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A finiteness condition for verbal conjugacy classes in a group

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Abstract. Given a group G and a word w, we denote by G_w the set of all w-values in G and by $w(G)$ the corresponding verbal subgroup. The main result of the paper is the following theorem. Let k be a positive integer and let w be either the word γ_k or the word δ_k. Suppose that G is a group in which $\langle x^{G_w}\rangle$ is Chernikov for all $x \in G$. Then $\langle x^{w(G)}\rangle$ is Chernikov for all $x \in G$ as well.

1. Introduction

Let w be a word in n variables, and let G be a group. The verbal subgroup $w(G)$ of G determined by w is the subgroup generated by the set G_w consisting of all values $w(g_1, \dots, g_n)$, where $g_1, \dots, g_n$ are elements of G. A word w is said to be concise if whenever G_w is finite for a group G, it always follows that $w(G)$ is finite. P. Hall asked whether every word is concise, but it was later proved that this problem has a negative solution in its general form (see [4], p. 439). On the other hand, many relevant words are known to be concise. For instance, Turner-Smith [7] showed that the lower central words γ_k and the derived words δ_k are concise; here the words γ_k and δ_k are defined by the positions $\gamma_1 = \delta_0 = x$, $\gamma_{k+1} = [\gamma_k, \gamma_1]$ and $\delta_{k+1} = [\delta_k, \delta_k]$. The corresponding verbal subgroups for these words are the familiar kth term of the lower central series of G denoted by $\gamma_k(G)$ and the kth derived group of G denoted by $G^{(k)}$.

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There are several natural ways to look at Hall's question from a different angle. The circle of problems arising in this context can be characterized as follows.

Given a word w and a group G, assume that certain restrictions are imposed on the set G_w. How does this influence the properties of the verbal subgroup $w(G)$?

If X and Y are non-empty subsets of a group G, we will write X^Y to denote the set $\{y^{-1}xy \mid x \in X,\ y \in Y\}$. In [2] groups G with the property that x^{G_w} is finite for all $x \in G$ were called $FC(w)$-groups. Recall that FC-groups are precisely groups with finite conjugacy classes. The main result of [2] tells us that if w is a concise word, then a group G is an $FC(w)$-group if and only if $x^{w(G)}$ is finite for all $x \in G$. In particular, it follows that if w is a concise word and G is an $FC(w)$-group, then the verbal subgroup $w(G)$ is FC. Later it was shown in [1] that there exists a function $f = f(m, w)$ such that if, under the hypothesis of the above theorem, x^{G_w} has at most m elements for all $x \in G$, then $x^{w(G)}$ has at most f elements for all $x \in G$. In view of these results we would like to consider the following question.

Given a concise word w and a group G, assume that for all $x \in G$ the subgroup $\langle x^{G_w} \rangle$ satisfies a certain finiteness condition. Is it true that a similar condition is also satisfied by $\langle x^{w(G)} \rangle$ for all $x \in G$?

Here and throughout the paper $\langle M \rangle$ denotes the subgroup generated by the set M. The main result of the present paper is as follows.

Theorem 1.1. *Let k be a positive integer and let w be either the word γ_k or the word δ_k. Suppose that G is a group in which $\langle x^{G_w} \rangle$ is Chernikov for all $x \in G$. Then $\langle x^{w(G)} \rangle$ is Chernikov for all $x \in G$ as well.*

Recall that a group G is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type C_{p^∞} for various primes p (quasicyclic p-groups). By a deep result obtained independently by Shunkov [6] and Kegel and Wehrfritz [3] Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup. The minimal subgroup of finite index of a Chernikov group G is called the radicable part of G. In general a group G is called radicable if the equation $x^n = a$ has a solution in G for every positive integer n and every $a \in G$. It is well-known that a periodic abelian radicable group is a direct product of quasicyclic p-subgroups.

A proof of Theorem 1.1 in the case where $w = \gamma_k$ can be obtained from the case $w = \delta_k$ by simply replacing everywhere in the proof the term "δ_k-commutators" by "γ_k-commutators". That is why we do not provide an explicit proof for the case $w = \gamma_k$ concentrating instead on proving Theorem 1.1 in the case $w = \delta_k$.

