

On (m, n) -injectivity and coherence of rings

By QIONGLING LIU (Nanjing) and JIANLONG CHEN (Nanjing)

Abstract. Let R be a ring. For two positive integers m and n , R is said to be left (m, n) -injective if every left R -homomorphism from an n -generated submodule of ${}_R R^m$ to ${}_R R$ extends to one from ${}_R R^m$ to ${}_R R$. The ring R is called left coherent if each of its finitely generated left ideals is finitely presented. The aim of this article is to investigate (m, n) -injectivity and the coherence of the ring $R[x]/(x^k)$ ($k \geq 1$). Various sufficient and necessary conditions are obtained for $R[x]/(x^2)$ to be left (m, n) -injective and for $R[x]/(x^k)$ ($k > 2$) to be left P -injective. Moreover, it is proved that R is left coherent if and only if $R[x]/(x^k)$ is left coherent for every $k \geq 1$ if and only if $R[x]/(x^k)$ is left coherent for some $k \geq 1$.

1. Introduction

Throughout this paper, R is an associative ring with identity. For two positive integers m and n , we write $R^{m \times n}$ for the set of all $m \times n$ matrices over R , and let $R^n = R^{1 \times n}$, $R_n = R^{n \times 1}$ and $M_n(R) = R^{n \times n}$. In 2001, (m, n) -injective modules were introduced and discussed in [3]. A left R -module M is called (m, n) -injective if every left R -homomorphism from an n -generated submodule of R^m to M extends to one from R^m to M . The ring R is said to be left (m, n) -injective if ${}_R R$ is (m, n) -injective. Some related notions are recalled here. A ring R is called left FP -injective if R is left (m, n) -injective for all positive integers m and n . If R is left $(1, n)$ -injective (resp., left $(1, 1)$ -injective), then R is called left n -injective (resp., left P -injective). A ring R is called left f -injective if R is left n -injective for every positive integer n . Right versions of these injectivities are defined analogously.

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The ring $R[x]/(x^k)$ ($k \geq 1$), as an important extension of R , has been discussed in many papers (see [5], [7], [8], [9] et al). In this paper, (m, n) -injectivity and coherence of $R[x]/(x^k)$ are studied. It is well known that $R[x]/(x^2)$ is isomorphic to the trivial extension of R by R , i.e., the ring $R \ltimes R = \{(a, b) : a, b \in R\}$ with addition defined componentwise and multiplication defined by $(a, b)(c, d) = (ac, ad + bc)$. By [6], $R \ltimes R$ is right self-injective if so is R . In [4], a sufficient but not necessary condition is given for $R \ltimes R$ to be right (m, n) -injective. In Section 2, we consider the left (m, n) -injectivity of $R \ltimes R$ and derive an equivalent condition for $R \ltimes R$ to be left (m, n) -injective. Some known results on (m, n) -injective rings in [4] are obtained as corollaries. The left (m, n) -injectivity of $R[x]/(x^k)$ ($k > 2$) is investigated in Section 3. For simplicity, we only consider the left P -injectivity and a sufficient and necessary condition for $R[x]/(x^k)$ to be left P -injective is given. A similar argument can be used to obtain an analogous result about the left (m, n) -injectivity of $R[x]/(x^k)$.

Another question we considered is about the coherence of $R[x]/(x^k)$ ($k \geq 1$). A ring R is said to be left coherent if each of its finitely generated left ideals is finitely presented [1], or equivalently if $l(a)$ is a finitely generated left ideal of R for any $a \in R$ and the intersection of two finitely generated left ideals of R is again finitely generated [10]. A sufficient and necessary condition for $R \ltimes R$ to be coherent was obtained by CHEN and ZHOU in [4] where they showed that $R \ltimes R$ is left coherent if and only if so is R . In Section 4, we generalize the result by showing that R is left coherent if and only if $R[x]/(x^k)$ is left coherent for every $k \geq 1$ if and only if $R[x]/(x^k)$ is left coherent for some $k \geq 1$.

In this paper, if $S \subseteq R^{m \times n}$, we set $l_{R^m}(S) = \{\alpha \in R^m : \alpha A = 0, \forall A \in S\}$ and $r_{R_n}(S) = \{\beta \in R_n : A\beta = 0, \forall A \in S\}$.

2. (m, n) -injectivity of $R \ltimes R$

Let R be a ring and m, n be two positive integers. In this section, we investigate the (m, n) -injectivity of the ring $R \ltimes R$, which is isomorphic to $R[x]/(x^2)$.

Recall that R is left (m, n) -injective [3] if and only if, for any $C \in R^{n \times m}$, every left R -homomorphism from $R^n C$ to R extends to one from R^m to R if and only if $r_{R_n} l_{R^n}(A) = AR_m$ for all $A \in R^{n \times m}$. For convenience, we fix some notations. Set $A = (a_{ij})$, $B = (b_{ij}) \in R^{m \times n}$. Denote $(R^m A : B) = \{\alpha \in R^m : \alpha B \in R^m A\}$, $(AR_n : B) = \{\alpha \in R_n : B\alpha \in AR_n\}$ and $A \ltimes B = ((a_{ij}, b_{ij})) \in (R \ltimes R)^{m \times n}$. By

calculation, it is clear that $A \propto B = 0$ if and only if $A = 0$, $B = 0$ and $(A \propto B)(C \propto D) = AC \propto (AD + BC)$ for any $A, B \in R^{m \times n}$ and any $C, D \in R^{n \times t}$.

Theorem 2.1. *Let m and n be two positive integers. The following are equivalent for a ring R :*

- (1) $R \propto R$ is a left (m, n) -injective ring;
- (2) $r_{R_n}(l_{R^n}(A) \cap (R^n A : B)) = AR_m + Br_{R_m}(A)$ for any $A, B \in R^{n \times m}$.

