3n}}\right)^{d/2} \exp\left(-\frac{B^2}{4A}\right) \\ &\quad + \sum_{r=1}^d (N_r - k_0) \log 2 - x_0 \left(R + Q - \frac{\log^2 2}{2x_0^2}\right) - \frac{1}{2} \sum_{r=1}^d (N_r - k_0)^2 dx_0. \end{aligned} \quad (5.8)$$

By formulae (2.11) and (2.12) of [6] and by (2.4), we have:

$$R + Q - \frac{\log^2 2}{2x_0^2} = \frac{dk_0}{2} + dk_0 \sum_{r=1}^d (N_r - k_0) + \frac{d}{2} \sum_{r=1}^d (N_r - k_0)^2 + o(\sqrt{n}).$$

Thus the argument of the exponential in (5.8) is

$$\exp\left(-\frac{B^2}{4A} + \dots\right) = \exp\left(-\frac{B^2}{4A} - \frac{\log 2}{2} - dx_0 \sum_{r=1}^d (N_r - k_0)^2 + o(1)\right).$$

Inserting this in (5.8) ends the proof of Theorem 1.1 for $\mathcal{D} = \{1, \dots, d\}$. □

6. First steps of the proof of Theorem 1.1 for $\mathcal{D} \neq \{1, \dots, d\}$

Like in Section 3 of [6], we apply Lemma 2.1 of [6], the Cauchy formula and write $z = x_0 + iy$:

$$\begin{aligned} \Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \prod_{r \in \mathcal{D}^c} h_r(x_0 + iy) \right\} \\ &\quad \times \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(d(x_0 + iy)) \right\} \exp((n - \mathcal{R}_{\mathcal{D}} - Q_{\mathcal{D}})(x_0 + iy)) dy, \quad (6.1) \end{aligned}$$

with

$$h_r(z) = \prod_{j=0}^{\infty} (1 + \exp(-(r + jd)z)).$$

When $\mathcal{D}^c \neq \emptyset$ and $\mathcal{D}^c \neq \{d\}$, the functions h_r are not $2\pi/d$ -periodic but we still split the integral in intervals of length $2\pi/d$ in order to use our previous work on the functions g_k .

$$\Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) = \frac{1}{2\pi} \sum_{|\lambda| \leq \lfloor \frac{d-1}{2} \rfloor} \int_{-\frac{\pi}{d} + \frac{2\lambda\pi}{d}}^{\frac{\pi}{d} + \frac{2\lambda\pi}{d}} \dots + B,$$

with

$$B = \begin{cases} 0 & \text{if } d \text{ is odd} \\ \int_{-\pi}^{-\pi + \frac{\pi}{d}} \dots + \int_{\pi - \frac{\pi}{d}}^{\pi} \dots = \int_{\pi - \frac{\pi}{d}}^{\pi + \frac{\pi}{d}} & \text{if } d \text{ is even.} \end{cases}$$

Next we do some convenient change of variables :

$$\begin{aligned} \Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) &= \frac{1}{2\pi} \sum_{-\frac{d}{2} < \lambda \leq \frac{d}{2} - \frac{\pi}{d}} \int_{-\frac{\pi}{d}}^{\frac{\pi}{d}} \left\{ \prod_{r \in \mathcal{D}^c} h_r \left(x_0 + iy + i \frac{2\lambda\pi}{d} \right) \right\} \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(d(x_0 + iy)) \right\} \\ &\quad \times \exp \left((n - \mathcal{R}_{\mathcal{D}} - Q_{\mathcal{D}})(x_0 + iy) + (n - \mathcal{R}_{\mathcal{D}}) \frac{2i\lambda\pi}{d} \right) dy = \sum_{-\frac{d}{2} < \lambda \leq \frac{d}{2}} S(\lambda), \end{aligned}$$

say. We will write the splitting

$$S(\lambda) = S_0(\lambda) + S_1(\lambda) + S_2(\lambda),$$

where in $S_0(\lambda)$ the range of integration for y is $|y| \leq y_1$, in $S_1(\lambda)$ it is for $y_1 \leq |y| \leq y_2$ (with $y_2 = 3\pi x_0$) and in $S_2(\lambda)$ we take $y_2 \leq |y| \leq \frac{\pi}{d}$ (cf. (4.11) of [6]).

7. Upper bounds of $S_1(\lambda)$ and $S_2(\lambda)$

To obtain a convenient upper bound of these terms we first remark that

$$\left| h_r \left(x_0 + iy + \frac{2i\pi\lambda}{d} \right) \right| \leq h_r(x_0). \quad (7.1)$$

Let $j \in \mathbb{N}$. To prove (7.1) it is enough to prove that each $T_j \leq 1$ with

$$T_j := \frac{|1 + \exp(-(r+jd)(x_0 + iy + \frac{2i\pi\lambda}{d}))|}{|1 + \exp(-(r+jd)x_0)|}.$$

By a simple computation we have

$$T_j^2 = 1 - \frac{4 \exp(-x_0(r+jd)) \sin^2\left(\frac{y}{2}(r+jd) + \frac{\pi\lambda r}{d}\right)}{(1 + \exp(-x_0(r+jd)))^2} \leq 1.$$

By Lemma 2.1 we have

$$\begin{aligned} |S_2(\lambda)| &\leq \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in \mathcal{D}^c} h_r(x_0) \right\} \\ &\quad \times \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0) \exp\left(\frac{|\mathcal{D}|(-1 + o(1))}{4dx_0}\right). \end{aligned}$$

By Lemma 3.1 we also have

$$\begin{aligned} |S_1(\lambda)| &\leq \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in \mathcal{D}^c} h_r(x_0) \right\} \\ &\quad \times \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0) \exp\left(-\frac{|\mathcal{D}|\sqrt{3}n^{2\varepsilon/3}}{27\pi^5 d}\right). \end{aligned}$$

If $|\mathcal{D}|$ is small, *i.e.*, if $|\mathcal{D}| \leq dn^{-\varepsilon/3}$ this last estimate for $S_1(\lambda)$ is not sufficient. However, by Lemma 3.1 (i), the contribution of the range $n^{-5/8+\varepsilon} \leq |y| \leq 3\pi x_0$ to $S_1(\lambda)$ is

$$\leq \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in \mathcal{D}^c} h_r(x_0) \right\} \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0) \exp\left(-\frac{|\mathcal{D}|\sqrt{3}n^{1/4+2\varepsilon}}{27\pi^5 d}\right).$$

Thus it remains to handle the case $|\mathcal{D}| \leq dn^{-\varepsilon/3}$ in the range $y_1 \leq |y| < n^{-5/8+\varepsilon}$.

• First we study the case $\lambda = 0$. We use a similar argument as in the proof of Lemma 3.1.