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2. Preliminary lemmas

We start the section with the following well-known lemma (see for example [5, Lemma 3.13]).

Lemma 2.1. *Suppose that R is a radicable abelian normal subgroup of the group G and suppose that H is a subgroup of G such that $[R, \underbrace{H, \dots, H}_{r}] = 1$ for some natural number r. If H/H' is periodic, then $[R, H] = 1$.*

From this we can easily deduce the following useful corollaries.

Corollary 2.2. *In a periodic nilpotent group G every radicable abelian subgroup Q is central.*

Proof. Arguing by induction on the nilpotency class of G we assume that the image of Q in $G/Z(G)$ is central. Therefore Q is contained in a normal abelian subgroup of G. In particular $\langle Q^G \rangle$ is a normal abelian radicable subgroup and the result is now immediate from Lemma 2.1. □

Let G be a group acted on by a group A. As usual, $[G, A]$ denotes the subgroup generated by all elements of the form $x^{-1}x^a$, where $x \in G$, $a \in A$. It is well-known that $[G, A]$ is a normal subgroup of G. If B is a normal subset of A such that $A = \langle B \rangle$, then $[G, A] = \langle [G, b];\ b \in B \rangle$.

Corollary 2.3. *Let A be a periodic group acting on a periodic radicable abelian group G. Then $[G, A, A] = [G, A]$.*

Proof. To show this, we can assume that $[G, A, A] = 1$. In this case Lemma 2.1 yields at once that $[G, A] = 1$. □

Lemma 2.4. *Let A be a finite group acting on a periodic radicable abelian group G. Then $[G, A]$ is radicable.*

Proof. Since $[G,A] = \prod_{a\in A}[G,a]$, it is sufficient to show that $[G,a]$ is radicable for every $a \in A$. Let $x \in [G,a]$ and let n be a positive integer. Then there exist $g \in G$ such that $x = [g,a]$ and $g_1 \in G$ such that $g_1^n = g$. Since G is abelian, we have $[g_1,a]^n = [g_1^n,a]$. Hence for every $x \in [G,a]$ and every positive integer n, there exists an element $[g_1,a] \in [G,a]$ such that $x = [g_1,a]^n$; that is, $[G,a]$ is radicable, as required. □

Lemma 2.5. *Let A be a radicable Chernikov group acting on a Chernikov group B. Then $[B,A,A]=1$.*

Proof. Denote by B_0 the radicable part of B. By [5, Theorem 3.29.2], $A/C_A(B_0)$ is finite. Since A is radicable, it follows that A has no subgroups of finite index and so $[B_0,A]=1$. On the other hand, B/B_0 is finite and therefore $A/C_A(B/B_0)$ is also finite. Again, since A A has no subgroups of finite index, it follows that $[B,A] \le B_0$. Hence $[B,A,A] \le [B_0,A] = 1$. □

Lemma 2.6. *Let G be a group and y an element of G. Suppose that $x_1,\dots,x_k \in G$ are δ_k-commutators for $k \ge 0$. Then $[y,x_1,\dots,x_k]$ is a δ_k-commutator as well.*

Proof. Note that $x_1,\dots,x_k$ can be viewed as δ_i-commutators for each $i \le k$. It is clear that $[y,x_1]$ is a δ_1-commutator. Arguing by induction on k assume that $k \ge 1$ and $[y,x_1,\dots,x_{k-1}]$ is a δ_{k-1}-commutator. Then $[y,x_1,\dots,x_k] = [[y,x_1,\dots,x_{k-1}],x_k]$ is a δ_k-commutator. □

Throughout the paper, whenever G is a Chernikov group we denote by G_0 the radicable part of G and by G^* the subgroup $[G_0,G]$.