PROOF. Denote $S = R \propto R$.

(1) $\Rightarrow$ (2). First we claim that $Br_{R_m}(A) \subseteq r_{R_n}((R^n A : B))$ for any $A, B \in R^{n \times m}$.

In fact, let $\alpha = B\bar{\alpha}$ with $\bar{\alpha} \in r_{R_m}(A)$. For any $\beta \in (R^n A : B)$, there exists $\gamma \in R^n$ such that $\beta B = \gamma A$. Then $\beta\alpha = \beta B\bar{\alpha} = \gamma A\bar{\alpha} = 0$, i.e., $\alpha \in r_{R_n}((R^n A : B))$. So

$$Br_{R_m}(A) \subseteq r_{R_n}((R^n A : B))$$

and

$$AR_m + Br_{R_m}(A) \subseteq AR_m + r_{R_n}((R^n A : B)) \subseteq r_{R_n}(l_{R^n}(A) \cap (R^n A : B)).$$

Next we show that $r_{R_n}(l_{R^n}(A) \cap (R^n A : B)) \subseteq AR_m + Br_{R_m}(A)$.

Set $A = (a_{ij}), B = (b_{ij}) \in R^{n \times m}$. Then $A \propto B = ((a_{ij}, b_{ij})) \in S^{n \times m}$. Since S is left (m, n) -injective, $r_{S_n}(l_{S^n}(A \propto B)) = (A \propto B)S_m$. Assume $\alpha \in r_{R_n}(l_{R^n}(A) \cap (R^n A : B))$. For any $P \propto Q \in l_{S^n}(A \propto B)$,

$$PA \propto (PB + QA) = (P \propto Q)(A \propto B) = 0.$$

So $PA = 0$ and $PB + QA = 0$, i.e., $P \in l_{R^n}(A) \cap (R^n A : B)$. Then $P\alpha = 0$. Thus $(P \propto Q)(0 \propto \alpha) = 0 \propto P\alpha = 0$ and $0 \propto \alpha \in r_{S_n}(l_{S^n}(A \propto B)) = (A \propto B)S_m$. So there exists $C \propto D \in S_m$ such that

$$0 \propto \alpha = (A \propto B)(C \propto D) = AC \propto (AD + BC).$$

Hence $AC = 0$ and $\alpha = AD + BC \in AR_m + Br_{R_m}(A)$. Thus $r_{R_n}(l_{R^n}(A) \cap (R^n A : B)) \subseteq AR_m + Br_{R_m}(A)$. Therefore $r_{R_n}(l_{R^n}(A) \cap (R^n A : B)) = AR_m + Br_{R_m}(A)$.

(2) $\Rightarrow$ (1). Assume (2). For any $A \in R^{n \times m}$, set $B = 0 \in R^{n \times m}$. Then $(R^n A : B) = R^n$. So the hypothesis implies that

$$r_{R_n}l_{R^n}(A) = AR_m.$$

Now for any $T = ((a_{ij}, b_{ij})) \in S^{n \times m}$, we shall show that $r_{S_n}l_{S^n}(T) = TS_m$.

Denote $A = (a_{ij}), B = (b_{ij})$. Then $A, B \in R^{n \times m}$ and $T = A \propto B$. Let $X \propto Y \in r_{S_n} l_{S^n}(T)$. For any $C \in l_{R^n}(A)$, we have $0 \propto C \in l_{S^n}(T)$ and $0 \propto CX = (0 \propto C)(X \propto Y) = 0$, so $CX = 0$. This implies that $X \in r_{R_n} l_{R^n}(A) = AR_m$. Write $X = AU$ with $U \in R_m$. For any $D \in l_{R^n}(A) \cap (R^n A : B)$, we have $DA = 0$ and $DB + HA = 0$ for some $H \in R^n$. Thus $(D \propto H)T = (D \propto H)(A \propto B) = DA \propto (DB + HA) = 0$. It follows that $DX \propto (DY + HX) = (D \propto H)(X \propto Y) = 0$, i.e., $DX = 0$ and $DY + HX = 0$. Consequently,

$$D(Y - BU) = DY - DBU = DY + HAU = DY + HX = 0.$$

This shows that $Y - BU \in r_{R_n}(l_{R^n}(A) \cap (R^n A : B)) = AR_m + Br_{R_m}(A)$, so

$$Y = AV + BU + BW$$

for some $V \in R_m$ and $W \in r_{R_m}(A)$. It is easy to see that

$$X \propto Y = AU \propto (AV + BU + BW) = (A \propto B)((U + W) \propto V) \in TS_m.$$

Thus $r_{S_n} l_{S^n}(T) \subseteq TS_m$. Note that the converse inclusion always holds. Therefore $S = R \propto R$ is left (m, n) -injective. $\square$

Corollary 2.2. *If $R \propto R$ is left (m, n) -injective, then $r_{R_n}(l_{R^n}(A) \cap (R^n A : B)) = AR_m + r_{R_n}((R^n A : B))$ for any $A, B \in R^{n \times m}$.*

PROOF. It is straightforward to verify that

$$AR_m + Br_{R_m}(A) \subseteq AR_m + r_{R_n}((R^n A : B)) \subseteq r_{R_n}(l_{R^n}(A) \cap (R^n A : B))$$

for any $A, B \in R^{n \times m}$. Therefore, the result follows immediately from Theorem 2.1. $\square$

Similarly, we can get the following theorem about the right (m, n) -injectivity of $R \propto R$.