For any $1 \leq r \leq d$, let J_r denote the set of the integers j such that

$$\frac{1}{2x_0} \leq r + jd \leq \frac{1}{x_0}. \quad (7.2)$$

Then for $j \in J_r$ we have

$$\frac{\exp(-x_0(r + jd))}{(1 + \exp(-x_0(r + jd)))^2} \geq \frac{1}{e(1 + e^{-1/2})^2}. \quad (7.3)$$

This gives for the correspondent T_j :

$$T_j^2 \leq 1 - \frac{4}{e(1 + e^{-1/2})^2} \sin^2 \left(\frac{y}{2}(r + jd) \right).$$

Next, quite like in the proof of Lemma 3.1, we have for n large enough

$$\begin{aligned} |h_r(x_0 + iy)| &\leq h_r(x_0) \left(1 - \frac{4}{e(1 + e^{-1/2})^2} \frac{y_1^2}{4\pi^2 x_0^2} \right)^{|J_r|/2} \\ &\leq h_r(x_0) \exp \left(- \frac{y_1^2}{48\pi^2 d x_0^3} \right) \leq h_r(x_0) \exp \left(- \frac{\sqrt{3}n^{2\varepsilon/3}}{2\pi^5 d} \right). \end{aligned}$$

This upper bound combined with Lemma 3.1 is sufficient to obtain

$$|S_1(0)| \leq \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in \mathcal{D}^c} h_r(x_0) \right\} \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0) \exp \left(- \frac{\sqrt{3}n^{2\varepsilon/3}}{27\pi^5} \right).$$

• Now we suppose that $\lambda \neq 0$. We write $\frac{\lambda}{d} = \frac{\lambda'}{d'}$ with $(\lambda', d') = 1$ and $d' > 0$.

First we suppose that $d' \geq n^{\varepsilon/4}$. Since $(\lambda', d') = 1$, there are $\frac{d'}{4} + O(1)$ integers $r_0 \in \{1, \dots, d'\}$ such that $\frac{\lambda' r_0}{d'} \bmod 1 \in [\frac{1}{4}, \frac{1}{2}]$. Thus there are $\frac{d}{4} + O(\frac{d}{d'})$ integers $r \in \{1, \dots, d\}$ such that $\frac{\lambda' r}{d'} \bmod 1 \in [\frac{1}{4}, \frac{1}{2}]$ (again for n large enough).

Since $|\mathcal{D}| \leq dn^{-\varepsilon/3} < \frac{d}{4} + O(\frac{d}{d'})$ for n large enough, there exists $r_1 \in \mathcal{D}^c$ such that $\frac{\lambda' r_1}{d'} \bmod 1 \in [\frac{1}{4}, \frac{1}{2}]$. For $j \in J_{r_1}$, $|(r_1 + jd)y| \ll n^{-1/8+\varepsilon}$. Thus $\sin^2 \left((r_1 + jd)\frac{y}{2} + \frac{\lambda r_1 \pi}{d} \right) \geq \sin^2 \frac{\pi}{6} = \frac{1}{4}$.

This gives

$$\prod_{j \in J_{r_1}} |T_j(r_1)|^2 \leq \left(\frac{11}{12} \right)^{|J_{r_1}|} \leq \exp \left(- \frac{1}{30dx_0} \right),$$

which is a sufficient upper bound.

Now we suppose that $2 \leq d' < n^{\varepsilon/4}$. There are $d - \frac{d}{d'}$ integers r such that $d' \nmid r$. For these integers r and $j \in J_r$, we have

$$\sin^2 \left((r + jd) \frac{y}{2} + \frac{\pi \lambda r}{d} \right) \geq \sin^2 \left(\frac{\pi}{2d'} \right) \geq \frac{1}{d^2}.$$

Since $|\mathcal{D}| \leq dn^{-\varepsilon/3}$, there are at least $d/3$ such integers $r \in \mathcal{D}^c$ such that $d' \nmid r$.

Thus we have:

$$\prod_{r \in \mathcal{D}^c} \frac{|h_r(x_0 + iy + \frac{2i\pi\lambda}{d})|}{h_r(x_0)} \leq \prod_{\substack{r \in \mathcal{D}^c \\ r \not\equiv 0 \pmod{d'}}} \prod_{j \in J_r} \left(1 - \frac{1}{6d^2} \right) \leq \exp \left(-\frac{d}{48d^3 x_0} \right),$$

which is a sufficient upper bound when $d \leq n^{1/4-2\varepsilon}$.

8. The terms $S_0(\lambda)$ for $\lambda \neq 0$

We have to consider the integrals

$$\begin{aligned} S_0(\lambda) &= \frac{1}{2\pi} \int_{|y| \leq y_1} \left\{ \prod_{r \in \mathcal{D}^c} h_r \left(x_0 + iy + \frac{2i\pi\lambda}{d} \right) \right\} \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(d(x_0 + iy)) \right\} \\ &\quad \times \exp \left((n - R_{\mathcal{D}} - Q_{\mathcal{D}}) \left(x_0 + iy + \frac{2i\pi\lambda}{d} \right) \right) dy. \end{aligned}$$

• First we suppose that there exists $r \in \mathcal{D}^c$ such that $\lambda r \not\equiv 0 \pmod{d}$. In the previous section we remarked that

$$\begin{aligned} &\left| h_r \left(x_0 + iy + \frac{2i\pi\lambda}{d} \right) \right| \\ &= h_r(x_0) \prod_{j=0}^{\infty} \left(1 - \frac{4 \exp(-x_0(r + jd)) \sin^2 \left(\frac{y}{2}(r + jd) + \frac{\pi \lambda r}{d} \right)}{(1 + \exp(-x_0(r + jd)))^2} \right)^{1/2}. \end{aligned}$$

We work again with the sets J_r defined by (7.2) in the previous section. For $j \in J_r$, $\lambda r \equiv a \pmod{d}$, $1 \leq |a| \leq d/2$ and $|y| \leq y_1$ we have:

$$\begin{aligned} &\sin^2 \left(\frac{y}{2}(r + jd) + \frac{\pi \lambda r}{d} \right) \\ &= \sin^2 \left(\frac{\pi |a|}{d} \pm \frac{|y|}{2}(r + jd) \right) \geq \sin^2 \left(\frac{\pi |a|}{d} - \frac{y_1}{x_0} \right) \geq \sin^2 \left(\frac{\pi}{d} - \frac{y_1}{x_0} \right) \geq \frac{3}{d^2}, \end{aligned}$$

for n large enough (we recall that $d \leq n^{1/4-2\varepsilon}$).

Since $|J_r| = \frac{1}{2dx_0} + O(1) \geq \frac{1}{3dx_0}$, we have:

$$\begin{aligned} \left| h_r \left(x_0 + iy + \frac{2i\pi\lambda}{d} \right) \right| &\leq h_r(x_0) \prod_{j \in J_r} \left(1 - \frac{12}{ed^2(1+e^{-1/2})^2} \right)^{1/2} \\ &\leq h_r(x_0) \exp \left(\frac{1}{6dx_0} \log \left(1 - \frac{12}{ed^2(1+e^{-1/2})^2} \right) \right) \leq h_r(x_0) \exp \left(-\frac{2}{ed^3x_0(1+e^{-1/2})^2} \right). \end{aligned}$$

Thus if there exists $r_0 \in \mathcal{D}^c$ such that $\lambda r_0 \not\equiv 0 \pmod{d}$ then

$$\begin{aligned} |S_0(\lambda)| &\leq \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in \mathcal{D}^c} h_r(x_0) \right\} \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0) \\ &\quad \times \exp \left(-\frac{2}{ed^3x_0(1+e^{-1/2})^2} \right). \quad (8.1) \end{aligned}$$

This upper bound is sufficient only for $d \leq n^{1/6-\varepsilon}$.

• We suppose now that

$$\lambda r \equiv 0 \pmod{d} \quad \text{for all } r \in \mathcal{D}^c. \quad (8.2)$$

If $\mathcal{D}^c = \{d\}$ then by (1.1), $n - R_{\mathcal{D}} - Q_{\mathcal{D}} \equiv 0 \pmod{d}$ and $S_0(\lambda) = S_0(0)$ for all λ .