Lemma 2.7. *Let G be a Chernikov group for which there exists a positive integer m such that G can be generated by elements of order dividing m. If $G^*=1$, then G is finite.*

Proof. Since $G^*=1$, it follows that G_0 is central. The Schur Theorem [5, Theorem 4.12] yields that G' is finite. Since G can be generated by elements of order dividing m, we conclude that G has finite exponent. In particular G has no subgroups of type C_{p^∞}. Thus, G must be finite. □

Lemma 2.8. *Let G be a group such that $\langle x^G\rangle$ is Chernikov for every $x \in G$. Then all abelian radicable subgroups of G generate an abelian radicable subgroup.*

Proof. Let T be the subgroup of G generated by all abelian radicable subgroups. Let A be an arbitrary abelian radicable subgroup in G and choose $x \in G$. Then Lemma 2.1 together with Lemma 2.5 shows that the product $\langle x^G\rangle_0 A$ is abelian. Thus, all subgroups of the form $\langle x^G\rangle_0$ lie in the center of T. Therefore $G/Z(T)$ is a periodic FC-group. Since T has no subgroups of finite index, it centralizes every finite normal subgroup and we conclude that the image of T in $G/Z(T)$ is central. Therefore T is nilpotent of class at most two. Corollary 2.2 now enables us to deduce that T is abelian, as required. □

We will also require the following lemma.

Lemma 2.9. *Let X be a normal set in a locally finite group G. Let $a \in G$ and assume that the set a^X is finite. Then the set $a^{\langle X\rangle}$ is likewise finite.*

Proof. Let $x_1, \dots, x_n$ be elements of X with the property that $a^X = \{a^{x_1}, \dots, a^{x_n}\}$ and let $Y = \langle x_1, \dots, x_n\rangle$. Since Y is finite, the class a^Y is finite as well. Let $N = \langle X\rangle$. We will show that $a^N = a^Y$. Choose $y \in N$. Then y can be written as a product $y = y_1 \dots y_m$, where $y_i \in X$. It is sufficient to show that $a^y \in a^Y$. If $m = 1$, then $y \in X$ and so $a^y \in \{a^{x_1}, \dots, a^{x_n}\} \subseteq a^Y$. Thus, assume that $m \geq 2$ and use induction on m. Suppose that $a^{y_1} = a^{x_1}$. Set $z_i = x_1 y_i {x_1}^{-1}$ for $i = 2, \dots, m$. Since X is a normal set of G, $z_i \in X$. Write

$$a^y = a^{x_1 y_2 \cdots y_m} = a^{x_1 y_2 \cdots y_m {x_1}^{-1} x_1} = a^{z_2 \cdots z_m x_1}.$$

By induction $a^{z_2 \cdots z_m} \in a^Y$. Since $x_1 \in Y$, it follows that $a^y \in a^Y$. This completes the proof. □

3. Proof of Theorem 1.1

Assume the hypothesis of Theorem 1.1 with $w = \delta_k$ and let X denote the set of all δ_k-commutators in G. By the hypothesis $\langle a^X\rangle$ is Chernikov for all $a \in G$. Set $H = G^{(k)}$. We wish to show that $\langle a^H\rangle$ is Chernikov for all $a \in G$. First we will deal with the particular case where $a \in X$. Thus, choose $a \in X$ and let $D = \langle a^X\rangle$. As usual, the normal closure of a subset $S \subseteq G$ is the minimal normal subgroup of G containing S.

Lemma 3.1. *With the above notation, the normal closure of D^* in G is an abelian radicable subgroup.*

Proof. By Corollary 2.3 $D^* = [D_0, \underbrace{D, \dots, D}_{k}]$. Since $a \in X$, every element of a^X is also a δ_k-commutator. It follows that D is generated by the normal set $X \cap D$. Therefore the subgroup D^* is generated by subgroups of the form $[D_0, b_1, \dots, b_k]$, where $b_1, \dots, b_k \in X \cap D$.

Let us show that for every choice of $b_1, \dots, b_k \in X \cap D$ the subgroup $[D_0, b_1, \dots, b_k]$ is contained in a normal abelian radicable subgroup of G. Thus, fix $b_1, \dots, b_k \in X \cap D$ and put $K = [D_0, b_1, \dots, b_k]$. Since D_0 is abelian, it is clear that for every $d_1, d_2 \in D_0$ we have

$$[d_1, b_1, \dots, b_k][d_2, b_1, \dots, b_k] = [d_1 d_2, b_1, \dots, b_k].$$

Now, Lemma 2.6 shows that every element of K is a δ_k-commutator and Lemma 2.4 yields that K is radicable. Since $\langle g^K \rangle$ is Chernikov for every $g \in G$, it follows from Lemma 2.5 that $[g^K, K, K] = 1$. In particular $[g, K, K] = 1$ and we conclude that K commutes with K^g for every $g \in G$. Therefore $\langle K^G \rangle$ is abelian. Since $\langle K^G \rangle$ is generated by radicable subgroups, it follows that $\langle K^G \rangle$ is radicable.