Theorem 2.3. *Let R be a ring and m, n be two positive integers. The following are equivalent for R :*

- (1) $R \propto R$ is a right (m, n) -injective ring;
- (2) $l_{R^n}(r_{R_n}(A) \cap (AR_n : B)) = R^m A + l_{R^m}(A)B$ for any $A, B \in R^{m \times n}$.

Corollary 2.4 ([4, Theorem 1]). *Let R be a ring. Suppose that, for any $A, B \in R^{m \times n}$, every right R -homomorphism from $AR_n + Br_{R_n}(A)$ to R extends to one from R_m to R . Then $R \propto R$ is a right (m, n) -injective ring.*

PROOF. First note that, if $B = 0$, then the hypothesis implies that every right R -homomorphism from AR_n to R extends to one from R_m to R for any $A \in R^{m \times n}$. This shows that R is right (m, n) -injective. As done in the proof of Theorem 2.1, we have $R^m A + l_{R^m}(A)B \subseteq l_{R^n}(r_{R_n}(A) \cap (AR_n : B))$. Assume $\alpha \in l_{R^n}(r_{R_n}(A) \cap (AR_n : B))$ and define:

$$f : AR_n + Br_{R_n}(A) \rightarrow R; \quad A\gamma_1 + B\gamma_2 \mapsto \alpha\gamma_2.$$

If $A\gamma_1 + B\gamma_2 = 0$, then $\gamma_2 \in r_{R_n}(A) \cap (AR_n : B)$, so $\alpha\gamma_2 = 0$. Thus f is well-defined. Moreover, it is easy to see that f is a right R -homomorphism. By hypothesis, f can be extended to a right R -homomorphism from R_m to R , i.e., there exists $\xi \in R^m$ such that, for any $A\gamma_1 + B\gamma_2 \in AR_n + Br_{R_n}(A)$,

$$f(A\gamma_1 + B\gamma_2) = \xi(A\gamma_1 + B\gamma_2).$$

Thus, for any $\gamma_1 \in R_n, \gamma_2 \in r_{R_n}(A)$,

$$\xi A\gamma_1 = f(A\gamma_1) = 0, \quad \xi B\gamma_2 = f(B\gamma_2) = \alpha\gamma_2.$$

Then $\xi A = \xi AI_n = \xi A(e_1, \dots, e_n) = 0$, where I_n is the identity of $R^{n \times n}$ and e_i is the i -th column of I_n . So $\xi \in l_{R^m}(A)$ and $\alpha - \xi B \in l_{R^n}r_{R_n}(A) = R^m A$. It follows that $\alpha = (\alpha - \xi B) + \xi B \in R^m A + l_{R^m}(A)B$ and $l_{R^n}(r_{R_n}(A) \cap (AR_n : B)) \subseteq R^m A + l_{R^m}(A)B$. Hence

$$l_{R^n}(r_{R_n}(A) \cap (AR_n : B)) = R^m A + l_{R^m}(A)B.$$

By Theorem 2.3, the result follows. $\square$

Corollary 2.5 ([4, Theorem 2]). *If $R \propto R$ is right (m, n) -injective, then so is R .*

PROOF. Set $B = 0$ in Theorem 2.3. We have $l_{R^n}r_{R_n}(A) = R^m A$ for all $A \in R^{m \times n}$. Therefore R is right (m, n) -injective. $\square$

Corollary 2.6. *$R \propto R$ is left P -injective if and only if $r_R(l_R(a) \cap (Ra : b)) = aR + br_R(a)$ for any $a, b \in R$.*

Corollary 2.7. *$R \propto R$ is left FP -injective if and only if $r_{R_n}(l_{R^n}(A) \cap (R^n A : B)) = AR_m + Br_{R_m}(A)$ for any positive integers m, n and any $A, B \in R^{n \times m}$.*

Corollary 2.8. *Let n be a fixed positive integer. Then $R \propto R$ is left n -injective if and only if $r_{R_n} \left(l_{R^n} \left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) \cap \left(R^n \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) \right) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} R + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} r_R \left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right)$ for any $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in R_n$.*

Corollary 2.9. *Let m be a fixed positive integer. Then every m -generated right ideal of $R \propto R$ is a right annihilator if and only if $r_R(l_R(K) \cap (R(a_1, \dots, a_m) : (b_1, \dots, b_m))) = K + (b_1, \dots, b_m)r_{R_m}((a_1, \dots, a_m))$ for any $(a_1, \dots, a_m), (b_1, \dots, b_m) \in R^m$, where $K = a_1R + \dots + a_mR$.*

3. P -injectivity of $R[x]/(x^k)$

It is well known that $R \propto R$ is isomorphic to $R[x]/(x^2)$. So it is natural to explore the left (m, n) -injectivity of $R[x]/(x^k)$ for an arbitrary positive integer k . For simplicity, we only consider the left P -injectivity of $R[x]/(x^k)$ and acquire an equivalent condition for it. Using a similar argument, an analogous result about the left (m, n) -injectivity of $R[x]/(x^k)$ can be obtained.

We regard $R[x]/(x^k)$ as a subring of $R^{k \times k}$ by identifying the element $a_0 + a_1x + \dots + a_{k-1}x^{k-1} \in R[x]/(x^k)$ with the matrix

$$\begin{pmatrix} a_0 & a_1 & \dots & a_{k-2} & a_{k-1} \\ & a_0 & a_1 & \dots & a_{k-2} \\ & & \ddots & \ddots & \vdots \\ & & & a_0 & a_1 \\ & & & & a_0 \end{pmatrix} \in R^{k \times k}.$$

Denote by $\psi : R[x]/(x^k) \rightarrow R^{k \times k}$ such ring inclusion and

$$S_{(k)} = \{\psi(a_0 + a_1x + \dots + a_{k-1}x^{k-1}) : a_0, a_1, \dots, a_{k-1} \in R\}.$$

Write $(R^k A : \alpha) = \{r \in R : r\alpha \in R^k A\}$ for any $A \in S_{(k)}$, $\alpha \in R^k$.