In the general case, if (8.2) holds then we must have $\frac{d}{(\lambda, d)} |r$. Thus $\frac{d}{(\lambda, d)} |\delta$ and again by (1.1), we have $n - R_{\mathcal{D}} - Q_{\mathcal{D}} \equiv 0 \pmod{\frac{d}{(\lambda, d)}}$. Thus $\exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}}) \frac{2i\pi\lambda}{d}) = 1$ and $S_0(\lambda) = S_0(0)$.

For $\mathcal{D}^c$ given, there exists δ integers λ modulo d such that $r\lambda \equiv 0 \pmod{d}$ for all $r \in \mathcal{D}^c$.

We summarize these observations in the following lemma.

Lemma 8.1. *For $d \leq n^{1/6-\varepsilon}$ we have:*

$$\sum_{-\frac{d}{2} < \lambda \leq \frac{d}{2}} S_0(\lambda) = \delta S_0(0)(1 + o(1)).$$

9. The function h_r^{-1} in the range $|y| \leq y_1$

The generating function associated to unequal partitions is $h(z) = \prod_{j=1}^{\infty} (1+z^j)$.

For $S_0(0)$ it remains to handle the integral

$$\frac{1}{2\pi} \int_{|y| \leq y_1} h(\exp(-(x_0 + iy))) \left\{ \prod_{r \in \mathcal{D}} \frac{g_{N_r}(d(x_0 + iy))}{h_r(x_0 + iy)} \right\} \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})(x_0 + iy)) dy.$$

In this section we will state an asymptotic estimate related to h_r^{-1} in the range $|y| \leq y_1$.

The following method looks like the general method of MEINARDUS [7] for studying generating functions associated to partitions (this method is presented in details in the book of ANDREWS [1]). In fact we were also inspired by the chapter on the application of saddle point method to the partitions function in the Master course of TENENBAUM [8].

In Lemma 4.1, we obtained an estimation of $g_k(dw)$ in function of $f(dw)$. This leads us to try in fact to obtain an estimation of $U_r(z) := f(dz)h_r^{-1}(z)$ for $|y| \leq y_1$ instead of $h_r^{-1}(z)$. This change will make some computations easier. Thus we consider

$$U_r(z) = \left\{ \prod_{j=1}^{\infty} (1 - \exp(-jdz))^{-1} \right\} \left\{ \prod_{j=0}^{\infty} (1 + \exp(-(r+jd)z))^{-1} \right\}.$$

The main result of this section is the following lemma.

Lemma 9.1. *Let $\eta > 0$. For $|y| \leq y_1$ and $1 \leq r \leq d$, we have*

$$U_r(z) = \exp \left(\frac{\pi^2}{12dz} + \left(\frac{r}{d} - 1 \right) \log 2 + \frac{1}{2} \log \left(\frac{dz}{\pi} \right) + O(d^{1+\eta} r^{-\eta} |z|) + O(d^{-1} n^{-2\varepsilon}) \right).$$

In the next section we will apply this lemma with $\eta > 0$ small enough such that $d^{2+\eta}|z| \leq n^{-\varepsilon}$.

PROOF. If $r = d$, there are quite no work to do since

$$U_d(z) = \prod_{j=1}^{\infty} (1 + \exp(-jdz))^{-1} (1 - \exp(-jdz))^{-1} = \prod_{j=1}^{\infty} (1 - \exp(-2jdz))^{-1} = f(2dz).$$

Thus

$$U_d(z) = \exp \left(\frac{\pi^2}{12dz} + \frac{1}{2} \log \left(\frac{2dz}{2\pi} \right) + O(d|z|) \right).$$

Now we suppose that $r \neq d$. We prefer to work with

$$\begin{aligned} \tilde{U}_r(z) &:= \prod_{j=1}^{\infty} (1 - \exp(-jdz))^{-1} (1 + \exp(-(r+jd)z))^{-1} \\ &= U_r(z)(1 + \exp(-rz)) = U_r(z)u_r(z). \end{aligned}$$

We easily see that

$$u_r(z) = (2 + O(r|z|)). \quad (9.1)$$

Let $F(v, s) = \sum_{k=1}^{\infty} \frac{\exp(-kv)}{k^s}$. If $\Re v > 0$, then $s \mapsto F(v, s)$ is analytic on $\mathbb{C}$.

The first step of the proof is the following result.

Lemma 9.2. *For $r \neq d$, $\eta > 0$, $z = x_0 + iy$ with $|y| \leq y_1$, we have*

$$\begin{aligned} \log(\tilde{U}_r(z)) &= \frac{1}{dz} \left(\frac{\pi^2}{6} + F(rz + i\pi, 2) \right) + \frac{1}{2} \log \left(\frac{dz}{2\pi} \right) \\ &\quad - \frac{1}{2} F(rz + i\pi, 1) + \frac{dz}{12} \left(-\frac{1}{2} + F(z + i\pi, 0) \right) + O(|z|r^{-\eta}d^{1+\eta}). \end{aligned}$$

The last term $\frac{dz}{12} \left(-\frac{1}{2} + F(z + i\pi, 0) \right)$ in the above formula could be removed because it is $O(|z|r^{-\eta}d^{1+\eta})$. The following proof uses Mellin formula and looks like the general method of Meinardus for studying generating functions associated to partitions.

We begin by some standard manipulations

$$\begin{aligned} \tilde{U}_r(z) &= \exp \left\{ - \sum_{j=1}^{\infty} (\log(1 + \exp(-(r + jd)z)) + \log(1 - \exp(-jdz))) \right\} \\ &= \exp \left\{ \sum_{j,k \geq 1} \frac{\exp(-jkdz)}{k} ((-1)^k \exp(-krz) + 1) \right\}. \end{aligned}$$

Let us write

$$\log(\tilde{U}_r(z)) = \sum_{m=1}^{\infty} \frac{\beta(m)}{m} \exp(-mdz),$$

with

$$\beta(m) = \sum_{jk=m} j((-1)^k \exp(-krz) + 1).$$

By the Mellin transform formula we have:

$$\log(\tilde{U}_r(z)) = \sum_{m \geq 1} \frac{\beta(m)}{m} \cdot \frac{1}{2i\pi} \int_{2-i\infty}^{2+i\infty} \frac{\Gamma(s)}{(mdz)^s} ds.$$

But the Dirichlet series is

$$\begin{aligned} \sum_{m \geq 1} \frac{\beta(m)}{m^{s+1}} &= \sum_{j \geq 1} \frac{1}{j^s} \sum_{k \geq 1} \frac{(1 + (-1)^k \exp(-krz))}{k^{s+1}} \\ &= \zeta(s)(\zeta(s+1) + F(rz + i\pi, s+1)). \end{aligned}$$

Thus we have

$$\log(\tilde{U}_r(z)) = \frac{1}{2i\pi} \int_{2-i\infty}^{2+i\infty} \frac{\Gamma(s)\zeta(s)(\zeta(s+1) + F(rz + i\pi, s+1))}{(dz)^s} ds.$$

Let $\eta \in]0, 1[$. We move the integral until the line $\Re s = -1 - \eta$. This gives

$$\begin{aligned} \log(\tilde{U}_r(z)) &= \frac{1}{2i\pi} \int_{-1-\eta-i\infty}^{-1-\eta+i\infty} \frac{\Gamma(s)\zeta(s)(\zeta(s+1) + F(rz + i\pi, s+1))}{(dz)^s} ds \\ &\quad + \text{Res}(1) + \text{Res}(0) + \text{Res}(-1) + E, \end{aligned}$$

where E is the error term arising from the horizontal branches.

