

The convergence of the sequences coding the ground model reals

By MILOŠ S. KURILIĆ (Novi Sad) and ALEKSANDAR PAVLOVIĆ (Novi Sad)

Abstract. We investigate the convergence λ_1 on a complete Boolean algebra $\mathbb{B}$ defined in the following way: a sequence $x = \langle x_n : n \in \omega \rangle$ in $\mathbb{B}$ converges to the point $\limsup x$ of $\mathbb{B}$, if in each generic extension $V_{\mathbb{B}}[G]$ the real coded by the name $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ belongs to the ground model V ; otherwise, x has no limit points. It is shown that λ_1 generates the same topology as the convergence $\bar{\lambda}_4$, generalizing the sequential convergence on the Aleksandrov cube and that for a c.B.a. $\mathbb{B}$ the following conditions are equivalent: (1) The algebra $\mathbb{B}$ is $(\omega, 2)$ -distributive; (2) The $(L2)$ -closure of λ_1 , $\bar{\lambda}_1$, is a topological convergence; (3) $\bar{\lambda}_1 = \bar{\lambda}_4$; (4) $\lambda_1 = \lambda_4$; and, for the algebras satisfying $\text{hcc}(\mathbb{B}) > \mathfrak{c}$, (5) $\bar{\lambda}_1$ is a weakly topological convergence. Also, it is shown that the convergence $\bar{\lambda}_1$ is not weakly topological, if forcing by $\mathbb{B}$ produces splitting reals.

1. Preliminaries

Topologies and convergence structures on Boolean algebras as well as the interplay between the topological, algebraic and forcing-related properties of Boolean algebras are extensively investigated. The results concerning this interplay are useful because, for example, they enable us to attack algebraic problems by topological methods (see e.g. [3]) or topological problems using the techniques of forcing [7].

In this paper we investigate the convergence λ_1 on an arbitrary complete

Mathematics Subject Classification: 54A20, 03E40, 03E75, 54D55, 54A10.

Key words and phrases: convergence structure, Boolean algebra, forcing.

The research supported by the Ministry of Education and Science of the Republic of Serbia under grants 174006.

Boolean algebra $\mathbb{B}$ defined in the following way: a sequence $x = \langle x_n : n \in \omega \rangle$ converges to $\limsup x$, if $1 \Vdash \tau_x \in V$, where $\tau_x = \{ \langle \check{n}, x_n \rangle : n \in \omega \}$ and, otherwise, x has no limit points. In addition, we compare this convergence with some convergences considered in [8]. One of them is the algebraic convergence [11], [1] related to the von Neumann and the Maharam problem and generalizing the convergence on the Cantor cube; another one is a generalization of the convergence on the Aleksandrov cube considered in [9].

Our notation is mainly standard. So, ω denotes the set of natural numbers, Y^X the set of all functions $f : X \rightarrow Y$ and $\omega^{\uparrow\omega}$ the set of all strictly increasing functions from ω into ω . A *sequence* in a set X is each function $x : \omega \rightarrow X$; instead of $x(n)$ we usually write x_n and also $x = \langle x_n : n \in \omega \rangle$. The *constant sequence* $\langle a, a, a, \dots \rangle$ is denoted by $\langle a \rangle$. If $f \in \omega^{\uparrow\omega}$, the sequence $y = x \circ f$ is said to be a *subsequence* of the sequence x and we write $y \prec x$.

If $\langle X, \mathcal{O} \rangle$ is a topological space, a point $a \in X$ is said to be a *limit point* of a sequence $x \in X^\omega$ (we will write: $x \rightarrow_{\mathcal{O}} a$) iff each neighborhood U of a contains all but finitely many members of the sequence. A space $\langle X, \mathcal{O} \rangle$ is called *sequential* iff a set $A \subset X$ is closed whenever it contains each limit of each sequence in A .

If X is a non-empty set, each mapping $\lambda : X^\omega \rightarrow P(X)$ is a *convergence* on X and the mapping $u_\lambda : P(X) \rightarrow P(X)$, defined by $u_\lambda(A) = \bigcup_{x \in A^\omega} \lambda(x)$, the *operator of sequential closure* determined by λ . If λ_1 is another convergence on X , then we will write $\lambda \leq \lambda_1$ iff $\lambda(x) \subset \lambda_1(x)$, for each sequence $x \in X^\omega$. Clearly, $\leq$ is a partial order on the set $\text{Conv}(X) = \{ \lambda : \lambda \text{ is a convergence on } X \}$.

If $\langle X, \mathcal{O} \rangle$ is a topological space, then the mapping $\lim_{\mathcal{O}} : X^\omega \rightarrow P(X)$ defined by $\lim_{\mathcal{O}}(x) = \{ a \in X : x \rightarrow_{\mathcal{O}} a \}$ is the *convergence on X determined by the topology $\mathcal{O}$* and for the operator $\lambda = \lim_{\mathcal{O}}$ we have (see [2])

- (L1) $\forall a \in X \quad a \in \lambda(\langle a \rangle)$;
- (L2) $\forall x \in X^\omega \quad \forall y \prec x \quad \lambda(x) \subset \lambda(y)$;
- (L3) $\forall x \in X^\omega \quad \forall a \in X \quad ((\forall y \prec x \quad \exists z \prec y \quad a \in \lambda(z)) \Rightarrow a \in \lambda(x))$.

A convergence $\lambda : X^\omega \rightarrow P(X)$ is called a *topological convergence* iff there is a topology $\mathcal{O}$ on X such that $\lambda = \lim_{\mathcal{O}}$. The following fact (see, for example, [8]) shows that each convergence has a minimal topological extension and connects topological and convergence structures.

Fact 1.1. *Let $\lambda : X^\omega \rightarrow P(X)$ be a convergence on a non-empty set X . Then*

- (a) *There is the maximal topology $\mathcal{O}_\lambda$ on X satisfying $\lambda \leq \lim_{\mathcal{O}}$;*
- (b) $\mathcal{O}_\lambda = \{ O \subset X : \forall x \in X^\omega \quad (O \cap \lambda(x) \neq \emptyset \Rightarrow \exists n_0 \in \omega \quad \forall n \geq n_0 \quad x_n \in O) \}$;

- (c) $\langle X, \mathcal{O}_\lambda \rangle$ is a sequential space;
- (d) $\mathcal{O}_\lambda = \{X \setminus F : F \subset X \wedge u_\lambda(F) = F\}$, if λ satisfies (L1) and (L2);
- (e) $\lim_{\mathcal{O}_\lambda} = \min\{\lambda' \in \text{Conv}(X) : \lambda' \text{ is topological and } \lambda \leq \lambda'\}$;
- (f) $\mathcal{O}_{\lim_{\mathcal{O}_\lambda}} = \mathcal{O}_\lambda$;
- (g) If $\lambda_1 : X^\omega \rightarrow P(X)$ and $\lambda_1 \leq \lambda$, then $\mathcal{O}_\lambda \subset \mathcal{O}_{\lambda_1}$;
- (h) λ is a topological convergence iff $\lambda = \lim_{\mathcal{O}_\lambda}$.

In our proofs we will mainly use the technique of *forcing* (see [4]). So, if $\mathbb{B}$ is a complete Boolean algebra belonging to the ground model V of ZFC, $V^\mathbb{B}$ will be the class of $\mathbb{B}$ -names. For a formula $\varphi(v_0, \dots, v_n)$ and $\tau_0, \dots, \tau_n \in V^\mathbb{B}$ the corresponding *Boolean value* will be denoted by $\|\varphi(\tau_0, \dots, \tau_n)\|$. If G is a $\mathbb{B}$ -generic filter over V and $\tau \in V^\mathbb{B}$, the G -evaluation of τ will be denoted by τ_G . For $A \in V$, the corresponding $\mathbb{B}$ -name will be $\check{A} = \{\langle a, 1 \rangle : a \in A\}$.