Lemma 3.1. *Let R be a ring and n be a fixed positive integer. If, for any $a \in R$, $\alpha = (a_1, \dots, a_n) \in R^n$, $r_R(l_R(a) \cap (R^n A : \alpha)) = aR + \alpha r_{R_n}(A)$, where $A = \psi(a + a_1x + \dots + a_{n-1}x^{n-1}) \in S_{(n)}$, then $r_R(l_R(b) \cap (R^m B : \beta)) = bR + \beta r_{R_m}(B)$ for each $1 \leq m \leq n$ and any $b \in R$, $\beta = (b_1, \dots, b_m) \in R^m$, $B = \psi(b + b_1x + \dots + b_{m-1}x^{m-1}) \in S_{(m)}$.*

PROOF. It suffices to prove the conclusion for $m = n - 1$. Suppose $b \in R$, $\beta = (b_1, \dots, b_{n-1}) \in R^{n-1}$ and $B = \psi(b + b_1x + \dots + b_{n-2}x^{n-2}) \in S_{(n-1)}$. Let $\bar{\beta} = (b, \beta) = (b, b_1, \dots, b_{n-1}) \in R^n$, $\bar{B} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \psi(bx + b_1x^2 + \dots + b_{n-2}x^{n-1}) \in S_{(n)}$. By hypothesis, $r_R((R^n \bar{B} : \bar{\beta})) = \bar{\beta} r_{R_n}(\bar{B})$.

Note that $x \in (R^n \bar{B} : \bar{\beta})$ iff there exists $\bar{\delta} = (\delta, r) \in R^n$, where $r \in R$, $\delta \in R^{n-1}$, such that

$$x(b, \beta) = x\bar{\beta} = \bar{\delta}\bar{B} = (\delta, r) \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = (0, \delta B)$$

iff $x \in l_R(b) \cap (R^{n-1}B : \beta)$. This implies that $(R^n \bar{B} : \bar{\beta}) = l_R(b) \cap (R^{n-1}B : \beta)$.

Moreover, it is easy to see that $r_{R_n}(\bar{B}) = \begin{pmatrix} R \\ r_{R_{n-1}}(B) \end{pmatrix}$. Therefore

$$\begin{aligned} r_R(l_R(b) \cap (R^{n-1}B : \beta)) &= r_R((R^n \bar{B} : \bar{\beta})) = \bar{\beta} r_{R_n}(\bar{B}) \\ &= (b, \beta) \begin{pmatrix} R \\ r_{R_{n-1}}(B) \end{pmatrix} = bR + \beta r_{R_{n-1}}(B). \quad \square \end{aligned}$$

Lemma 3.2. *Let m be a positive integer. If $S_{(m)}$ is left P -injective and $r_R(l_R(a) \cap (R^m A : \alpha)) = aR + \alpha r_{R_m}(A)$ for any $a \in R$, $\alpha = (a_1, \dots, a_m) \in R^m$ and $A = \psi(a + a_1x + \dots + a_{m-1}x^{m-1}) \in S_{(m)}$, then $S_{(m+1)}$ is left P -injective.*

PROOF. Suppose $\bar{A} = \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} \in S_{(m+1)}$ with $a \in R$, $\alpha = (a_1, \dots, a_m) \in R^m$ and $A = \psi(a + a_1x + \dots + a_{m-1}x^{m-1}) \in S_{(m)}$, then $r_{S_{(m)}}l_{S_{(m)}}(A) = AS_{(m)}$ because $S_{(m)}$ is left P -injective. We will show that $r_{S_{(m+1)}}l_{S_{(m+1)}}(\bar{A}) = \bar{A}S_{(m+1)}$.

Assume $\bar{Z} = \begin{pmatrix} z & \xi \\ 0 & Z \end{pmatrix} \in r_{S_{(m+1)}}l_{S_{(m+1)}}(\bar{A})$, where $z \in R$, $\xi = (z_1, \dots, z_m) \in R^m$ and $Z = \psi(z + z_1x + \dots + z_{m-1}x^{m-1}) \in S_{(m)}$. Since $\begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \in l_{S_{(m+1)}}(\bar{A})$ for any $Y \in l_{S_{(m)}}(A)$,

$$\begin{pmatrix} 0 & YZ \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z & \xi \\ 0 & Z \end{pmatrix} = 0,$$

i.e., $YZ = 0$. Thus $Z \in r_{S_{(m)}}l_{S_{(m)}}(A) = AS_{(m)}$, whence $Z = AH$ for some $H = \psi(h + h_1x + \dots + h_{m-1}x^{m-1}) \in S_{(m)}$.

For any $t \in l_R(a) \cap (R^m A : \alpha)$, we have $ta = 0$ and $t\alpha + \beta A = 0$ for some $\beta = (b_1, \dots, b_m) \in R^m$, i.e., $\begin{pmatrix} t & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} = 0$. Let $B = \psi(t + b_1x + \dots + b_{m-1}x^{m-1}) \in S_{(m)}$. Then $\begin{pmatrix} t & \beta \\ 0 & B \end{pmatrix} \bar{A} = \begin{pmatrix} t & \beta \\ 0 & B \end{pmatrix} \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} = 0$, i.e., $\begin{pmatrix} t & \beta \\ 0 & B \end{pmatrix} \in l_{S_{(m+1)}}(\bar{A})$. It follows that

$$\begin{pmatrix} tz & t\xi + \beta Z \\ 0 & BZ \end{pmatrix} = \begin{pmatrix} t & \beta \\ 0 & B \end{pmatrix} \begin{pmatrix} z & \xi \\ 0 & Z \end{pmatrix} = 0.$$