For $\Re s \in [-1 - \eta, 2]$ we have:

$$\begin{aligned} F(zr + i\pi, s+1) &= \sum_{\ell \geq 1} e^{-2\ell zr} \left(\frac{1}{(2\ell)^{s+1}} - \frac{e^{rz}}{(2\ell-1)^{s+1}} \right) \\ &= \sum_{\ell \geq 1} e^{-2\ell zr} \left(\frac{1}{(2\ell)^{s+1}} - \frac{1}{(2\ell-1)^{s+1}} \right) + O\left(r|z| \sum_{\ell \geq 1} \frac{e^{-2\ell rx_0}}{\ell^{1+\Re s}}\right). \end{aligned}$$

When ℓ is small, the term $e^{-\ell rx_0}$ is $O(1)$. Thus for $-1 - \eta \leq \Re s \leq 2$, we have :

$$\sum_{\ell \geq 1} \frac{e^{-2\ell rx_0}}{\ell^{1+\Re s}} \ll \sum_{\ell \leq 100/rx_0} \ell^\eta + \sum_{\ell > 100/rx_0} \ell^\eta e^{-2\ell rx_0} \ll (rx_0)^{-1-\eta},$$

since for the second sum we have

$$\sum_{\ell > 100/rx_0} \ell^\eta e^{-2\ell rx_0} \ll \sum_{\ell > 100/rx_0} \ell e^{-2\ell rx_0} (rx_0)^{1-\eta} \ll \frac{(rx_0)^{1-\eta}}{(1 - \exp(-2rx_0))^2}.$$

For the other term we have

$$\left| \sum_{\ell \geq 1} e^{-2\ell rz} \left(\frac{1}{(2\ell)^{1+s}} - \frac{1}{(2\ell-1)^{1+s}} \right) \right| \ll (1+|s|) \sum_{\ell \geq 1} \frac{e^{-2\ell rx_0}}{\ell^{1-\eta}} \ll \frac{1+|s|}{(rx_0)^\eta}.$$

Thus for $-1 - \eta \leq \Re s \leq 2$, we have

$$|F(zr + i\pi, s+1)| \ll |s|(rx_0)^{-\eta} + r|z|(rx_0)^{-1-\eta}. \quad (9.2)$$

We will also use the following classical results for the functions ζ and Γ in vertical strips:

– there exists $H > 0$ such that $|\zeta(s)| \ll |\Im s|^H$ for $-3 \leq \Re s \leq 3$ (in fact more generally for $\Re s \in [\sigma_1, \sigma_2]$) and $|\Im s| \geq 1$ (cf. [2] Theorem 12.23 p. 270 for a more precise formulation);

– for $-3 \leq \Re s \leq 3$ (or $\Re s \in [\sigma_1, \sigma_2]$) and $|\Im s| \rightarrow +\infty$, ([9] Corollaire II.0.13 p. 182) we have

$$\Gamma(s) = (1 + O(|\Im s|^{-1})) \sqrt{2\pi} |\Im s|^{\Re s - \frac{1}{2}} e^{-\pi |\Im s|/2} e^{i\alpha(s)},$$

with $\alpha(s) = (\Im s) \log |\Im s| - \Im s + \frac{1}{2}\pi(\Re s - \frac{1}{2}) \operatorname{sgn}(\Im s)$.

With this two formulae and by (9.2) we easily see that

$$\lim_{T \rightarrow +\infty} \int_{-1-\eta \pm iT}^{2 \pm iT} \frac{\Gamma(s) \zeta(s) (\zeta(s+1) + F(rz + i\pi, s+1))}{(dz)^s} ds = 0, \quad (9.3)$$

and

$$\left| \int_{\Re s = -1-\eta} \frac{\Gamma(s) \zeta(s) (\zeta(s+1) + F(rz + i\pi, s+1))}{(dz)^s} ds \right| \ll r^{-\eta} d^{1+\eta} |z|. \quad (9.4)$$

Now we compute the different residues:

We have

$$\operatorname{Res}(1) = \frac{\Gamma(1) (\zeta(2) + F(rz + i\pi, 2))}{dz} = \frac{1}{dz} \left(\frac{\pi^2}{6} + F(rz + i\pi, 2) \right). \quad (9.5)$$

In $s = 0$, we have two poles, one from Γ , the other from the function $s \mapsto \zeta(s+1)$.

We use the well known results $\Gamma'(1) = -\gamma$, $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$.

Thus $\operatorname{Res}(0)$ is the coefficient in s^{-1} in the following formula:

$$\begin{aligned} & \left(\frac{1 - \gamma s + O(|s|^2)}{s} \right) (\zeta(0) + s\zeta'(0) + O(|s|^2)) \\ & \times \left[\frac{1}{s} + \gamma + O(|s|) + F(rz + i\pi, 1) \right] (1 - s \log(dz) + O(|s|^2)). \end{aligned} \quad (9.6)$$

Since $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\frac{1}{2} \log(2\pi)$, we find

$$\operatorname{Res}(0) = \frac{1}{2} \left(\log \left(\frac{dz}{2\pi} \right) - F(rz + i\pi, 1) \right). \quad (9.7)$$

In $s = -1$, Γ has a simple pole with residue -1 thus we have:

$$\operatorname{Res}(-1) = -\zeta(-1) (\zeta(0) + F(rz + i\pi, 0)) dz.$$

Since $\zeta(-1) = -\frac{1}{12}$, we obtain

$$\operatorname{Res}(-1) = \frac{dz}{12} \left(-\frac{1}{2} + F(rz + i\pi, 0) \right). \quad (9.8)$$

Formulae (9.5), (9.7), (9.8), (9.3), (9.4) end the proof of Lemma 9.2. $\square$

Now we have to study the terms $F(*, s)$ with $s = 0, 1, 2$. First we have

$$F(rz + i\pi, 0) = \sum_{m=1}^{\infty} (-1)^m \exp(-mrz) = -\frac{1}{1 + \exp(rz)} = -\frac{1}{2} + O(rn^{-1/2}),$$

for $|y| \leq y_1$. For the contribution of $F(rz + i\pi, 1)$ we have

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m} \exp(-mrz) = \frac{1}{2} \log \left(1 + \exp(-rz) \right) = \frac{1}{2} \log 2 + O(r|z|). \quad (9.9)$$

The contribution of $F(rz + i\pi, 2)$ is more difficult to handle because of the factor $(dz)^{-1}$. We will prove the following lemma

Lemma 9.3. *For $|y| \leq y_1$, we have for $d \leq n^{1/4-2\varepsilon}$*

$$\frac{1}{dz} F(rz + i\pi, 2) = -\frac{\pi^2}{12dz} + \frac{r}{d} \log 2 + O(d^{-1}n^{-2\varepsilon}).$$