Subsets of ω are called *reals* and can be coded by convenient names. Namely, each real belonging to a generic extension has a nice name of the form $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$, where $x_n = \|\check{n} \in \tau\|$, for each $n \in \omega$.

A real $r \in [\omega]^\omega \cap V_\mathbb{B}[G]$ will be called: *new* iff $r \notin V$; *old* iff $r \in V$; *dependent* iff there is $A \in [\omega]^\omega \cap V$ such that $A \subset r$ or $A \subset \omega \setminus r$; *independent* or a *splitting real* iff it is not dependent [6]; *supported* iff there is $A \in [\omega]^\omega \cap V$ such that $A \subset r$; *unsupported* iff it is not supported [5]. Using the elementary properties of forcing it is easy to prove the following two facts (see [9])

Fact 1.2. Let $x = \langle x_n : n \in \omega \rangle$ be a sequence in a complete Boolean algebra $\mathbb{B}$ and $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ the corresponding $\mathbb{B}$ -name for a subset of ω . Then

- (a) $\|\tau_x = \check{\omega}\| = \bigwedge_{n \in \omega} x_n$;
- (b) $\|\tau_x \text{ is cofinite}\| = \bigvee_{k \in \omega} \bigwedge_{n \geq k} x_n (= \liminf x)$;
- (c) $\|\tau_x \text{ is old infinite}\| = \bigvee_{A \in [\omega]^\omega} \bigwedge_{n \in \omega} x_n^{\chi_A(n)}$; where $x_n^1 = x_n$, $x_n^0 = x'_n$.
- (d) $\|\tau_x \text{ is supported}\| = \bigvee_{A \in [\omega]^\omega} \bigwedge_{n \in A} x_n$;
- (e) $\|\tau_x \text{ is dependent}\| = \bigvee_{A \in [\omega]^\omega} (\bigwedge_{n \in A} x_n \vee \bigwedge_{n \in A} x'_n)$;
- (f) $\|\tau_x \text{ is infinite}\| = \bigwedge_{k \in \omega} \bigvee_{n \geq k} x_n (= \limsup x)$;
- (g) $\|\tau_x = \check{\omega}\| \leq \|\tau_x \text{ is cofinite}\| \leq \|\tau_x \text{ is old infinite}\| \leq \|\tau_x \text{ is supported}\| \leq \|\tau_x \text{ is infinite dependent}\| \leq \|\tau_x \text{ is infinite}\|$.

PROOF. We prove (c) and the rest of the proof is similar.

$$\begin{aligned} \|\tau_x \text{ is old infinite}\| &= \|\exists A \in ([\omega]^\omega)^{V^*} (\forall n \in A (n \in \tau_x) \wedge \forall n \in \check{\omega} \setminus A (n \notin \tau_x))\| = \\ &= \bigvee_{A \in [\omega]^\omega} (\bigwedge_{n \in A} x_n \wedge \bigwedge_{n \in \omega \setminus A} x'_n) = \bigvee_{A \in [\omega]^\omega} \bigwedge_{n \in \omega} x_n^{\chi_A(n)}. \quad \square \end{aligned}$$

Fact 1.3. *If $x = \langle x_n : n \in \omega \rangle$ is a sequence in a c.B.a. $\mathbb{B}$ and $f \in \omega^{\uparrow\omega}$, then $y = x \circ f$ is a subsequence of x and for the $\mathbb{B}$ -names τ_x and τ_y we have*

- (a) $1 \Vdash \tau_y = f^{-1}[\tau_x]$;
- (b) $\limsup y = \|\|f[\omega]^\sim \cap \tau_x\| = \check{\omega}\|$;
- (c) $\liminf y = \|f[\omega]^\sim \subset^* \tau_x\|$;
- (d) $\liminf x \leq \liminf y \leq \limsup y \leq \limsup x$.

2. The convergence λ_1

First, choosing a convenient notation, we present this research in the context of some previous results. Let $\mathbb{B}$ be a complete Boolean algebra and let the convergences $\lambda_i : \mathbb{B}^\omega \rightarrow P(\mathbb{B})$, for $i \in \{0, 1, 2, 3, 4\}$, be defined by

$$\lambda_i(x) = \begin{cases} \{b_4(x)\} & \text{if } b_i(x) = b_4(x), \\ \emptyset & \text{if } b_i(x) < b_4(x), \end{cases} \quad (1)$$

where

$$\begin{aligned} b_0(x) &= \|\tau_x \text{ is cofinite}\| = \liminf x, \\ b_1(x) &= \|\tau_x \text{ is old infinite}\|, \\ b_2(x) &= \|\tau_x \text{ is supported}\|, \\ b_3(x) &= \|\tau_x \text{ is infinite dependent}\|, \\ b_4(x) &= \|\tau_x \text{ is infinite}\| = \limsup x. \end{aligned}$$

Then λ_0 is the well known *algebraic convergence* [11] generating the *sequential topology* $\mathcal{O}_{\lambda_0}$ on $\mathbb{B}$ [1] related to the von-Neumann and the Maharam problem, λ_1 will be considered in this paper and the convergences λ_2, λ_3 and λ_4 were investigated in [8] and [9] and are related in the following way (see [8]).

Fact 2.1. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

- (a) $\lambda_2 \leq \lambda_3 \leq \lambda_4$;
- (b) λ_2, λ_3 and λ_4 satisfy condition (L1), but do not satisfy (L2);
- (c) $\lambda_2 = \lambda_3$ iff $\lambda_2 = \lambda_4$ iff the algebra $\mathbb{B}$ is $(\omega, 2)$ -distributive;
- (d) $\lambda_3 = \lambda_4$ iff forcing by $\mathbb{B}$ does not produce splitting reals.

Thus $\lambda_1(x) = \{\limsup x\}$, if $\|\tau_x \text{ is old infinite}\| = \|\tau_x \text{ is infinite}\|$ and $\lambda_1(x) = \emptyset$, otherwise. Preliminarily we have

Theorem 2.2. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

(a) *For each sequence x in $\mathbb{B}$ we have*

$$\lambda_1(x) = \begin{cases} \{\limsup x\} & \text{if } 1 \Vdash \tau_x \text{ is old,} \\ \emptyset & \text{otherwise;} \end{cases}$$

(b) $\lambda_0 \leq \lambda_1 \leq \lambda_2$;

(c) *The convergence λ_1 satisfies condition (L1), but does not satisfy (L2);*

(d) $\lambda_1 = \lambda_2$ *iff the algebra $\mathbb{B}$ is $(\omega, 2)$ -distributive;*

(e) $\lambda_1 = \lambda_4$ *iff the algebra $\mathbb{B}$ is $(\omega, 2)$ -distributive.*

PROOF. (a)

$$\begin{aligned} \lambda_1(x) \neq \emptyset &\Leftrightarrow \|\tau_x \text{ is infinite}\| \wedge \|\tau_x \text{ is old}\| = \|\tau_x \text{ is infinite}\| \\ &\Leftrightarrow \|\tau_x \text{ is infinite}\| \leq \|\tau_x \text{ is old}\| \\ &\Leftrightarrow 1 \Vdash \tau_x \text{ is infinite} \Rightarrow \tau_x \text{ is old} \\ &\Leftrightarrow 1 \Vdash \tau_x \text{ is finite} \vee \tau_x \text{ is old} \\ &\Leftrightarrow 1 \Vdash \tau_x \text{ is old.} \end{aligned}$$

(b) follows from Fact 1.2(g).