So $t\xi + \beta Z = 0$. Since $Z = AH$, we have $t(\xi - \alpha H) = t\xi - t\alpha H = t\xi + \beta AH = t\xi + \beta Z = 0$. Then $t \left(z_m - \alpha \begin{pmatrix} h_{m-1} \\ \vdots \\ h_1 \\ h \end{pmatrix} \right) = 0$. Thus $z_m - \alpha \begin{pmatrix} h_{m-1} \\ \vdots \\ h_1 \\ h \end{pmatrix} \in$

$r_R(l_R(a) \cap (R^m A : \alpha)) = aR + \alpha r_{R_m}(A)$. Write

$$z_m - \alpha \begin{pmatrix} h_{m-1} \\ \vdots \\ h_1 \\ h \end{pmatrix} = ar + \alpha \begin{pmatrix} g_{m-1} \\ \vdots \\ g_1 \\ g \end{pmatrix}$$

with $r \in R$, $\begin{pmatrix} g_{m-1} \\ \vdots \\ g_1 \\ g \end{pmatrix} \in r_{R_m}(A)$. Then $z_m = ar + \alpha \begin{pmatrix} h_{m-1} + g_{m-1} \\ \vdots \\ h_1 + g_1 \\ h + g \end{pmatrix}$. Set $G =$

$\psi(g + g_1x + \cdots + g_{m-1}x^{m-1})$. Since $A \begin{pmatrix} g_{m-1} \\ \vdots \\ g_1 \\ g \end{pmatrix} = 0$, $AG = 0$. So $Z = AH = A(H + G)$. Then $z = a(h + g)$ and

$$\begin{aligned} (z_1, \dots, z_{m-1}) &= (a, a_1, \dots, a_{m-1}) \begin{pmatrix} h_1 + g_1 & h_2 + g_2 & h_3 + g_3 & \dots & h_{m-1} + g_{m-1} \\ h + g & h_1 + g_1 & h_2 + g_2 & \dots & h_{m-2} + g_{m-2} \\ & h + g & h_1 + g_1 & \dots & h_{m-3} + g_{m-3} \\ & & \ddots & \ddots & \vdots \\ & & & h + g & h_1 + g_1 \\ & & & & h + g \end{pmatrix} \\ &= a(h_1 + g_1, \dots, h_{m-1} + g_{m-1}) \\ &\quad + (a_1, \dots, a_{m-1}) \begin{pmatrix} h + g & h_1 + g_1 & h_2 + g_2 & \dots & h_{m-2} + g_{m-2} \\ & h + g & h_1 + g_1 & \dots & h_{m-3} + g_{m-3} \\ & & \ddots & \ddots & \vdots \\ & & & h + g & h_1 + g_1 \\ & & & & h + g \end{pmatrix} \\ &= a(h_1 + g_1, \dots, h_{m-1} + g_{m-1}) \\ &\quad + (a_1, \dots, a_{m-1}, a_m) \begin{pmatrix} h + g & h_1 + g_1 & h_2 + g_2 & \dots & h_{m-2} + g_{m-2} \\ & h + g & h_1 + g_1 & \dots & h_{m-3} + g_{m-3} \\ & & \ddots & \ddots & \vdots \\ & & & h + g & h_1 + g_1 \\ 0 & 0 & 0 & \dots & h + g \\ & & & & 0 \end{pmatrix} \\ &= a(h_1 + g_1, \dots, h_{m-1} + g_{m-1}) \\ &\quad + \alpha \begin{pmatrix} h + g & h_1 + g_1 & h_2 + g_2 & \dots & h_{m-2} + g_{m-2} \\ & h + g & h_1 + g_1 & \dots & h_{m-3} + g_{m-3} \\ & & \ddots & \ddots & \vdots \\ & & & h + g & h_1 + g_1 \\ 0 & 0 & 0 & \dots & h + g \\ & & & & 0 \end{pmatrix}. \end{aligned}$$

Since $z_m = ar + \alpha \begin{pmatrix} h_{m-1} + g_{m-1} \\ \vdots \\ h_1 + g_1 \\ h + g \end{pmatrix}$,

$$\xi = (z_1, \dots, z_{m-1}, z_m) = a(h_1 + g_1, \dots, h_{m-1} + g_{m-1}, r)$$

$$+ \alpha \begin{pmatrix} h+g & h_1+g_1 & h_2+g_2 & \dots & h_{m-2}+g_{m-2} & h_{m-1}+g_{m-1} \\ & h+g & h_1+g_1 & \dots & h_{m-3}+g_{m-3} & h_{m-2}+g_{m-2} \\ & & & \ddots & \vdots & \vdots \\ & & & & h+g & h_1+g_1 \\ 0 & 0 & 0 & \dots & 0 & h_2+g_2 \\ & & & & & h_1+g_1 \\ & & & & & h+g \end{pmatrix} = a\eta + \alpha(H + G),$$

where $\eta = (h_1 + g_1, \dots, h_{m-1} + g_{m-1}, r) \in R^m$. From this, we can see that $\begin{pmatrix} h+g & \eta \\ 0 & H+G \end{pmatrix} \in S_{(m+1)}$ and

$$\begin{aligned} \bar{Z} &= \begin{pmatrix} z & \xi \\ 0 & Z \end{pmatrix} = \begin{pmatrix} a(h+g) & a\eta + \alpha(H+G) \\ 0 & A(H+G) \end{pmatrix} \\ &= \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} \begin{pmatrix} h+g & \eta \\ 0 & H+G \end{pmatrix} \in \bar{A}S_{(m+1)}. \end{aligned}$$