PROOF. Let M be an odd integer. We have

$$\frac{1}{dz} F(rz + i\pi, 2) = \frac{1}{dz} \sum_{m=1}^M \frac{(-1)^m \exp(-mrz)}{m^2} + \sum_{m=M+1}^{+\infty} \frac{(-1)^m \exp(-mrz)}{dz m^2} = T_1 + T_2,$$

say. We begin with T_2 . Since the sum is absolutely convergent, we can regroup the terms $m = 2\ell$ with the terms $m = 2\ell + 1$

$$\begin{aligned} |T_2| &\leq \frac{1}{d|z|} \sum_{\ell=\frac{M+1}{2}}^{+\infty} \exp(-2\ell x_0) \left| \frac{1}{4\ell^2} - \frac{\exp(-rz)}{(2\ell+1)^2} \right| \leq \frac{1}{d|z|} \sum_{\ell=\frac{M+1}{2}}^{+\infty} \left| \frac{1}{4\ell^2} - \frac{1+O(r|z|)}{(2\ell+1)^2} \right| \\ &\leq \frac{1}{d|z|} \sum_{\ell=\frac{M+1}{2}}^{+\infty} \left(\frac{4\ell+1}{4\ell^2(2\ell+1)^2} + O\left(\frac{r|z|}{(2\ell+1)^2}\right) \right) \ll \frac{1}{d|z|M^2} + \frac{r}{dM}. \end{aligned}$$

This leads us to choose M the smallest odd integer $\geq n^{1/4+\varepsilon}$. For T_1 we use the fact that $\exp(-mrz)$ is near 1 if M is not too large. Let $J \in \mathbb{N}$ to be specified later:

$$T_1 = \frac{1}{dz} \sum_{m=1}^M \frac{(-1)^m}{m^2} (1 - mrz + \sum_{j=2}^{J-1} \frac{(-1)^j m^j r^j z^j}{j!} + O(m^J r^J |z|^J)).$$

By the same type of argument as for T_2 (which are standard manipulations on alternating series) we show that

$$\frac{1}{dz} \sum_{m=1}^M \frac{(-1)^m}{m^2} = \frac{1}{dz} \sum_{m=1}^{+\infty} \frac{(-1)^m}{m^2} + O\left(\frac{1}{d|z|M^2}\right) = -\frac{\pi^2}{12dz} + O\left(\frac{1}{d|z|M^2}\right),$$

and

$$\frac{r}{d} \sum_{m=1}^M \frac{(-1)^m}{m} = \frac{r}{d} \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} + O\left(\frac{r}{dM}\right) = -\frac{r}{d} \log 2 + O\left(\frac{r}{dM}\right).$$

Next, for each $2 \leq j \leq J-1$ we have (recall that $r \leq d \leq n^{1/4-2\varepsilon}$)

$$\frac{r^j |z|^{j-1}}{d} \left| \sum_{m=1}^M (-1)^m m^{j-2} \right| \ll \frac{r^j |z|^{j-1} M^{j-2}}{d} \ll \frac{n^{-(2+j)\varepsilon}}{d}.$$

Finally for the last error term we have:

$$r^J |z|^{J-1} d^{-1} \sum_{m=1}^M m^{J-2} \ll (M|z|)^{J-1} r^J d^{-1} \ll n^{\frac{1}{4}-J\varepsilon} d^{-1},$$

which is small enough for $J = \lceil \frac{5}{\varepsilon} \rceil$.

This ends the proof of Lemma 9.3.

If we insert (9.9) and Lemma 9.3 in Lemma 9.2 and don't forget (9.1) we obtain Lemma 9.1 in the case $r \neq d$. $\square$

10. The term $S_0(0)$, end of the proof of Theorem 1.1

Recall that

$$h(\exp(-z)) = \exp\left(\frac{\pi^2}{12z} - \frac{1}{2} \log 2 + O(|z|)\right),$$

if $z \rightarrow 0$ with $|\arg z| \leq \kappa < \pi/2$. By Lemma 9.1 we have for $|y| \leq y_1$:

$$\begin{aligned} h(\exp(-z)) \prod_{r \in \mathcal{D}} U_r(z) &= \exp\left(\frac{\pi^2}{12z} \left(1 + \frac{|\mathcal{D}|}{d}\right) + \frac{|\mathcal{D}|}{2} \log z - \frac{\log 2}{2}\right. \\ &\quad \left.+ \sum_{r \in \mathcal{D}} \left(\frac{r}{d} - 1\right) \log 2 + \frac{|\mathcal{D}|}{2} \log\left(\frac{d}{\pi}\right) + O(n^{-\varepsilon})\right). \end{aligned}$$

As in the case $\mathcal{D} = \{1, \dots, d\}$ we insert above the two formulae:

$$\frac{1}{z} = \frac{1}{x_0} - \frac{iy}{x_0^2} - \frac{y^2}{x_0^3} + O\left(\frac{|y|^3}{x_0^4}\right) \quad \text{and} \quad \log z = \log x_0 + O\left(\frac{|y|}{x_0}\right),$$

and we apply Lemma 4.1:

$$S_0(0) = \frac{1}{2\pi} \int_{-y_1}^{y_1} \exp(C_{\mathcal{D}} + iy\tilde{B}_{\mathcal{D}} - y^2 A_{\mathcal{D}}) dy,$$

with

$$\begin{aligned} C_{\mathcal{D}} = & \frac{\pi^2}{12x_0} \left(1 + \frac{|\mathcal{D}|}{d}\right) + \frac{|\mathcal{D}|}{2} \log x_0 - \frac{\log 2}{2} + \sum_{r \in \mathcal{D}} \left(\frac{r}{d} - 1\right) \log 2 + \frac{|\mathcal{D}|}{2} \log \left(\frac{d}{\pi}\right) \\ & - \frac{C_2 |\mathcal{D}|}{dx_0} + \frac{|\mathcal{D}| \log 2}{2} + \sum_{r \in \mathcal{D}} (N_r - k_0) \log 2 - \frac{dx_0}{2} \sum_{r \in \mathcal{D}} (N_r - k_0)^2 \\ & + (n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0 + O(n^{-\varepsilon}), \end{aligned} \quad (10.1)$$

$$\begin{aligned} \tilde{B}_{\mathcal{D}} = & -\frac{\pi^2}{12x_0^2} \left(1 + \frac{|\mathcal{D}|}{d}\right) + \frac{C_2 |\mathcal{D}|}{dx_0^2} + \frac{|\mathcal{D}| k_0 \log 2}{x_0} - dk_0 \sum_{r \in \mathcal{D}} (N_r - k_0) \\ & + n - R_{\mathcal{D}} - Q_{\mathcal{D}}, \end{aligned} \quad (10.2)$$

$$A_{\mathcal{D}} = \frac{\pi^2}{12x_0^3} \left(1 + \frac{|\mathcal{D}|}{d}\right) - \frac{C_2 |\mathcal{D}|}{dx_0^3} - \frac{|\mathcal{D}| k_0 \log 2}{x_0^2} - \frac{d|\mathcal{D}| k_0^2}{2x_0}. \quad (10.3)$$

Since $k_0 dx_0 = \log 2$, and $C_2 = \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}$, $A_{\mathcal{D}}$, $\tilde{B}_{\mathcal{D}}$ and $C_{\mathcal{D}}$ may be simplified:

$$A_{\mathcal{D}} = \frac{\pi^2}{12x_0^3} - \frac{|\mathcal{D}|(\log 2)^2}{dx_0^3} = \frac{2\sqrt{3}n^{\frac{3}{2}}}{\pi} \left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right).$$

Recall that $k_0 = \frac{2\sqrt{3n} \log 2}{\pi d}$ and $|N_r - k_0| \leq \frac{n^{\frac{1}{4}} \sqrt{\log n}}{d^{1/3} |\mathcal{D}|^{2/3} w(n)}$.