(c) For a constant sequence $x = \langle a \rangle$ we have $a \Vdash \tau_x = \check{\omega}$ and $a' \Vdash \tau_x = \check{\emptyset}$, which implies $1 \Vdash \tau_x \text{ is old}$. Since $\limsup x = a$, by (a) we have $a \in \lambda_1(\langle a \rangle)$ and (L1) holds. For the sequence $x = \langle 1, 0, 1, 0, \dots \rangle$ we have $1 \Vdash \tau_x = \{0, 2, 4, \dots\} \in V$ and, by (a), $\lambda_1(x) = \{\limsup x\} = \{1\}$. But $y = \langle 0, 0, 0, \dots \rangle \prec x$ and $1 \Vdash \tau_y = \check{\emptyset}$, which, by (a), implies $\lambda_1(y) = \{0\} \not\subset \lambda_1(x)$.

(d) By Theorem 7.5 of [8] for each sequence x in $\mathbb{B}$ we have

$$\lambda_2(x) = \begin{cases} \{\limsup x\} & \text{if } 1 \Vdash \tau_x \text{ is finite or supported,} \\ \emptyset & \text{otherwise.} \end{cases} \quad (2)$$

($\Leftarrow$) If $\mathbb{B}$ is $(\omega, 2)$ -distributive, it does not produce new reals and, hence, for each sequence x in $\mathbb{B}$ we have $1 \Vdash \tau_x \text{ is old}$ and, clearly, $1 \Vdash \tau_x \text{ is finite or supported}$. So, by (a) and (2), $\lambda_1(x) = \{\limsup x\} = \lambda_2(x)$.

($\Rightarrow$) Suppose that the algebra $\mathbb{B}$ is not $(\omega, 2)$ -distributive. Then there is an extension $V_{\mathbb{B}}[G]$ containing a new set $X \subset \omega$. Let σ be a $\mathbb{B}$ -name such that $X = \sigma_G$ and $1 \Vdash \sigma \subset \check{\omega}$ and let $b \in G$, where

$$b \Vdash \sigma \text{ is new.} \quad (3)$$

If $y = \langle y_n : n \in \omega \rangle$, where $y_n = \|\check{n} \in \sigma\|$, $n \in \omega$, for $\tau_y = \{\langle \check{n}, y_n \rangle : n \in \omega\}$ we have

$$1 \Vdash \sigma = \tau_y. \quad (4)$$

For $x = \langle y_0, 1, y_1, 1, y_2, 1 \dots \rangle$ we have $1 \Vdash \{1, 3, 5, \dots\} \subset \tau_x$ and, hence, $1 \Vdash \tau_x$ is supported, which, by (2) implies $\lambda_2(x) \neq \emptyset$.

On the other hand $y = x \circ f$, where $f : \omega \rightarrow \omega$ is defined by $f(k) = 2k$, so, by Fact 1.3(a), $1 \Vdash \tau_y = \check{f}^{-1}[\tau_x]$, which, together with (3) and (4), implies $b \Vdash \check{f}^{-1}[\tau_x]$ is new. Now, since $f \in V$, we have $b \Vdash \tau_x$ is new and, by (a), $\lambda_1(x) = \emptyset$. So $\lambda_1 \neq \lambda_2$.

(e) follows from (d) and Fact 2.1(c). $\square$

Remark 2.3. Imitating the proof of the part (a) of the previous theorem one can easily show that, if $\mathbb{B}$ is a complete Boolean algebra, x a sequence in $\mathbb{B}$ and τ_x the corresponding name for a real, then the real determined by τ_x is

- always old iff $\lambda_1(x) \neq \emptyset$;
- sometimes new, but always supported iff $\lambda_1(x) = \emptyset$ and $\lambda_2(x) \neq \emptyset$;
- sometimes unsupported, but always unsplitting iff $\lambda_2(x) = \emptyset$ and $\lambda_3(x) \neq \emptyset$;
- sometimes splitting iff $\lambda_3 = \emptyset$ and $\lambda_4(x) \neq \emptyset$.

(Here “always” means in each and “sometimes” in some generic extension.)

By Fact 2.1(a) and Theorem 2.2(b) we have $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$; by Fact 2.1(c), $\lambda_2 = \lambda_3 < \lambda_4$ is impossible and, by Fact 2.1(c) and Theorem 2.2(d), $\lambda_1 = \lambda_2$ iff $\lambda_2 = \lambda_3$. Now, using Fact 2.1(c), (d) and Theorem 2.2(d), we show that, up to these restrictions, everything is possible.

Example 2.4. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ holds in each $(\omega, 2)$ -distributive and, in particular, each atomic complete Boolean algebra.

$\lambda_1 < \lambda_2 < \lambda_3 = \lambda_4$ holds in each complete Boolean algebra which produces new reals, but does not produce splitting reals, for example in $\text{r.o.}(\mathbb{P})$, where $\mathbb{P}$ is the Sacks or the Miller forcing.

$\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ holds in each complete Boolean algebra which produces splitting reals, for example in $\text{r.o.}(\mathbb{P})$, where $\mathbb{P}$ is the Cohen or the random forcing.

3. The closure of λ_1 under (L2)

By Theorem 2.2(c), the convergence λ_1 does not satisfy (L2) and, hence, it is never a topological convergence. The minimal closures of a convergence under (L2) and (L3) are described in the following general fact (see [8]).

Fact 3.1. *Let $\lambda : X^\omega \rightarrow P(X)$ be a convergence satisfying condition (L1). Then*

- (a) *The mapping $\bar{\lambda} : X^\omega \rightarrow P(X)$ defined by $\bar{\lambda}(y) = \bigcup_{x \in X^\omega, f \in \omega^{\uparrow\omega}, y = x \circ f} \lambda(x)$ is the minimal convergence bigger than λ and satisfying (L1) and (L2);*
- (b) *$\bar{\lambda}^* : X^\omega \rightarrow P(X)$ defined by $\bar{\lambda}^*(y) = \bigcap_{f \in \omega^{\uparrow\omega}} \bigcup_{g \in \omega^{\uparrow\omega}} \bar{\lambda}(y \circ f \circ g)$ is the minimal convergence bigger than $\bar{\lambda}$ and satisfying (L1)–(L3);*
- (c) $\lambda \leq \bar{\lambda} \leq \bar{\lambda}^* \leq \lim_{\mathcal{O}_\lambda}$;
- (d) $\mathcal{O}_\lambda = \mathcal{O}_{\bar{\lambda}} = \mathcal{O}_{\bar{\lambda}^*}$.

For a subset A of a complete Boolean algebra $\mathbb{B}$ let $A \uparrow = \{b \in \mathbb{B} : \exists a \in A, a \leq b\}$. The (L2)-closures of the convergences λ_2 , λ_3 and λ_4 are described in the following fact (see [8] and [9]).

Fact 3.2. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

- (a) $\bar{\lambda}_4(y) = \{\limsup y\} \uparrow$, for each sequence y in $\mathbb{B}$;
- (b) $\bar{\lambda}_2 = \bar{\lambda}_3 = \bar{\lambda}_4$;
- (c) *The convergence $\bar{\lambda}_4$ generalizes the convergence on the Aleksandrov cube.*

Now, concerning the convergence λ_1 we have

Theorem 3.3. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

- (a) *The closure of λ_1 under (L2) is given by*

$$\bar{\lambda}_1(y) = \begin{cases} \{\limsup y\} \uparrow & \text{if } 1 \Vdash \tau_y \text{ is old,} \\ \emptyset & \text{otherwise;} \end{cases} \quad (5)$$

- (b) $\bar{\lambda}_1 = \bar{\lambda}_4$ *iff the algebra $\mathbb{B}$ is $(\omega, 2)$ -distributive.*

PROOF. (a)

Claim 1. $\bar{\lambda}_1(y) = \{\limsup y\} \uparrow$ if and only if $1 \Vdash \tau_y$ is old.