Hence $r_{S_{(m+1)}} l_{S_{(m+1)}}(\bar{A}) = \bar{A}S_{(m+1)}$, and this shows that $S_{(m+1)}$ is left P -injective. $\square$

Theorem 3.3. *Let n be a positive integer. The following are equivalent for a ring R :*

- (1) $R[x]/(x^n)$ is a left P -injective ring;
- (2) $S_{(n)}$ is a left P -injective ring;
- (3) $r_R(l_R(a) \cap (R^{n-1}A : \alpha)) = aR + \alpha r_{R_{n-1}}(A)$ for any $a \in R$, $\alpha = (a_1, \dots, a_{n-1}) \in R^{n-1}$ and $A = \psi(a + a_1x + \dots + a_{n-2}x^{n-2}) \in S_{(n-1)}$.

PROOF. We only need to show (2) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (3). Suppose $a \in R$, $\alpha = (a_1, \dots, a_{n-1}) \in R^{n-1}$. Set $A = \psi(a + a_1x + \dots + a_{n-2}x^{n-2}) \in S_{(n-1)}$ and $\bar{A} = \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} \in S_{(n)}$. Then $r_{S_{(n)}} l_{S_{(n)}}(\bar{A}) = \bar{A}S_{(n)}$ by hypothesis.

Let $t = \alpha\mu$ with $\mu \in r_{R_{n-1}}(A)$. For any $r \in (R^{n-1}A : \alpha)$, $r\alpha + \gamma A = 0$ for some $\gamma \in R^{n-1}$ and hence

$$rt = r\alpha\mu = r\alpha\mu + \gamma A\mu = (r\alpha + \gamma A)\mu = 0.$$

This shows $t \in r_R((R^{n-1}A : \alpha))$ and $\alpha r_{R_{n-1}}(A) \subseteq r_R((R^{n-1}A : \alpha))$. So

$$aR + \alpha r_{R_{n-1}}(A) \subseteq aR + r_R((R^{n-1}A : \alpha)) \subseteq r_R(l_R(a) \cap (R^{n-1}A : \alpha)).$$

Conversely, assume $z \in r_R(l_R(a) \cap (R^{n-1}A : \alpha))$. For any $\bar{B} = \begin{pmatrix} b & \beta \\ 0 & B \end{pmatrix} \in l_{S_{(n)}}(\bar{A})$,

$$\begin{pmatrix} ba & b\alpha + \beta A \\ 0 & BA \end{pmatrix} = \begin{pmatrix} b & \beta \\ 0 & B \end{pmatrix} \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} = \bar{B}\bar{A} = 0.$$

So $ba = 0$ and $b\alpha + \beta A = 0$, i.e., $b \in l_R(a) \cap (R^{n-1}A : \alpha)$. Thus $bz = 0$. Let $\xi = (0, \dots, 0, z) \in R^{n-1}$. Then $b\xi = 0$ and

$$\begin{pmatrix} b & \beta \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b\xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \in r_{S_{(n)}} l_{S_{(n)}}(\bar{A}) = \bar{A} S_{(n)}$. Write

$$\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} \begin{pmatrix} d & \eta \\ 0 & D \end{pmatrix}$$

with $d \in R$, $\eta = (y_1, \dots, y_{n-1}) \in R^{n-1}$ and $\begin{pmatrix} d & \eta \\ 0 & D \end{pmatrix} \in S_{(n)}$. Then $AD = 0$ and

$\xi = a\eta + \alpha D$. Let $\lambda = \begin{pmatrix} y_{n-2} \\ \vdots \\ y_1 \\ d \end{pmatrix}$ be the last column of D . Then $A\lambda = 0$ and

$z = ay_{n-1} + \alpha\lambda \in aR + \alpha r_{R_{n-1}}(A)$. This implies that $r_R(l_R(a) \cap (R^{n-1}A : \alpha)) \subseteq aR + \alpha r_{R_{n-1}}(A)$. Hence $r_R(l_R(a) \cap (R^{n-1}A : \alpha)) = aR + \alpha r_{R_{n-1}}(A)$.

(3) $\Rightarrow$ (2). Assume (3). By Lemma 3.1, we get that $r_R(l_R(a) \cap (R^m A : \alpha)) = aR + \alpha r_{R_m}(A)$ for each $1 \leq m \leq n-1$ and any $a \in R$, $\alpha = (a_1, \dots, a_m) \in R^m$, $A = \psi(a + a_1x + \dots + a_{m-1}x^{m-1}) \in S_{(m)}$. In particular, $r_R(l_R(a) \cap (Ra : b)) = aR + br_R(a)$ for any $a, b \in R$. So $S_{(2)} = R \propto R$ is left P -injective by Corollary 2.6. Hence, by Lemma 3.2, $S_{(3)}$ is left P -injective. Proceeding in this manner, we can get that $S_{(m)}$ is left P -injective for all $2 \leq m \leq n$. In particular, $S_{(n)}$ is left P -injective, and the proof is completed. $\square$

Corollary 3.4. *If $R[x]/(x^n)$ is a left P -injective ring, then $R[x]/(x^m)$ is left P -injective for all $1 \leq m \leq n$.*

PROOF. By Theorem 3.3 and Lemma 3.1. $\square$

4. Coherence of $R[x]/(x^n)$

Let n be a positive integer. In this section, we explore the interplay between the coherence of a ring R and the coherence of $R[x]/(x^n)$ ($n \geq 1$). We denote $S_{(n)} = \{\psi(a_0 + a_1x + \dots + a_{n-1}x^{n-1}) \in R^{n \times n} : a_0, a_1, \dots, a_{n-1} \in R\}$ as in Section 3 and write $(R_n a : \alpha) = \{H \in S_{(n)} : H\alpha \in R_n a\}$ for any $a \in R$, $\alpha \in R_n$.