Now choose other elements $b'_1, \dots, b'_k \in X \cap D$ and set $K_1 = [D_0, b'_1, \dots, b'_k]$. Repeating the above argument we conclude that $\langle K_1^G \rangle$ is abelian and radicable. Thus, the product $\langle K^G \rangle \langle K_1^G \rangle$ is nilpotent of class at most two and Corollary 2.2 tells us that $\langle K^G \rangle \langle K_1^G \rangle$ is abelian. Thus, all subgroups of the form $\langle [D_0, x_1, \dots, x_k]^G \rangle$, where $x_1, \dots, x_k \in X \cap D$, commute and the lemma follows. □

Set $R = \langle \langle y^X \rangle^*;\ y \in X \rangle$. This notation will be kept throughout the rest of the paper.

Corollary 3.2. *The subgroup R is abelian and radicable.*

Proof. Choose $y_1, y_2 \in X$. Let R_1 be the normal closure of $\langle {y_1}^X \rangle^*$ and R_2 that of $\langle {y_2}^X \rangle^*$. By Lemma 3.1 both R_1 and R_2 are abelian radicable subgroups. We conclude that the product $R_1 R_2$ is nilpotent of class at most two and Corollary 2.2 shows that $R_1 R_2$ is abelian. The result follows. □

In the next lemma we use terminology and some results from the paper [2]. For the reader's convenience we will briefly explain it. Let w be a word, G a group and H a subgroup of $w(G)$. We say that H has finite w-index if the elements of G_w lie in finitely many right cosets of H in $w(G)$. A group G is an $FC(w)$-group if and only if the subgroup $C_{w(G)}(x)$ has finite w-index for every element x of G.

Lemma 3.3. *The group G is locally finite.*

Proof. First of all we notice that G is torsion since $\langle y^X \rangle$ is torsion for every $y \in G$. Since R is abelian (Corollary 3.2), it is sufficient to prove the local finiteness of G under the assumption that $R = 1$. Choose $y \in X$. By the above assumption $\langle y^X \rangle^* = 1$. Since $\langle y^X \rangle$ is generated by conjugates of y, it follows from Lemma 2.7 that $\langle y^X \rangle$ is finite. This implies that $C_G(y) \cap H$ has finite δ_k-index.

This happens for every choice of $y \in X$. Since every element of H is a product of finitely many δ_k-commutators, [2, Lemma 2.1] shows that H is an $FC(\delta_k)$-group. In particular, the main result of [2] tells us that the kth derived group of H is an FC-group. Now the local finiteness of G is obvious. $\square$

Lemma 3.4. *The subgroup $\langle a^H \rangle$ is Chernikov.*

Proof. Recall that $a \in X$ and $D = \langle a^X \rangle$. Set $E = \langle a^H \rangle$. We wish to show that E is Chernikov. Let us show first that ER/R is finite. It suffices to show this under the additional assumption that $R = 1$. In this case $D^* = 1$ and so D is finite by Lemma 2.7. In particular a^X is finite and since G is locally finite, we use Lemma 2.9 to conclude that E is finite. Thus, indeed ER/R is finite. Set $R_1 = E \cap R$. Choose elements $e_1, \ldots, e_s \in X$ such that every conjugate of a in H belong to a coset e_iR_1 for some $i = 1, \ldots, s$. We have $E = \langle R_1, e_1, \ldots, e_s \rangle$. Since the set $e_1, \ldots, e_s$ generates E and is normal in H modulo R_1, it follows that