Proof of Claim 1. $(\Rightarrow)$ Let $\bar{\lambda}_1(y) = \{\limsup y\} \uparrow$. Then, by Fact 3.1(a) the set $\bar{\lambda}_1(y) = \bigcup_{x \in \mathbb{B}^\omega, f \in \omega^{\uparrow\omega}, y = x \circ f} \lambda_1(x)$ is nonempty and, hence there are $x \in \mathbb{B}^\omega$ and $f \in \omega^{\uparrow\omega}$ such that $y = x \circ f$ and $\lambda_1(x) \neq \emptyset$. By Theorem 2.2(a), $1 \Vdash \tau_x$ is old” and by Fact 1.3(a), $1 \Vdash \tau_y = f^{-1}[\tau_x]$, which implies $1 \Vdash \tau_y$ is old”.

$(\Leftarrow)$ Let $1 \Vdash \tau_y$ is old”. According to Fact 3.1(a) we show that

$$\bigcup_{x \in \mathbb{B}^\omega, f \in \omega^{\uparrow\omega}, y = x \circ f} \lambda_1(x) = \{\limsup y\} \uparrow.$$

$(\subset)$ Suppose that $x \in \mathbb{B}^\omega$, $f \in \omega^{\uparrow\omega}$, $y = x \circ f$ and $b \in \lambda_1(x)$. Then $b = \limsup x$ and, since $y \prec x$, by Fact 1.3(d) we have $\limsup y \leq \limsup x = b$.

($\supset$) Let $b \geq \limsup y$. Let $x = \langle y_0, b, y_1, b, y_2, \dots \rangle$ and $f, g \in \omega^{\uparrow\omega}$, where $f(k) = 2k$ and $g(k) = 2k + 1$. Then $y = x \circ f$ and, if $z = x \circ g$, using Facts 1.2(f) and 1.3(b) we have

$$\begin{aligned} \limsup x &= |||\tau_x| = \check{\omega}|| = |||\tau_x \cap \check{f}[\omega]| = \check{\omega}|| \vee |||\tau_x \cap \check{g}[\omega]| = \check{\omega}|| \\ &= |||\tau_y| = \check{\omega}|| \vee |||\tau_z| = \check{\omega}|| = |||\tau_y| = \check{\omega}|| \vee b = b. \end{aligned}$$

So, by Theorem 2.2(a), for a proof that $b \in \lambda_1(x)$ it remains to be shown that $1 \Vdash \text{“}\tau_x \text{ is old”}$, which follows from $1 \Vdash \text{“}\tau_y \text{ is old”}$ and the following subclaim.

Subclaim 1. (i) $b' \Vdash \tau_x = \check{f}[\tau_y]$; (ii) $b \Vdash \tau_x = \check{f}[\tau_y] \cup \{1, 3, 5, \dots\}$.

Proof of Subclaim 1. By Fact 1.3(a) we have $1 \Vdash \tau_y = \check{f}^{-1}[\tau_x]$ and, hence,

$$1 \Vdash \check{f}[\tau_y] \subset \tau_x. \quad (6)$$

Let G be a $\mathbb{B}$ -generic filter over V .

(i) If $b' \in G$, then for $n \in (\tau_x)_G$ we have $x_n \in G$ and, since $b \notin G$, there is $k \in \omega$ such that $x_n = x_{2k} = y_k$. Hence $k \in (\tau_y)_G$ and $n = f(k) \in f[(\tau_y)_G]$. So $b' \Vdash \tau_x \subset \check{f}[\tau_y]$ and, by (6), $b' \Vdash \tau_x = \check{f}[\tau_y]$.

(ii) Clearly, $b \Vdash \{1, 3, 5, \dots\} \subset \tau_x$ and, by (6), $b \Vdash \check{f}[\tau_y] \subset \tau_x$. On the other hand, let $b \in G$ and $n \in (\tau_x)_G$, that is $x_n \in G$. If n is odd, we are done. Otherwise, as in (a) we show that $n \in f[(\tau_y)_G]$. Claim 1 is proved.

Claim 2. $\bar{\lambda}_1(y) \neq \emptyset \Leftrightarrow 1 \Vdash \tau_y \text{ is old}$.

Proof of Claim 2. ($\Rightarrow$) Suppose that $a \in \bar{\lambda}_1(y)$. Then, by Fact 3.1(a), there are $x \in \mathbb{B}^\omega$ and $f \in \omega^{\uparrow\omega}$ such that $y = x \circ f$ and $a \in \lambda_1(x)$, which, by Theorem 2.2(a), implies $1 \Vdash \text{“}\tau_x \text{ is old”}$. By Fact 1.3(a) we have $1 \Vdash \tau_y = f^{-1}[\tau_x]$ and, consequently, $1 \Vdash \text{“}\tau_y \text{ is old”}$.

($\Leftarrow$) If $\bar{\lambda}_1(y) = \emptyset$, then, since $\lambda_1 \leq \bar{\lambda}_1$, we have $\lambda_1(y) = \emptyset$ and, by Theorem 2.2(a), $\neg 1 \Vdash \tau_y \text{ is old}$.

(b) It is well known [4] that $\mathbb{B}$ is $(\omega, 2)$ -distributive iff forcing by $\mathbb{B}$ does not produce new reals, that is $1 \Vdash \tau_y \text{ is old}$, for each sequence y in $\mathbb{B}$. So we apply (a) and Fact 3.2(a). $\square$

4. The topology generated by λ_1

By Theorem 3.3(b) and Fact 3.1(d), if $\mathbb{B}$ is an $(\omega, 2)$ -distributive algebra, then $\mathcal{O}_{\lambda_1} = \mathcal{O}_{\lambda_4}$. In this section we show more, that on each complete Boolean algebra the convergences $\lambda_1, \lambda_2, \lambda_3$ and λ_4 generate the same topology, investigated in [9]. Concerning the convergences $\lambda_0, \lambda_2, \lambda_3$ and λ_4 we have (see [8] and [9])

Fact 4.1. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

- (a) $\mathcal{O}_{\lambda_2} = \mathcal{O}_{\lambda_3} = \mathcal{O}_{\lambda_4}$;
- (b) $\mathcal{O}_{\lambda_4}$ is a sequential T_0 connected compact topology on $\mathbb{B}$;
- (c) $\mathcal{O}_{\lambda_4}$ and its dual generate the sequential topology, $\mathcal{O}_{\lambda_0}$, when $\mathbb{B}$ is a Maharam algebra.

We will use the following general fact (see [8]).

Fact 4.2. *Let $\lambda : X^\omega \rightarrow P(X)$ be a convergence satisfying (L1) and (L2) and let the mappings $u_\lambda^\alpha : P(X) \rightarrow P(X)$, $\alpha \leq \omega_1$, be defined by recursion in the following way: for $A \subset X$*

$$\begin{aligned} u_\lambda^0(A) &= A, \\ u_\lambda^{\alpha+1}(A) &= u_\lambda(u_\lambda^\alpha(A)) \text{ and} \\ u_\lambda^\gamma(A) &= \bigcup_{\alpha < \gamma} u_\lambda^\alpha(A), \text{ for a limit } \gamma \leq \omega_1. \end{aligned}$$

Then $u_\lambda^{\omega_1}$ is the closure operator in the space $\langle X, \mathcal{O}_\lambda \rangle$.

We will say that a subset A of a c.B.a. $\mathbb{B}$ is *upward closed* iff $A = A \uparrow$. A sequence x in $\mathbb{B}$ will be called *decreasing* if $x_0 \geq x_1 \geq x_2 \geq \dots$.

Lemma 4.3. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

- (a) *The set $\bar{\lambda}_1(x)$ is upward closed, for each sequence x in $\mathbb{B}$;*
- (b) *If x is a decreasing sequence in $\mathbb{B}$, then $\lambda_1(x) = \{\bigwedge_{n \in \omega} x_n\}$;*
- (c) *If $A \subset \mathbb{B}$ is an upward closed set, then $u_{\bar{\lambda}_1}(A) = u_{\bar{\lambda}_2}(A)$;*
- (d) *The set $u_{\bar{\lambda}_1}(A)$ is upward closed, for each $A \subset \mathbb{B}$.*

PROOF. (a) follows from Theorem 3.3.