By Lemma 2.2 of [6], for $|y| \leq y_1$, we have

$$y\tilde{B}_{\mathcal{D}} = -2ydk_0 \sum_{r \in \mathcal{D}} (N_r - k_0) + O(n^{-\varepsilon/3}) = yB_{\mathcal{D}} + O(n^{-\varepsilon/3}),$$

say.

We end the computations as in the case $\mathcal{D} = \{1, \dots, d\}$. Finally we obtain

$$\Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) = (1 + o(1)) \frac{\delta}{2\pi} \sqrt{\frac{\pi}{A_{\mathcal{D}}}} \exp\left(C_{\mathcal{D}} - \frac{B_{\mathcal{D}}^2}{4A_{\mathcal{D}}}\right). \quad (10.4)$$

To simplify $C_{\mathcal{D}}$ we need some more precise estimations of $R_{\mathcal{D}}$ and $Q_{\mathcal{D}}$:

$$\begin{aligned} x_0 R_{\mathcal{D}} &= x_0 k_0 \sum_{r \in \mathcal{D}} r + O(d^{2/3} |\mathcal{D}|^{1/3} n^{-1/4+\varepsilon}), \\ Q_{\mathcal{D}} x_0 &= \frac{dx_0}{2} \sum_{r \in \mathcal{D}} (N_r - k_0)^2 + k_0 dx_0 \sum_{r \in \mathcal{D}} (N_r - k_0) \\ &\quad + \frac{x_0 d k_0^2 |\mathcal{D}|}{2} - \frac{d |\mathcal{D}| k_0 x_0}{2} + O(d n^{-1/4+\varepsilon}). \end{aligned}$$

It remains to insert these different formulae in (10.4) to finish the proof of Theorem 1.1.

11. Local stability of $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}})$

In this section we settle a result analogous to Corollary 9.1 of [5]. If $\mathcal{R}_{\mathcal{D}} = \{N_r : r \in \mathcal{D}\}$ and $\mathcal{R}_{\mathcal{D}}^* = \{N_r^* : r \in \mathcal{D}\}$ are two sets and such that N_r^* is near N_r on average then in the estimation of $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}})$ given by Theorem 1.1, we may replace the N_r by N_r^* . Like in [5] this corollary will be useful for the proofs of the different corollaries announced in the introduction of [6].

Corollary 11.1. *Let $0 < \varepsilon < 10^{-2}$, $n \geq n_0$, $d^3 |\mathcal{D}| \leq n^{1/2-3\varepsilon}$ and two sets $\mathcal{R}_{\mathcal{D}} = \{N_r : r \in \mathcal{D}\} \in \mathbb{Z}^{|\mathcal{D}|}$, $\mathcal{R}_{\mathcal{D}}^* = \{N_r^* : r \in \mathcal{D}\} \in \mathbb{R}^{|\mathcal{D}|}$ such that*

- (i) (1.1) is satisfied for $\mathcal{R}_{\mathcal{D}}$;
- (ii) $|N_r - k_0| \leq \frac{n^{1/4} \sqrt{\log n}}{d^{1/3} |\mathcal{D}|^{2/3} w(n)}$ for all $r \in \mathcal{D}$ where $w(n)$ is a non-decreasing function such that $\lim_{u \rightarrow \infty} w(u) = \infty$;
- (iii) $\sum_{r \in \mathcal{D}} |N_r - N_r^*| \leq \delta + |\mathcal{D}| - 1$, $\sum_{r \in \mathcal{D}} |N_r - N_r^*|^2 \leq \delta^2 + |\mathcal{D}| - 1$.

Then we have

$$\begin{aligned} \Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) &= q(n) \frac{\delta(1+o(1))}{\sqrt{1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}} \left(\frac{d}{2\sqrt{3n}} \right)^{|\mathcal{D}|/2} \\ &\times \exp \left(- \frac{2\sqrt{3}(\log 2)^2}{\pi \left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right) \sqrt{n}} \left(\sum_{r \in \mathcal{D}} (N_r^* - k_0) \right)^2 - \frac{\pi d}{2\sqrt{3n}} \sum_{r \in \mathcal{D}} (N_r^* - k_0)^2 \right). \end{aligned}$$

PROOF. By (iii), $\sum_{r \in \mathcal{D}} |N_r - N_r^*| \leq 2d$ and we have

$$\left| \left(\sum_{r \in \mathcal{D}} (N_r - k_0) \right)^2 - \left(\sum_{r \in \mathcal{D}} (N_r^* - k_0) \right)^2 \right|$$

$$\begin{aligned}
&\leq \left(\sum_{r \in \mathcal{D}} |N_r - N_r^*| \right) \left(2 \sum_{r \in \mathcal{D}} |N_r - k_0| + \sum_{r \in \mathcal{D}} |N_r - N_r^*| \right) \\
&= O \left(d^{2/3} |\mathcal{D}|^{1/3} \frac{n^{1/4} \sqrt{\log n}}{w(n)} + d^2 \right) = o(n^{1/2}).
\end{aligned}$$

Similarly we have using also $\delta \leq |\mathcal{D}| + 1$ (since $\delta \leq \min_{a \in \mathcal{D}^c} a$ if $\mathcal{D}^c \neq \emptyset$)

$$\begin{aligned}
\left| \sum_{r \in \mathcal{D}} (N_r - k_0)^2 - \sum_{r \in \mathcal{D}} (N_r^* - k_0)^2 \right| &\leq \sum_{r \in \mathcal{D}} |N_r - N_r^*| (2|N_r - k_0| + |N_r - N_r^*|) \\
&\ll (\delta + |\mathcal{D}| - 1) |\mathcal{D}|^{-2/3} d^{-1/3} \frac{n^{1/4} \sqrt{\log n}}{w(n)} + \delta^2 + |\mathcal{D}| \\
&\ll |\mathcal{D}|^{1/3} d^{-1/3} \frac{n^{1/4} \sqrt{\log n}}{w(n)} + \delta^2 + |\mathcal{D}| = o \left(\frac{\sqrt{n}}{d} \right). \quad \square
\end{aligned}$$

This ends the proof of Corollary 11.1.

12. On the normal order of the numbers of parts: proof of Corollary 1.2 of [6]

Let $C_{\mathcal{D}} = \lceil \frac{2\sqrt{3}\log 2}{\pi} \frac{\sqrt{n}}{d^2} - \frac{n^{1/4} \sqrt{\log n}}{d^{4/3} |\mathcal{D}|^{2/3} w(n)} \rceil d$ and $D_{\mathcal{D}} = \lfloor \frac{2\sqrt{3}\log 2}{\pi} \frac{\sqrt{n}}{d^2} + \frac{n^{1/4} \sqrt{\log n}}{d^{4/3} |\mathcal{D}|^{2/3} w(n)} \rfloor d$.