It was proved in [4] that a ring R is left coherent if and only if $R \propto R$ is left coherent. This is a special case of the main result of this section. We first give the following lemma which appears in the proof of [4, Theorem 12].

Lemma 4.1. *If R is a left coherent ring, then $(Ra : b)$ is a finitely generated left ideal of R for any $a, b \in R$.*

Lemma 4.2. *Let R be a ring and n be a positive integer. If $S_{(n)}$ is left coherent, then $(R_na : \alpha)$ is a finitely generated left ideal of $S_{(n)}$ for any $a \in R, \alpha \in R_n$.*

PROOF. Suppose $a \in R, \alpha = \begin{pmatrix} a_n \\ \vdots \\ a_1 \end{pmatrix} \in R_n$. Let $A = \psi(a), B = \psi(a_1 + a_2x + \cdots + a_nx^{n-1})$. Then $A, B \in S_{(n)}$. Denote $(S_{(n)}A : B) = \{H \in S_{(n)} : HB \in S_{(n)}A\}$. Note that $H \in (R_na : \alpha)$ iff $H\alpha = \gamma a$ for some $\gamma = \begin{pmatrix} r_n \\ \vdots \\ r_1 \end{pmatrix} \in R_n$ iff $HB = GA$, where $G = \psi(r_1 + r_2x + \cdots + r_nx^{n-1}) \in S_{(n)}$ iff $H \in (S_{(n)}A : B)$. This implies that $(R_na : \alpha) = (S_{(n)}A : B)$. Since $S_{(n)}$ is left coherent, $(S_{(n)}A : B)$ is a finitely generated left ideal of $S_{(n)}$ by Lemma 4.1. So $(R_na : \alpha)$ is a finitely generated left ideal of $S_{(n)}$. $\square$

Theorem 4.3. *The following are equivalent for a ring R :*

- (1) R is left coherent;
- (2) $R[x]/(x^n)$ is left coherent for all $n \geq 1$;
- (3) $R[x]/(x^n)$ is left coherent for some $n \geq 1$.

PROOF. Since $R[x]/(x^n) \cong S_{(n)}$, we proceed the proof for $S_{(n)}$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Assume (3). We first show that R is left P -coherent, i.e., $l_R(a)$ is a finitely generated left ideal of R for any $a \in R$.

Set $A = \psi(ax^{n-1}) \in S_{(n)}$. Note that $l_{S_{(n)}}(A) = \{\psi(b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) : b_0 \in l_R(a), b_1, \dots, b_{n-1} \in R\}$. Since $S_{(n)}$ is left coherent, $l_{S_{(n)}}(A)$ is a finitely generated left ideal of $S_{(n)}$. Write

$$l_{S_{(n)}}(A) = S_{(n)}\psi(a_1 + a_{11}x + \cdots + a_{1(n-1)}x^{n-1}) + \cdots \\ + S_{(n)}\psi(a_m + a_{m1}x + \cdots + a_{m(n-1)}x^{n-1})$$

with all $a_i, a_{ij} \in R$. It follows that

$$l_R(a) = Ra_1 + \cdots + Ra_m,$$

so R is left P -coherent.

Now since $R[x]/(x^n)$ is left coherent, $M_k(R[x]/(x^n))$ is left coherent for each $k \geq 1$. So $(M_k(R))[x]/(x^n) \cong M_k(R[x]/(x^n))$ is left coherent. Thus, as above,

$M_k(R)$ is left P -coherent for each $k \geq 1$, and so R is left coherent by [2, Proposition 2.7].

(1) $\Rightarrow$ (2). We prove the conclusion by induction on n . Since $S_{(1)} \cong R$, $S_{(n)}$ is left coherent for $n = 1$.

Assume $S_{(n)}$ is left coherent for some $n \geq 1$, we show that $S_{(n+1)}$ is left coherent. First we verify that $S_{(n+1)}$ is left P -coherent, i.e., for any $\bar{A} \in S_{(n+1)}$, $l_{S_{(n+1)}}(\bar{A})$ is a finitely generated left ideal of $S_{(n+1)}$.

Write $\bar{A} = \begin{pmatrix} A & \alpha \\ 0 & a \end{pmatrix}$ with $a \in R$, $\alpha = \begin{pmatrix} a_n \\ \vdots \\ a_1 \end{pmatrix} \in R_n$, $A = \psi(a + a_1x + \cdots + a_{n-1}x^{n-1}) \in S_{(n)}$. Since $S_{(n)}$ is left coherent, $l_{S_{(n)}}(A)$ is a finitely generated left ideal of $S_{(n)}$ and $l_R(a)$ is a finitely generated left ideal of R . By Lemma 4.2, $(R_na : \alpha)$ is a finitely generated left ideal of $S_{(n)}$. So $l_{S_{(n)}}(A) \cap (R_na : \alpha)$ is finitely generated. Write

$$l_{S_{(n)}}(A) \cap (R_na : \alpha) = S_{(n)}G_1 + \cdots + S_{(n)}G_m, \quad l_R(a) = Rt_1 + \cdots + Rt_m,$$

where all $G_i = \psi(g_i + g_{i1}x + \cdots + g_{i(n-1)}x^{n-1}) \in S_{(n)}$, $t_i \in R$. Then $G_iA = 0$, $G_i\alpha + \eta_i a = 0$ for some $\eta_i = \begin{pmatrix} d_{in} \\ \vdots \\ d_{i1} \end{pmatrix} \in R_n$, and $t_i a = 0$. Then $\psi(t_i x^n) \in l_{S_{(n+1)}}(\bar{A})$ for all $1 \leq i \leq m$. Let $\theta_i = \begin{pmatrix} d_{in} \\ g_{i(n-1)} \\ \vdots \\ g_{i1} \end{pmatrix}$. Then $\begin{pmatrix} G_i & \theta_i \\ 0 & g_i \end{pmatrix} \in S_{(n+1)}$. Since $G_iA = 0$ and $G_i\alpha + \eta_i a = 0$, we get $G_i\alpha + \theta_i a = 0$. This implies that $\begin{pmatrix} G_i & \theta_i \\ 0 & g_i \end{pmatrix} \in l_{S_{(n+1)}}(\bar{A})$, $\forall 1 \leq i \leq m$.