(b) If $x = \langle x_n : n \in \omega \rangle$ is decreasing, then $\limsup x = \bigwedge_{n \in \omega} \bigvee_{k \geq n} x_k = \bigwedge_{n \in \omega} x_n$ and, by Theorem 2.2(a), it remains to be shown that $1 \Vdash \tau_x$ is old. If G is a $\mathbb{B}$ -generic filter over V , then $(\tau_x)_G = \{n : x_n \in G\}$, so if $m < n \in (\tau_x)_G$, then $x_m \geq x_n \in G$, which implies $x_m \in G$ and, hence, $m \in (\tau_x)_G$. Thus $(\tau_x)_G$ is either a finite set or equal to ω and, consequently, belongs to V .

(c) Let $A \subset \mathbb{B}$ be an upward closed set.

($\subset$) Since $\lambda_1 \leq \lambda_2 \leq \bar{\lambda}_2$, by the minimality of $\bar{\lambda}_1$ (see Fact 3.1(a)) we have $\bar{\lambda}_1 \leq \bar{\lambda}_2$ and, hence, $u_{\bar{\lambda}_1}(A) = \bigcup_{x \in A^\omega} \bar{\lambda}_1(x) \subset \bigcup_{x \in A^\omega} \bar{\lambda}_2(x) = u_{\bar{\lambda}_2}(A)$.

($\supset$) By Fact 3.2 we have $\bar{\lambda}_2(x) = \{\limsup x\} \uparrow$. So, for $x \in A^\omega$ we show that $\{\limsup x\} \uparrow \subset u_{\bar{\lambda}_1}(A)$. Let $\limsup x = b$. Then the sequence $t = \langle t_n : n \in \omega \rangle$ defined by

$$t_n = b \vee \bigvee_{k \geq n} x_k$$

is decreasing and, since $t_n \geq x_n \in A$, we have $t \in A^\omega$. Since

$$\bigwedge_{n \in \omega} t_n = b \vee \bigwedge_{n \in \omega} \bigvee_{k \geq n} x_k = b \vee \limsup x = b$$

by (b) we have $b \in \lambda_1(t) \subset \bar{\lambda}_1(t)$ and, by (a), $\{b\} \uparrow \subset \bar{\lambda}_1(t) \uparrow = \bar{\lambda}_1(t) \subset u_{\bar{\lambda}_1}(A)$.

(d) We prove that $u_{\bar{\lambda}_1}(A) \uparrow \subset u_{\bar{\lambda}_1}(A)$. If $b \geq a \in u_{\bar{\lambda}_1}(A)$, then there is $x \in A^\omega$ such that $a \in \bar{\lambda}_1(x)$. By (a) we have $b \in \bar{\lambda}_1(x)$, which implies $b \in u_{\bar{\lambda}_1}(A)$. $\square$

Theorem 4.4. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

- (a) $u_{\bar{\lambda}_1}^{\omega_1}(A) = u_{\bar{\lambda}_2}^{\omega_1}(A)$, for each $A \subset \mathbb{B}$;
- (b) $\mathcal{O}_{\lambda_1} = \mathcal{O}_{\lambda_2} = \mathcal{O}_{\lambda_3} = \mathcal{O}_{\lambda_4}$.

PROOF. (a) ($\subset$) Since $\bar{\lambda}_1 \leq \bar{\lambda}_2$, we have $u_{\bar{\lambda}_1}^{\omega_1}(A) \subset u_{\bar{\lambda}_2}^{\omega_1}(A)$, for each $A \subset \mathbb{B}$.

($\supset$) First, for $A \subset \mathbb{B}$ using induction we show that for each $\alpha \leq \omega_1$

$$u_{\bar{\lambda}_2}^\alpha(u_{\bar{\lambda}_1}(A)) = u_{\bar{\lambda}_1}^\alpha(u_{\bar{\lambda}_1}(A)) \text{ and this set is upward closed.} \quad (7)$$

By Lemma 4.3(d), (7) is true for $\alpha = 0$.

Let $\beta \leq \omega_1$ and suppose that (7) holds for each $\alpha < \beta$.

If β is a limit ordinal, then, by the induction hypothesis, we have

$$u_{\bar{\lambda}_2}^\beta(u_{\bar{\lambda}_1}(A)) = \bigcup_{\alpha < \beta} u_{\bar{\lambda}_2}^\alpha(u_{\bar{\lambda}_1}(A)) = \bigcup_{\alpha < \beta} u_{\bar{\lambda}_1}^\alpha(u_{\bar{\lambda}_1}(A)) = u_{\bar{\lambda}_1}^\beta(u_{\bar{\lambda}_1}(A))$$

and, since the union of upward closed sets is upward closed, (7) is true for β .

If $\beta = \alpha + 1$, then, by the induction hypothesis we have

$$u_{\bar{\lambda}_2}^{\alpha+1}(u_{\bar{\lambda}_1}(A)) = u_{\bar{\lambda}_2}(u_{\bar{\lambda}_2}^\alpha(u_{\bar{\lambda}_1}(A))) = u_{\bar{\lambda}_2}(u_{\bar{\lambda}_1}^\alpha(u_{\bar{\lambda}_1}(A))). \quad (8)$$

By the hypothesis the set $u_{\bar{\lambda}_1}^\alpha(u_{\bar{\lambda}_1}(A))$ is upward closed and, by Lemma 4.3(c),

$$u_{\bar{\lambda}_2}(u_{\bar{\lambda}_1}^\alpha(u_{\bar{\lambda}_1}(A))) = u_{\bar{\lambda}_1}(u_{\bar{\lambda}_1}^\alpha(u_{\bar{\lambda}_1}(A))) = u_{\bar{\lambda}_1}^{\alpha+1}(u_{\bar{\lambda}_1}(A)) \quad (9)$$

and $u_{\bar{\lambda}_2}^\beta(u_{\bar{\lambda}_1}(A)) = u_{\bar{\lambda}_1}^\beta(u_{\bar{\lambda}_1}(A))$ follows from (8) and (9). By Lemma 4.3(d) and (9) this set is upward closed and the proof of (7) is over.

Since $A \subset u_{\bar{\lambda}_1}(A) \subset u_{\bar{\lambda}_1}^{\omega_1}(A)$, by Fact 4.2 we have $u_{\bar{\lambda}_1}^{\omega_1}(u_{\bar{\lambda}_1}(A)) = u_{\bar{\lambda}_1}^{\omega_1}(A)$. Using (7) we obtain $u_{\bar{\lambda}_2}^{\omega_1}(A) \subset u_{\bar{\lambda}_2}^{\omega_1}(u_{\bar{\lambda}_1}(A)) = u_{\bar{\lambda}_1}^{\omega_1}(u_{\bar{\lambda}_1}(A)) = u_{\bar{\lambda}_1}^{\omega_1}(A)$.

(b) By (a) and Fact 4.2 we have $\mathcal{O}_{\bar{\lambda}_1} = \mathcal{O}_{\bar{\lambda}_2}$ and, by Fact 3.1(d), $\mathcal{O}_{\lambda_1} = \mathcal{O}_{\lambda_2}$. By Fact 4.1(a), the other two equalities hold as well. $\square$

Thus the topology $\mathcal{O}_{\lambda_1}$, generated by the convergence λ_1 , has the properties given in Fact 4.1(b) and (c).

5. Topological and weakly topological convergences

In this section we investigate the classes of complete Boolean algebras on which the convergence $\bar{\lambda}_1$ (satisfying conditions (L1) and (L2)) is topological or weakly topological. According to [8], a convergence $\lambda : X^\omega \rightarrow P(X)$ will be called *weakly topological* iff it satisfies conditions (L1) and (L2) and its (L3)-closure, λ^* , is a topological convergence. The following general fact can be found in [8].