To prove Corollary 1.2 of [6], it is sufficient to show that

$$S^* := \sum_{\substack{N_r \in [C_{\mathcal{D}}, D_{\mathcal{D}}[\\ r \in \mathcal{D} \\ \mathcal{R}_{\mathcal{D}} \equiv n \pmod{\delta}}} \Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) = q(n)(1 + o(1)). \quad (12.1)$$

As in [5] p. 82, we have to remove the dependence between n and the N_r given by the congruence condition modulo δ . If $1 \notin \mathcal{D}$ then $\delta = 1$, there are no difficulty and we take $N_r^* = N_r$ for all $r \in \mathcal{D}$. If $1 \in \mathcal{D}$ we write $N_1^* = \lfloor \frac{N_1}{\delta} \rfloor \delta$ and $N_r^* = N_r$ for $r \in \mathcal{D} \setminus \{1\}$. Now we will suppose that $1 \in \mathcal{D}$, the proof of the other case being similar. Next we apply Corollary 11.1 with these two sets:

$$\begin{aligned}
S^* &= (1 + o(1)) q(n) \frac{\delta}{\sqrt{1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}} \left(\frac{d}{2\sqrt{3}n} \right)^{|\mathcal{D}|/2} \\
&\times \sum_{\substack{C_{\mathcal{D}} \leq N_1 \delta < D_{\mathcal{D}} \\ N_r \in [C_{\mathcal{D}}, D_{\mathcal{D}}[\\ r \in \mathcal{D} \setminus \{1\}}} \exp \left(- \frac{2\sqrt{3}(\log 2)^2}{\pi \left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2} \right) \sqrt{n}} \left(\delta N_1 - k_0 + \sum_{r \in \mathcal{D} \setminus \{1\}} (N_r - k_0) \right)^2 \right)
\end{aligned}$$

$$- \frac{\pi d}{2\sqrt{3n}} \left((\delta N_1 - k_0)^2 + \sum_{r \in \mathcal{D} \setminus \{1\}} (N_r - k_0)^2 \right).$$

We apply again Corollary 11.1 with $N_i^* = t_i$ so that $|t_i - N_i| \leq 1$ for $i \in \mathcal{D}$, we easily see that

$$|\delta t'_1 - \delta N_1|^2 + \sum_{r \in \mathcal{D} \setminus \{1\}} (t_r - N_r)^2 \leq \delta^2 + |\mathcal{D}| - 1.$$

Thus we have (after replacing $\delta t'_1$ by t_1 in the integral):

$$\begin{aligned} S^* &= (1 + o(1))q(n) \frac{1}{\sqrt{1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}} \\ &\times \left(\frac{d}{2\sqrt{3n}} \right)^{|\mathcal{D}|/2} \int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \dots \int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \exp \left(- \frac{2\sqrt{3}(\log 2)^2}{\pi \left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right) \sqrt{n}} \left(\sum_{r \in \mathcal{D}} (t_r - k_0) \right)^2 \right. \\ &\left. - \frac{\pi d}{2\sqrt{3n}} \left(\sum_{r \in \mathcal{D}} (t_r - k_0)^2 \right) \right) \prod_{r \in \mathcal{D}} dt_r. \end{aligned}$$

Lemma 12.1. *We have*

$$\begin{aligned} S^* &= (1 + o(1))q(n) \frac{1}{\sqrt{1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}} \\ &\times \left(\frac{d}{2\sqrt{3n}} \right)^{|\mathcal{D}|/2} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp \left(- \frac{2\sqrt{3}(\log 2)^2}{\pi \left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right) \sqrt{n}} \left(\sum_{r \in \mathcal{D}} (t_r - k_0) \right)^2 \right. \\ &\left. - \frac{\pi d}{2\sqrt{3n}} \left(\sum_{r \in \mathcal{D}} (t_r - k_0)^2 \right) \right) \prod_{r \in \mathcal{D}} dt_r \\ &+ O \left(q(n) \frac{w(n)|\mathcal{D}|^{5/3}}{d^{1/6}\sqrt{\log n}} \exp \left(- \frac{\pi d^{1/3} \log n}{2\sqrt{3}|\mathcal{D}|^{4/3}w^2(n)} \right) \right). \end{aligned}$$

PROOF. We have to consider the contribution of terms of type $\int_{D_{\mathcal{D}}}^{\infty} \int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \dots \int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \dots$. Clearly it is less than

$$\int_{D_{\mathcal{D}}}^{\infty} \int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \dots \int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \exp \left(- \frac{\pi d}{2\sqrt{3n}} \left(\sum_{r \in \mathcal{D}} (t_r - k_0)^2 \right) \right) \prod_{r \in \mathcal{D}} dt_r.$$

But

$$\int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \exp \left(- \frac{\pi d}{2\sqrt{3n}} (t_r - k_0)^2 \right) dt_r \leq \int_{-\infty}^{+\infty} \exp \left(- \frac{\pi dt_r^2}{2\sqrt{3n}} \right) dt_r = \sqrt{\frac{2\sqrt{3n}}{d}},$$

and

$$\begin{aligned} \int_{D_{\mathcal{D}}}^{\infty} \exp\left(-\frac{\pi d}{2\sqrt{3n}}(t_r - k_0)^2\right) dt_r &\leq \int_{D_{\mathcal{D}}}^{\infty} \exp\left(-\frac{\pi d}{2\sqrt{3n}}(D_{\mathcal{D}} - k_0)(t_r - k_0)\right) dt_r \\ &= \frac{2\sqrt{3n}}{\pi d(D_{\mathcal{D}} - k_0)} \exp\left(-\frac{\pi d}{2\sqrt{3n}}(D_{\mathcal{D}} - k_0)^2\right). \end{aligned}$$

We have $d^{4/3}|\mathcal{D}|^{2/3} = d^{8/6}|\mathcal{D}|^{1/2}|\mathcal{D}|^{1/6} \leq d^{9/6}|\mathcal{D}|^{1/2} = (d^3|\mathcal{D}|)^{1/2} \leq n^{1/4-\varepsilon}$. Thus the contribution of some term of type $\int_{D_{\mathcal{D}}}^{\infty} \int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \dots \int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \dots$ to S^* is

$$O\left(q(n) \frac{w(n)|\mathcal{D}|^{2/3}}{d^{1/6}\sqrt{\log n}} \exp\left(-\frac{\pi d^{1/3} \log n}{2\sqrt{3}|\mathcal{D}|^{4/3}w^2(n)}\right)\right).$$

Since there are $2|\mathcal{D}|$ such error terms we obtain Lemma 12.1. It remains to compute the main term to finish the proof of Corollary 1.2 of [6]. We remark that

$$\begin{aligned} \exp\left(-\frac{2\sqrt{3}(\log 2)^2}{\pi\left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right)\sqrt{n}}\left(\sum_{r \in \mathcal{D}}(t_r - k_0)\right)^2 - \frac{\pi d}{2\sqrt{3n}}\left(\sum_{r \in \mathcal{D}}(t_r - k_0)^2\right)\right) \\ = \exp\left(-\frac{1}{2}T^t M T\right), \end{aligned} \quad (12.2)$$

with $T = \begin{pmatrix} t_1 - k_0 \\ t_2 - k_0 \\ \vdots \\ t_{|\mathcal{D}|} - k_0 \end{pmatrix} \in \mathbb{R}^{|\mathcal{D}|}$ and M is the symmetric matrix $M = (m_{ij})_{1 \leq i, j \leq |\mathcal{D}|}$ defined by

$$\begin{aligned} \frac{m_{ii}}{2} &= \frac{2\sqrt{3}(\log 2)^2}{\pi\left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right)\sqrt{n}} + \frac{\pi d}{2\sqrt{3n}} =: V + U \quad (1 \leq i \leq |\mathcal{D}|) \\ \frac{m_{ij}}{2} &= \frac{m_{ji}}{2} = \frac{2\sqrt{3}(\log 2)^2}{\pi\left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right)\sqrt{n}} =: V \quad (1 \leq i < j \leq |\mathcal{D}|). \end{aligned}$$

