

On the diophantine equation $xy + yz + zx = m$

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Let m, n be arbitrary fixed positive integers. K. KOVÁCS [1] investigated the diophantine equation

$$(1) \quad \sum_{1 \leq i < j \leq n} x_i x_j = m$$

for x_i positive integers. The case $n = 1$ or $n = 2$ is trivial. For $n > 3$, KOVÁCS proved that (1) has a solution if $m \geq 136n^2$. In the case $n = 3$, the problem is still open. KOVÁCS examined that the equation

$$(2) \quad xy + yz + zx = m$$

has a solution for all $m \leq 10^7$ except $m = 1, 2, 4, 18, 22, 30, 42, 58, 70, 78, 102, 130, 190, 210, 330$ and 462 .

In this note we prove the following theorem using the properties of quadratic residues and the Chinese remainder theorem.

Theorem. *Let $E(X)$ be the number of $m \leq X$ for which (2) has no solutions. Then for any $\varepsilon > 0$*

$$E(X) = O\left(X 2^{-(1-\varepsilon)(\log X)/\log \log X}\right).$$

PROOF. It is easy to see that the equation (2) has a solution if and only if there exist $x, y \geq 1, xy < m$ such that

$$(3) \quad m \equiv xy \pmod{x+y}.$$

Replacing $x + y = t$ the congruence (3) goes over into

$$(4) \quad x^2 \equiv -m \pmod{t}, \quad 1 \leq x < t, \quad x(t-x) < m.$$

For any $Y \leq \sqrt{x}$ and prime $p \leq Y$ let $S_p(X, Y) = \{Y^2 \leq m \leq X \mid (4) \text{ has a solution for } p\}$.

Each reduced residue system mod p contains exactly $(p-1)/2$ quadratic residues which yields that

$$(5) \quad |S_p(X, Y)| = \frac{1}{2}(1 - 1/p)X + O(Y^2) + O((p-1)/2).$$

Using the Chinese remainder theorem for the primes $p, q, r, \dots \leq Y$ and a simple sieve we have

$$\begin{aligned} X - E(X) &\geq \sum_{p \leq Y} |S_p(X, Y)| - \sum_{p < q \leq Y} |S_p(X, Y) \cap S_q(X, Y)| \\ &\quad + \sum_{p < q < r \leq Y} |S_p(X, Y) \cap S_q(X, Y) \cap S_r(X, Y)| - \dots \\ &= X \left(1 - \prod_{p \leq Y} \left(1 - \frac{1}{2}(1 - 1/p) \right) \right) + O(Y^2 2^{\pi(Y)}) \\ &\quad + O \left(\prod_{p \leq Y} (1 + (p-1)/2) \right) \\ &= X + O(X(\log Y)/2^{\pi(Y)}) + O(Y^2 2^{\pi(Y)}) \\ &\quad + O(e^{\psi(Y)}(\log Y)/2^{\pi(Y)}). \end{aligned}$$

Using the well-known results $\prod_{p \leq Y} (1 + 1/p) = O(\log Y)$, $\pi(Y) = \sum_{p \leq Y} 1 = Y/\log Y(1+o(1))$ and $\psi(Y) = \sum_{p \leq Y} \log p = Y(1+o(1))$ the choice $X = e^{\psi(Y)}$ implies

$$E(X) = O(X 2^{-(1-\varepsilon)(\log X)/\log \log X}).$$

References

- [1] K. KOVÁCS, About some positive solutions of the diophantine equation $\sum_{1 \leq i < j \leq n} a_i a_j = m$, *Publ. Math. Debrecen* **40** (1992), 207–210.

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