Fact 5.1. *A convergence $\lambda : X^\omega \rightarrow P(X)$ satisfying (L1) and (L2) is weakly topological iff $\lambda^* = \lim_{\mathcal{O}_\lambda}$, that is for each $x \in X^\omega$ and $a \in X$*

$$a \in \lim_{\mathcal{O}_\lambda}(x) \Leftrightarrow \forall y \prec x \exists z \prec y \ a \in \lambda(z).$$

By [9], for the convergence $\bar{\lambda}_4$ we have

Fact 5.2. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

- (a) $\bar{\lambda}_4$ is a topological convergence iff the algebra $\mathbb{B}$ is $(\omega, 2)$ -distributive;
- (b) If the algebra $\mathbb{B}$ satisfies $(\bar{h})$, then $\bar{\lambda}_4$ is a weakly topological convergence.

We note that, according to [7], a complete Boolean algebra satisfies condition $(\bar{h})$ iff $\forall x \in \mathbb{B}^\omega \exists y \prec x \forall z \prec y \limsup z = \limsup y$. More about condition $(\bar{h})$ (implied by the ccc) can be found in [10].

For the convergence $\bar{\lambda}_1$ we have the following analogue of Fact 5.2(a).

Theorem 5.3. *$\bar{\lambda}_1$ is a topological convergence iff the algebra $\mathbb{B}$ is $(\omega, 2)$ -distributive.*

PROOF. $(\Rightarrow)$ Let $\bar{\lambda}_1$ be a topological convergence. Then, by Fact 1.1(h), $\bar{\lambda}_1 = \lim_{\mathcal{O}_{\bar{\lambda}_1}}$. By Fact 3.1(d) and Theorem 4.4(b) we have $\mathcal{O}_{\bar{\lambda}_1} = \mathcal{O}_{\lambda_1} = \mathcal{O}_{\lambda_2}$ thus $\bar{\lambda}_1 = \lim_{\mathcal{O}_{\lambda_2}} \geq \lambda_2$. Since $\lambda_1 \leq \lambda_2 \leq \bar{\lambda}_1$, by Fact 3.1(a) we have $\bar{\lambda}_1 = \bar{\lambda}_2$ and, by Fact 3.2(b) and Theorem 3.3(b) the algebra $\mathbb{B}$ is $(\omega, 2)$ -distributive.

$(\Leftarrow)$ follows from Theorem 3.3(b) and Fact 5.2(a). $\square$

Now we deal with the question on which algebras the convergence $\bar{\lambda}_1$ is weakly topological. First we describe its (L3)-closure, $\bar{\lambda}_1^*$, in terms of forcing.

Theorem 5.4. *Let $\mathbb{B}$ be a complete Boolean algebra. Then for $y \in \mathbb{B}^\omega$ we have*

- (a) $\bar{\lambda}_1^*(y) = \bigcap_{A \in [\omega]^\omega} \bigcup_{B \in [A]^\omega \wedge \|\tau_y \cap \check{B} \text{ is old}\| = 1} \|\tau_y \cap \check{B}\| = \check{\omega} \uparrow;$
- (b) $\bar{\lambda}_1^*(y) \neq \emptyset$ iff $\mathcal{D}_y = \{B \in [\omega]^\omega : \|\tau_y \cap \check{B} \text{ is old}\| = 1\}$ is a dense set in the poset $\langle [\omega]^\omega, \subset \rangle$.

PROOF. (a) By Fact 3.1(b), for $y \in \mathbb{B}^\omega$ we prove that

$$\bigcap_{f \in \omega^{\uparrow\omega}} \bigcup_{g \in \omega^{\uparrow\omega}} \bar{\lambda}_1(y \circ f \circ g) = \bigcap_{A \in [\omega]^\omega} \bigcup_{B \in [A]^\omega \wedge \|\tau_y \cap \check{B} \text{ is old}\| = 1} \|\tau_y \cap \check{B}\| = \check{\omega} \uparrow$$

(C) Suppose that for each $f \in \omega^{\uparrow\omega}$ there is $g \in \omega^{\uparrow\omega}$ such that $a \in \bar{\lambda}_1(y \circ f \circ g)$, which, by Theorem 3.3 and Fact 1.3, means that

$$\|\tau_{y \circ f \circ g} \text{ is old}\| = 1 \quad \text{and} \quad a \in \|\tau_y \cap (f \circ g)[\omega]^\sim\| = \check{\omega} \uparrow. \quad (10)$$

Let $A \in [\omega]^\omega$ and let $f \in \omega^{\uparrow\omega}$, where $A = f[\omega]$. By the assumption, there is $g \in \omega^{\uparrow\omega}$ such that (10) holds. Then $B = f[g[\omega]] \subset A$ and, since f and g are injections, $B \in [A]^\omega$. By (10), $a \in \|\tau_y \cap \check{B}\| = \check{\omega} \uparrow$. By Fact 1.3, in each generic extension $V_{\mathbb{B}}[G]$ we have $(\tau_{y \circ f \circ g})_G = (f \circ g)^{-1}[(\tau_y)_G] = (f \circ g)^{-1}[(\tau_y)_G \cap B]$ and, hence, $(\tau_y)_G \cap B = f[g[(\tau_{y \circ f \circ g})_G]]$. Thus

$$\|\tau_{y \circ f \circ g} \text{ is old}\| \Leftrightarrow \|\tau_y \cap \check{B} \text{ is old}\| = 1, \quad (11)$$

which together with (10) implies $\|\tau_y \cap \check{B} \text{ is old}\| = 1$.

(D) Suppose that for each $A \in [\omega]^\omega$ there is $B \in [A]^\omega$ such that

$$\|\tau_y \cap \check{B} \text{ is old}\| = 1 \quad \text{and} \quad a \in \|\tau_y \cap \check{B}\| = \check{\omega} \uparrow. \quad (12)$$

Let $f \in \omega^{\uparrow\omega}$ and $A = f[\omega]$. By the assumption, there is $B \in [A]^\omega$ such that (12) holds. If $g \in \omega^{\uparrow\omega}$ where $g[\omega] = f^{-1}[B]$, then $B = (f \circ g)[\omega]$ and, by (12), we have $a \in \|\tau_y \cap (f \circ g)[\omega]^\sim\| = \check{\omega} \uparrow$. By (11), $\|\tau_{y \circ f \circ g} \text{ is old}\| = 1$, thus $a \in \bar{\lambda}_1(y \circ f \circ g)$.

(b) ($\Rightarrow$) Let $a \in \bar{\lambda}_1^*(y)$ and $A \in [\omega]^\omega$. By (a) there is $B \in [A]^\omega$ such that $\|\tau_y \cap \check{B} \text{ is old}\| = 1$ and $a \geq \|\tau_y \cap \check{B}\| = \check{\omega}$. Thus $B \subset A$ and $B \in \mathcal{D}_y$.

($\Leftarrow$) Let $\mathcal{D}_y$ be a dense set in $\langle [\omega]^\omega, \subset \rangle$ and $a = \|\tau_y\| = \check{\omega}$. Since for each $A \in [\omega]^\omega$ there is $B \in [A]^\omega$ such that $\|\tau_y \cap \check{B} \text{ is old}\| = 1$ and, clearly, $a \geq \|\tau_y \cap \check{B}\| = \check{\omega}$, by (a) we have $a \in \bar{\lambda}_1^*(y)$. $\square$

Theorem 5.5. *If there is a sequence y in $\mathbb{B}$ such that $\|\tau_y \text{ is splitting}\| > 0$, then $\bar{\lambda}_1^*(y) = \emptyset$ and the convergence $\bar{\lambda}_1$ is not weakly topological.*

PROOF. Let $\|\tau_y \text{ is splitting}\| = b > 0$ and suppose that $\bar{\lambda}_1^*(y) \neq \emptyset$. Then, by (b), there is $B \in [\omega]^\omega$ such that $1 \Vdash \tau_y \cap \check{B} \text{ is old}$. But then $b \Vdash \tau_y \cap \check{B} \text{ is old} \wedge \tau_y \text{ is splitting}$, which is impossible. Thus $\bar{\lambda}_1^*(y) = \emptyset$. By Theorem 4.4, $\lim_{\lambda_1}(y) = \lim_{\lambda_4}(y) \supset \limsup y$ and, hence, $\bar{\lambda}_1^*(y) \neq \lim_{\lambda_1}(y)$ so $\bar{\lambda}_1$ is not a weakly topological convergence. $\square$

Concerning the previous theorem we remark that it is possible that the convergence $\bar{\lambda}_1$ is not weakly topological, although forcing by $\mathbb{B}$ does not produce splitting reals (see Example 5.8). In contrast to Fact 5.2(b) we have

Example 5.6. The ccc (and, consequently, condition $(\bar{h})$) does not imply that the convergence $\bar{\lambda}_1$ is weakly topological. The Cohen algebra is ccc, produces splitting reals and, by Theorem 5.5, $\bar{\lambda}_1$ is not a weakly topological convergence.