A classical result on determinant announces that

$$\det M = 2^{|\mathcal{D}|} U^{|\mathcal{D}|-1} (U + |\mathcal{D}|V) = \left(\frac{\pi d}{\sqrt{3n}}\right)^{|\mathcal{D}|} \left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right)^{-1}.$$

Thus the function (12.2) is proportional to the density of the law of a Gaussian vector with covariance matrix M^{-1} . We deduce that

$$\begin{aligned} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left(-\frac{2\sqrt{3}(\log 2)^2}{\pi\left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right)\sqrt{n}}\left(\sum_{r \in \mathcal{D}}(t_r - k_0)\right)^2 \right. \\ \left. - \frac{\pi d}{2\sqrt{3n}}\left(\sum_{r \in \mathcal{D}}(t_r - k_0)^2\right)\right) \prod_{r \in \mathcal{D}} dt_r = (2\pi)^{|\mathcal{D}|/2} \sqrt{\det M^{-1}}. \end{aligned}$$

This ends the proof of Corollary 1.2 of [6]. $\square$

**13. Unequal partitions with equilibrated residue classes:
proof of Corollary 1.3 of [6]**

For $1 \leq a < b \leq d$, let $E^*(a, b)$ denote the number of unequal partitions of n such that $N_a = N_b$:

$$E^*(a, b) = \sum_{\substack{N_a = N_b \\ n \equiv aN_a + bN_b \pmod{\delta}}} \Pi_d^*(n, \mathcal{R}_{\{a, b\}}) = \sum_{\substack{N_a = N_b \\ N_a \in [C, D] \\ n \equiv aN_a + bN_b \pmod{\delta}}} \Pi_d^*(n, \mathcal{R}_{\{a, b\}}) + o(q(n)),$$

by Corollary 1.2 of [6] applied with $w(n) = 2^{-2/3} \log \log n$,

$$C = \left\lfloor \frac{2\sqrt{3} \log 2}{\pi} \frac{\sqrt{n}}{d^2} - \frac{n^{1/4} \sqrt{\log n}}{d^{4/3} \log \log n} \right\rfloor d \quad \text{and} \quad D = \left\lceil \frac{2\sqrt{3} \log 2}{\pi} \frac{\sqrt{n}}{d^2} + \frac{n^{1/4} \sqrt{\log n}}{d^{4/3} \log \log n} \right\rceil d.$$

Next we apply Theorem 1.1 and Corollary 11.1. Here again we have to remove the condition $n \equiv aN_a + bN_b \pmod{\delta}$ otherwise N_a, N_b, n wouldn't be independent. If $d \geq 5$ then $\delta = 1$. For $2 \leq d \leq 4$, the problematic cases are $(a, b, d) \in \{(1, 2, 2), (1, 2, 3), (1, 3, 4)\}$. We will handle these cases later. Suppose now that $\delta = 1$.

$$\begin{aligned} E^*(a, b) &= q(n) \frac{1}{\sqrt{1 - \frac{24(\log 2)^2}{d\pi^2}}} \left(\frac{d}{2\sqrt{3n}} \right) \\ &\quad \times \int_C^D \exp \left[- \left(\frac{8\sqrt{3}(\log 2)^2}{\pi \left(1 - \frac{24(\log 2)^2}{d\pi^2}\right) \sqrt{n}} + \frac{\pi d}{\sqrt{3n}} \right) (t - k_0)^2 \right] dt + o(q(n)) \\ &= \sqrt{\frac{d}{4\sqrt{3n}}} q(n) + o(q(n)). \end{aligned}$$

As in the proof of Corollary 1.2 of [6] we show that the contributions of $\int_D^\infty \dots$ and $\int_{-\infty}^C$ are small enough. When $\delta > 1$, as explained in [5] we fix some congruence conditions on N_a and N_b and next do quite the same computations.

**14. Comparison between the number of parts in two residue classes:
proof of Corollary 1.4 of [6]**

We proceed in the same way as in the previous section. Let $\Delta \in \{0, 1\}$. We have to estimate

$$T^*(a, b) = \sum_{\substack{N_a \geq N_b + \Delta \\ n \equiv R_{\{a, b\}} \pmod{\delta}}} \Pi_d^*(n, \mathcal{R}_{\{a, b\}}).$$

We take $\Delta = 0$ to examine the unequal partitions with $N_a \geq N_b$ and $\Delta = 1$ for the unequal partitions with $N_a > N_b$. In the same way as in the proof of Corollary 1.3 of [6], we obtain

$$\begin{aligned} T^*(a, b) = & q(n) \frac{1}{\sqrt{1 - \frac{24(\log 2)^2}{d\pi^2}}} \left(\frac{d}{2\sqrt{3n}} \right) \\ & \times \int_{-\infty}^{\infty} \int_{t_b}^{\infty} \exp \left[- \frac{2\sqrt{3}(\log 2)^2}{\pi \left(1 - \frac{24(\log 2)^2}{d\pi^2}\right) \sqrt{n}} (t_a + t_b - 2k_0)^2 \right. \\ & \left. - \frac{\pi d}{2\sqrt{3n}} ((t_a - k_0)^2 + (t_b - k_0)^2) \right] dt_a dt_b + o(q(n)). \end{aligned}$$

In the same way we find a similar formula for $T^*(b, a)$. This proves that $T^*(a, b) = q(n)/2 + o(q(n))$.

15. On the d regularity of the unequal partitions: proof of Corollary 1.5 of [6]

Now we suppose that d is fixed in order to use Corollary 1.3 of [6] uniformly. We now study for $\Delta = 0$ or 1 :

$$W^*(a) := \sum_{\substack{N_1, \dots, N_d \\ n \equiv R \pmod{d} \\ N_a \geq \Delta + \max_{b \neq a} N_b}} \Pi_d^*(n, \mathcal{R}).$$

We proceed as [5] Sections 12, 13 and like the previous paragraphs of this present paper:

$$W^*(a) = o(q(n)) + \frac{d}{\sqrt{1 - \frac{12(\log 2)^2}{\pi^2}}} \left(\frac{d}{2\sqrt{3n}} \right)^{d/2} \int \dots \int_{t_a \geq \max_{b \neq a} t_b} f(t_1, \dots, t_d) dt_1 \dots dt_d,$$

with

$$f(t_1, \dots, t_d) = \exp \left\{ - \frac{2\sqrt{3} \log^2 2}{\pi \left(1 - \frac{12(\log 2)^2}{\pi^2}\right) \sqrt{n}} \left(\sum_{r=1}^d (t_r - k_0) \right)^2 - \frac{\pi d}{2\sqrt{3n}} \sum_{r=1}^d (t_r - k_0)^2 \right\}.$$

By Corollary 1.3 of [6] applied d times (it is why d is fixed), we have

$$\sum_{a=1}^d W^*(a) = q(n) + o(q(n)).$$

Since $f(t_1, \dots, t_d)$ is symmetrical the above terms are asymptotically equal:

$$W^*(a) = \left(\frac{1}{d} + o(1) \right) q(n),$$

as it was conjectured in [4] p. 334 and in the introduction of [3].

The proof of the second assertion of Corollary 1.5 of [6] is similar. We have only to replace the condition $N_a \geq \max_{b \neq a} N_b$ by $N_{\sigma(1)} \geq N_{\sigma(2)} \geq \dots \geq N_{\sigma(d)}$.

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