$$[R_1, E] = [R_1, e_1] \ldots [R_1, e_s].$$

By Lemma 2.4 $[R_1, e_i] = [R_1, e_i, e_i]$ and Lemma 2.6 shows that $[R_1, e_i] \subseteq X$. We conclude that $[R_1, e_i] = [R_1, e_i, e_i] \leq [X, e_i] \leq \langle {e_i}^X \rangle$ and so the subgroups $[R_1, e_i]$ are Chernikov for every $i = 1, \ldots, s$. Therefore $[R_1, E]$ is Chernikov and we can pass to the quotient $H\langle a \rangle / [R_1, E]$. Without loss of generality we assume that $[R_1, E] = 1$. In this case, $R_1 \leq Z(E)$ and $E/Z(E)$ is finitely generated. Lemma 3.3 shows that $E/Z(E)$ is finite. The Schur Theorem now tells us that the derived group E' is finite. We see that D is a Chernikov group generated by elements of the same order and its derived group D' is finite. It follows that D is finite. Now Lemma 2.9 enables us to deduce that E is finite. This completes the proof. $\square$

Corollary 3.5. *If $g \in H$, then $\langle g^H \rangle$ is Chernikov.*

Proof. This follows directly from Lemma 3.4 and the fact that every element of H is a product of finitely many elements from X. $\square$

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Combining Corollary 3.5 with Lemma 2.8 we deduce that all abelian radicable subgroups of H generate an abelian radicable subgroup. This will be denoted by T.

To complete the proof of Theorem 1.1 we need to show that $\langle b^H \rangle$ is Chernikov for every $b \in G$. Thus, let $b \in G$. Set $B = \langle b^X \rangle$ and $C = \langle b^H \rangle$. By the hypothesis, B is Chernikov. Since T contains all the abelian radicable subgroups of H, the image of B in $H\langle b\rangle/T$ is finite. Therefore Lemma 2.9 shows that also the image of C is finite. Let us define

$$S = \langle [T, b_1, \ldots, b_k] \mid b_i \in X\rangle.$$

For every choice $b_1, \ldots, b_k \in X$ the subgroup $[T, b_1, \ldots, b_k]$ is a radicable subgroup (Lemma 2.4) contained in X (Lemma 2.6). Thus, S is a normal radicable subgroup of G. Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be the list of the radicable subgroups contained in $S \cap X$. Then $S = \langle S_\lambda \mid \lambda \in \Lambda\rangle$. Since $S_\lambda \subseteq X$, we have $[S_\lambda, b] \leq [X, b] \leq B$ for every λ and so we deduce that $[S, b] \leq B$. In particular, $[S, x]$ is Chernikov for every $x \in G$. Set $T_1 = C \cap T$. Now choose in C finitely many conjugates of b, say $c_1, \ldots, c_n$, such that $C = \langle T_1, c_1, \ldots, c_n\rangle$ and the set $c_1T_1, \ldots, c_nT_1$ is normal in C/T_1. Then $[S, C] = [S, c_1] \ldots [S, c_n]$. Since every subgroup $[S, c_i]$ is Chernikov, so is $[S, C]$. Moreover the subgroup $[S, C]$ is normal in $H\langle b\rangle$ and so we can consider the quotient $H\langle b\rangle/[S, C]$. Thus, we assume that $[S, C] = 1$.

Suppose temporarily that $S = 1$. Then T is contained in the kth term of the upper central series of H and Lemma 2.1 shows that actually $T \leq Z(H)$. In this case B_0, the radicable part of B, is normal in $H\langle b\rangle$ and so we can consider the quotient $H\langle b\rangle/B_0$. The image of B in the quotient is finite. By Lemma 2.9 the image of C must be finite as well. This proves the theorem in the particular case where $S = 1$.

Let us now drop the assumption that $S = 1$. The above argument shows that the image of C in $H\langle b\rangle/S$ is Chernikov. Taking into account that $[S, C] = 1$ we deduce that $C/Z(C)$ is Chernikov. Polovickii's Theorem [5, p. 129] now tells us that C', the derived group of C, is Chernikov and we can pass to the quotient $H\langle b\rangle/C'$. Thus, we assume that $C' = 1$. Now C is an abelian group generated by elements of the same order (namely, of order equal to that of b). We deduce that C has finite exponent. Taking into account that $B \leq C$ we observe that B is a Chernikov group of finite exponent. Hence, B is finite. But then by Lemma 2.9, C must be finite as well. The proof is now complete. □

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