Now, inside a wide class of complete Boolean algebras, we characterize the algebras on which the convergence $\bar{\lambda}_1$ is weakly topological.

Theorem 5.7. *Let $\mathbb{B}$ be a Boolean algebra such that $\text{hcc}(\mathbb{B}) > \mathfrak{c}$ (i.e. below each $b \in \mathbb{B}^+$ there is an antichain of size $\mathfrak{c}$). Then*

$$\bar{\lambda}_1 \text{ is a weakly topological convergence} \Leftrightarrow \mathbb{B} \text{ is } (\omega, 2)\text{-distributive.} \quad (13)$$

PROOF. ($\Leftarrow$) This implication follows from Theorem 5.3.

($\Rightarrow$) If $\mathbb{B}$ is not $(\omega, 2)$ -distributive, then $b = \|\exists r \subset \check{\omega} (r \text{ is new})\| > 0$ and, by the Maximum Principle, there is a name π such that

$$b \Vdash \pi \subset \check{\omega} \wedge \pi \text{ is new.} \quad (14)$$

We choose an enumeration $[\omega]^\omega = \{S_\alpha : \alpha < \mathfrak{c}\}$, bijections $f_\alpha : S_\alpha \rightarrow \omega$, $\alpha < \mathfrak{c}$, and a maximal antichain under b , $\{b_\alpha : \alpha < \mathfrak{c}\}$. Now, for the $\mathbb{B}$ -name σ defined by $\sigma = \{\langle \check{n}, \bigvee_{\alpha < \mathfrak{c}} (b_\alpha \wedge \|f_\alpha(n)^\sim \in \pi\|) \rangle : n \in \omega\}$ it is easy to prove that $b_\alpha \Vdash \sigma = f_\alpha^{-1}[\pi]$, for $\alpha < \mathfrak{c}$, (see [7, Th. 4, Cl. 1]) and, clearly, $1 \Vdash \sigma = \tau_x$, where $x = \langle x_n : n \in \omega \rangle$ and $x_n = \bigvee_{\alpha < \mathfrak{c}} b_\alpha \wedge \|f_\alpha(n)^\sim \in \pi\|$, $n \in \omega$. Thus

$$b_\alpha \Vdash \tau_x = f_\alpha^{-1}[\pi]. \quad (15)$$

Let us prove

$$\forall B \in [\omega]^\omega \quad \|\tau_x \cap \check{B} \text{ is new}\| > 0. \quad (16)$$

Let $B \in [\omega]^\omega$ and $\alpha < \mathfrak{c}$, where $B = S_\alpha$. Let G be a $\mathbb{B}$ -generic filter over V containing b_α . Since $b_\alpha < b$ we have $b \in G$ and, by (14), $\pi_G \notin V$. By (15), $(\tau_x)_G = f_\alpha^{-1}[\pi_G] \subset B$. Now, $f_\alpha^{-1}[\pi_G] \in V$ would imply $f_\alpha[f_\alpha^{-1}[\pi_G]] = \pi_G \in V$, which is false. Thus $f_\alpha^{-1}[\pi_G] = (\tau_x)_G = (\tau_x)_G \cap B \notin V$ and (16) is proved.

By (16) we have $\mathcal{D}_x = \{B \in [\omega]^\omega : 1 \Vdash \tau_x \cap \check{B} \text{ is old}\} = \emptyset$ so, by Theorem 5.4(b), $\bar{\lambda}_1^*(x) = \emptyset$. But, by Theorem 4.4, $\limsup x \in \lim_{\mathcal{O}_{\lambda_1}}(x) = \lim_{\mathcal{O}_{\bar{\lambda}_1}}(x)$ and, by Fact 5.1, $\bar{\lambda}_1$ is not a weakly topological convergence. $\square$

Example 5.8. $\bar{\lambda}_1$ is not a weakly topological convergence on the Sacks algebra. Namely, if $\mathbb{B}$ is the Boolean completion of the Sacks forcing, $\mathbb{B}$ is homogeneous, has antichains of size $\mathfrak{c}$, adds new reals and we apply Theorem 5.7.

Is the equivalence (13) a theorem of ZFC? In the following theorem, using a result of VELIČKOVIĆ [12], we show that under the CH, a possible counterexample can not be nicely definable.

Theorem 5.9. (CH) *If $\mathbb{B} = r.o.(\mathbb{P})$, where $\mathbb{P}$ is a Suslin forcing notion, then*

$$\bar{\lambda}_1 \text{ is a weakly topological convergence} \Leftrightarrow \mathbb{B} \text{ is } (\omega, 2)\text{-distributive.}$$

PROOF. ($\Leftarrow$) This implication follows from Theorem 5.3.

($\Rightarrow$) If the algebra $\mathbb{B}$ is not $(\omega, 2)$ -distributive, then $b = \|\exists r \subset \check{\omega} \ (r \text{ is new})\| > 0$. If there exists an uncountable antichain below b , then, as in the proof of Theorem 5.7, we show that $\bar{\lambda}_1$ is not a weakly topological convergence. Otherwise, $\mathbb{B}|b$ is a non-atomic ccc forcing, clearly, $\mathbb{P} \cap b \downarrow$ is a non-atomic ccc Suslin forcing and, by a result of VELIČKOVIĆ [12], produces splitting reals. Now, by Theorem 5.5, $\bar{\lambda}_1$ is not a weakly topological convergence again. $\square$

6. A diagram

Here we describe the relations between the convergence structures considered in this paper.

Theorem 6.1. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

- (a) *If $\mathcal{A} \subset [\omega]^\omega$ is a mad family, y a sequence in $\mathbb{B}$ and $1 \Vdash \tau_y$ kills $\check{\mathcal{A}}$, then $\bar{\lambda}_1^*(y) = \mathbb{B}$ and $\bar{\lambda}_4(y) = \{1\}$;*
- (b) *If forcing by $\mathbb{B}$ produces a splitting real in each extension, then the convergences $\bar{\lambda}_1^*$ and $\bar{\lambda}_4$ are not comparable;*

PROOF. (a) Suppose that $1 \Vdash |\tau_y| = \check{\omega} \wedge \forall A \in \check{\mathcal{A}} \ |\tau_y \cap A| < \check{\omega}$. Then $\|\tau_y \text{ is infinite}\| = 1$ and, by Facts 1.2 and 3.2(a), we have $\bar{\lambda}_4(y) = \{1\} \upharpoonright = \{1\}$.