Assume $\begin{pmatrix} B & \beta \\ 0 & b \end{pmatrix} \in l_{S_{(n+1)}}(\bar{A})$, where $b \in R$, $\beta = \begin{pmatrix} b_n \\ \vdots \\ b_1 \end{pmatrix} \in R_n$, $B = \psi(b + b_1x + \cdots + b_{n-1}x^{n-1}) \in S_{(n)}$. Then

$$\begin{pmatrix} BA & B\alpha + \beta a \\ 0 & ba \end{pmatrix} = \begin{pmatrix} B & \beta \\ 0 & b \end{pmatrix} \begin{pmatrix} A & \alpha \\ 0 & a \end{pmatrix} = 0.$$

So $BA = 0$, and $B\alpha + \beta a = 0$, thus $B \in l_{S_{(n)}}(A) \cap (R_na : \alpha) = S_{(n)}G_1 + \cdots + S_{(n)}G_m$. Write

$$B = Z_1G_1 + \cdots + Z_mG_m$$

with all $Z_i = \psi(z_i + z_{i1}x + \cdots + z_{i(n-1)}x^{n-1}) \in S_{(n)}$. Then

$$(\beta - (Z_1\theta_1 + \cdots + Z_m\theta_m))a = \beta a + Z_1G_1\alpha + \cdots + Z_mG_m\alpha = B\alpha + \beta a = 0.$$

Hence

$$(b_n - (\xi_1\theta_1 + \cdots + \xi_m\theta_m))a = 0,$$

where ξ_i is the first row of Z_i . It follows that $b_n - (\xi_1\theta_1 + \cdots + \xi_m\theta_m) \in l_R(a) = Rt_1 + \cdots + Rt_m$. Set

$$b_n - (\xi_1\theta_1 + \cdots + \xi_m\theta_m) = r_1t_1 + \cdots + r_mt_m$$

with all $r_i \in R$. Then $b_n = \xi_1\theta_1 + \cdots + \xi_m\theta_m + r_1t_1 + \cdots + r_mt_m$. Let $\lambda_i = \begin{pmatrix} 0 \\ z_{i(n-1)} \\ \vdots \\ z_{i1} \end{pmatrix}$. Then $\begin{pmatrix} Z_i & \lambda_i \\ 0 & z_i \end{pmatrix} \in S_{(n+1)}$. Since $B = \sum_{i=1}^m Z_i G_i$ and $b_n = \sum_{i=1}^m \xi_i \theta_i + \sum_{i=1}^m r_i t_i$,

$$\beta = \sum_{i=1}^m (Z_i \theta_i + \lambda_i g_i) + \begin{pmatrix} \sum_{i=1}^m r_i t_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} \begin{pmatrix} B & \beta \\ 0 & b \end{pmatrix} &= \begin{pmatrix} \sum_{i=1}^m Z_i G_i & \sum_{i=1}^m (Z_i \theta_i + \lambda_i g_i) \\ 0 & \sum_{i=1}^m z_i g_i \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & \sum_{i=1}^m r_i t_i \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \\ &= \sum_{i=1}^m \begin{pmatrix} Z_i & \lambda_i \\ 0 & z_i \end{pmatrix} \begin{pmatrix} G_i & \theta_i \\ 0 & g_i \end{pmatrix} + \sum_{i=1}^m \psi(r_i) \psi(t_i x^n). \end{aligned}$$

This means that $l_{S_{(n+1)}}(\bar{A}) = \sum_{i=1}^m S_{(n+1)} \begin{pmatrix} G_i & \theta_i \\ 0 & g_i \end{pmatrix} + \sum_{i=1}^m S_{(n+1)} \psi(t_i x^n)$ is finitely generated. Thus we have proved that $S_{(n)}$ being left coherent implies that $S_{(n+1)}$ is left P -coherent, i.e., $R[x]/(x^n)$ being left coherent implies that $R[x]/(x^{n+1})$ is left P -coherent.

Since $R[x]/(x^n)$ is left coherent, $(M_k(R))[x]/(x^n) \cong M_k(R[x]/(x^n))$ is left coherent for each $k \geq 1$. As above, $M_k(R[x]/(x^{n+1})) \cong (M_k(R))[x]/(x^{n+1})$ is left P -coherent for each $k \geq 1$. So $R[x]/(x^{n+1})$ is left coherent by [2, Proposition 2.7]. Thus, by induction, R being left coherent implies that $R[x]/(x^n)$ is left coherent for each $n \geq 1$ and the result follows. $\square$

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QIONGLING LIU
DEPARTMENT OF MATHEMATICS
SOUTHEAST UNIVERSITY
NANJING, JIANGSU, 210096
P.R. CHINA

E-mail: ahlql123@163.com

JIANLONG CHEN
DEPARTMENT OF MATHEMATICS
SOUTHEAST UNIVERSITY
NANJING, JIANGSU, 210096
P.R. CHINA

E-mail: jlchen@seu.edu.cn

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