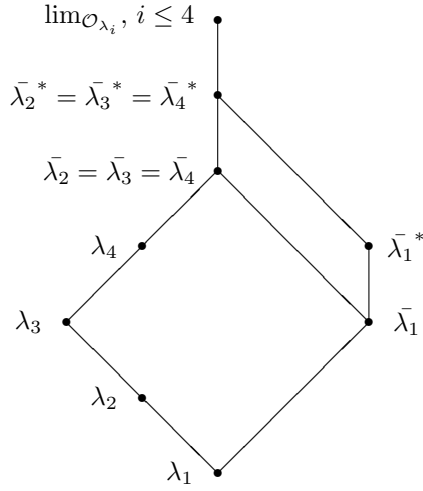
Using Theorem 5.4(a) we prove that $0 \in \bar{\lambda}_1^*(y)$ (which implies $\bar{\lambda}_1^*(y) = \mathbb{B}$). For $A \in [\omega]^\omega$, by the maximality of $\mathcal{A}$, there is $A_1 \in \mathcal{A}$ such that $B = A \cap A_1 \in [A]^\omega$. Since $1 \Vdash |\tau_y \cap A_1| < \check{\omega}$, we have $\|\tau_y \cap \check{B}\| < \check{\omega} = 1$ which implies $\|\tau_y \cap \check{B} \text{ is old}\| = 1$ and $\|\tau_y \cap \check{B}\| = \check{\omega} = 0$.

(b) ($\bar{\lambda}_4 \not\leq \bar{\lambda}_1^*$). By the assumption, there is $y \in \mathbb{B}^\omega$ such that $\|\tau_y \text{ is splitting}\| > 0$ so, by Theorem 5.5, $\bar{\lambda}_1^*(y) = \emptyset$ and $\bar{\lambda}_4(y) \neq \emptyset$.

($\bar{\lambda}_1^* \not\leq \bar{\lambda}_4$). It is known (see [7, Lemma 1]) that there is a mad family $\mathcal{A} \subset [\omega]^\omega$ which is killed in each generic extension of the ground model containing new reals. By the assumption, forcing by $\mathbb{B}$ produces new reals in each extension and, hence,

we have $1 \Vdash \exists x \subset \tilde{\omega}$ (x kills $\check{\mathcal{A}}$) and, by the Maximum Principle, there is a $\mathbb{B}$ -name σ such that $1 \Vdash \sigma \subset \tilde{\omega}$ and $1 \Vdash \sigma$ kills $\check{\mathcal{A}}$. If $y_n = \|\check{n} \in \sigma\|$, $n \in \omega$, then $1 \Vdash \tau_y = \sigma$ and $1 \Vdash \tau_y$ kills $\check{\mathcal{A}}$. By (a) we have $\bar{\lambda}_1^*(y) = \mathbb{B}$ and $\bar{\lambda}_4(y) = \{1\}$. $\square$

In the following diagram $\lambda' \leq \lambda''$ denotes that for each c.B.a. $\mathbb{B}$ and each sequence x in $\mathbb{B}$, $\lambda'(x) \subset \lambda''(x)$.



In the sequel we show that the diagram is correct. By Fact 2.1(a) and Theorem 2.2(b) we have $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ and, by Example 2.4, all the inequalities can be strict. By Fact 3.2(b) we have $\bar{\lambda}_2 = \bar{\lambda}_3 = \bar{\lambda}_4$, which implies $\bar{\lambda}_2^* = \bar{\lambda}_3^* = \bar{\lambda}_4^*$ and $\lim_{\mathcal{O}_{\lambda_2}} = \lim_{\mathcal{O}_{\lambda_3}} = \lim_{\mathcal{O}_{\lambda_4}}$. By Theorem 4.4(b) we have $\lim_{\mathcal{O}_{\lambda_1}} = \lim_{\mathcal{O}_{\lambda_2}}$. By Fact 3.1, $\lambda_1 \leq \lambda_4$ implies $\bar{\lambda}_1 \leq \bar{\lambda}_4$ and $\bar{\lambda}_1^* \leq \bar{\lambda}_4^*$.

The convergence $\bar{\lambda}_1^*$ is not comparable with $\lambda_2, \lambda_3, \lambda_4$ and $\bar{\lambda}_4$. The relation $\bar{\lambda}_1^* \not\leq \bar{\lambda}_4$ is proved in (b) of Theorem 6.1. For a proof that $\bar{\lambda}_1^* \not\geq \lambda_2$ we follow Theorem 5.5. If $y \in \mathbb{B}^\omega$, where $\|\tau_y$ is splitting $\| > 0$, then $\bar{\lambda}_1^*(y) = \emptyset$. For the sequence $z = \langle y_0, 1, y_1, 1, \dots \rangle$ we have $y \prec z$ and, since $\bar{\lambda}_1^*$ fulfills (L2), we have $\bar{\lambda}_1^*(z) = \emptyset$. But $1 \Vdash \{1, 3, 5, \dots\} \subset \tau_z$, thus $1 \Vdash \text{"}\tau_z \text{ is supported"}$ which, by (2), implies $\lambda_2(z) \neq \emptyset$.

The convergence $\bar{\lambda}_1$ is not comparable with λ_2, λ_3 , and λ_4 . Namely, $\bar{\lambda}_1(\langle 0 \rangle) = \{0\} \uparrow$ and $\lambda_4(\langle 0 \rangle) = \{0\}$ implies $\bar{\lambda}_1 \not\leq \lambda_4$. The relation $\bar{\lambda}_1 \not\geq \lambda_2$ follows from $\bar{\lambda}_1^* \not\geq \lambda_2$, proved above. $\square$

Remark 6.2. For the $(\omega, 2)$ -distributive algebras the diagram collapses to the diagram containing two elements, e.g. λ_1 and $\bar{\lambda}_1$ (see Example 2.4 and Fact 5.2(a)).

References

- [1] B. BALCAR, W. GLÓWCZYŃSKI and T. JECH, The sequential topology on complete Boolean algebras, *Fund. Math.* **155** (1998), 59–78.
- [2] R. ENGELKING, General Topology, *P.W.N., Warszawa*, 1985.
- [3] I. FARAH and S. SOLECKI, Two $F_{\sigma\delta}$ ideals, *Proc. Amer. Math. Soc.* **131**, no. 6 (2003), 1971–1975.
- [4] T. JECH, Set Theory, 2. corr. ed., *Springer, Berlin*, 1997.
- [5] M. S. KURILIĆ, Unsupported Boolean algebras and forcing, *Math. Logic Quart.* **50**, no. 6 (2004), 594–602.
- [6] M. S. KURILIĆ, Independence of Boolean algebras and forcing, *Ann. Pure Appl. Logic* **124** (2003), 179–191.
- [7] M. S. KURILIĆ and A. PAVLOVIĆ, A posteriori convergence in complete Boolean algebras with the sequential topology, *Ann. Pure Appl. Logic* **148**, no. 1–3 (2007), 49–62.
- [8] M. S. KURILIĆ and A. PAVLOVIĆ, Some forcing related convergence structures on complete Boolean algebras, *Novi Sad J. Math.* **40**, no. 2 (2010), 77–94.
- [9] M. S. KURILIĆ and A. PAVLOVIĆ, A convergence on Boolean algebras generalizing the convergence on the Aleksandrov cube, submitted.
- [10] M. S. KURILIĆ and S. TODORČEVIĆ, Property (h) and cellularity of complete Boolean algebras, *Arch. Math. Logic* **48**, no. 8 (2009), 705–718.
- [11] D. MAHARAM, An algebraic characterization of measure algebras, *Ann. of Math.* **48** (1947), 154–167.
- [12] B. VELIČKOVIĆ, ccc forcing and splitting reals, *Israel J. Math.* **147** (2005), 209–220.

MILOŠ S. KURILIĆ
DEPARTMENT OF MATHEMATICS
AND INFORMATICS
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4
21000 NOVI SAD
SERBIA

E-mail: milos@dmi.uns.ac.rs

ALEKSANDAR PAVLOVIĆ
DEPARTMENT OF MATHEMATICS
AND INFORMATICS
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4
21000 NOVI SAD
SERBIA

E-mail: apavlovic@dmi.uns.ac.rs

(Received April 4, 2011)