

## On the Diophantine equation $cy^l = \frac{x^p-1}{x-1}$

By MOHAMMAD SADEK (Cairo)

**Abstract.** Let  $p, c$  be distinct odd primes, and  $l \geq 2$  an integer. We find sufficient conditions for the Diophantine equation

$$cy^l = \Phi_p(x) = \frac{x^p-1}{x-1} = x^{p-1} + x^{p-2} + \cdots + 1$$

not to have integer solutions.

### 1. Introduction

The solutions of the Nagell–Ljunggren equation  $y^q = \frac{x^n-1}{x-1}$ , where  $q, n \geq 2$  are integers, have been the source for many conjectures. One of these is the following:

**Conjecture 1.1.** *The only solutions to the Diophantine equation  $y^q = \frac{x^n-1}{x-1}$  in integers  $x, y > 1$ ,  $n > 2$ ,  $q \geq 2$  are given by*

$$\frac{3^5-1}{3-1} = 11^2, \quad \frac{7^4-1}{7-1} = 20^2, \quad \text{and} \quad \frac{18^3-1}{18-1} = 7^3.$$

The above conjecture has been solved completely for  $q = 2$ . Furthermore, it has been proved if one of the following assumptions holds:

$$3 \mid n, \text{ or } 4 \mid n, \text{ or } q = 3 \text{ and } n \not\equiv 5 \pmod{6}.$$

---

*Mathematics Subject Classification:* 11D61.

*Key words and phrases:* Diophantine equations, Ljunggren–Nagell equation, cyclotomic polynomials.

We moreover know that the Nagell–Ljunggren equation has no solutions with  $x$  square. The main tools used to attack this Diophantine equation are effective Diophantine approximation, linear forms in  $p$ -adic logarithms, and Cyclotomic fields theory. For these results and more see [1], [5] and [6].

In [3] the Diophantine equation  $y^l = c \frac{x^n-1}{x-1}$  has been treated. A complete list of such Diophantine equations with integer solutions has been given, under the condition that  $1 \leq c \leq x \leq 100$ . A more general equation  $a \frac{x^n-1}{x-1} = cy^l$  where  $ac > 1$  has been considered in [7]. Our interest in the latter equation is when  $a = 1$ .

In this note we will be concerned with the Diophantine equation  $cy^l = \frac{x^p-1}{x-1}$ , where  $c, p$  are distinct odd primes and  $l \geq 2$ . We exhibit the existence of an infinite set of triples  $(p, c, l)$  for which the mentioned Diophantine equation has no integer solutions. For example, this infinite set contains the set of triples  $(p, c, l)$  where the Legendre symbol  $\left(\frac{c}{p}\right) = -1$  and  $l$  is even.

The key idea is exploiting the following identity satisfied by the cyclotomic polynomial  $\Phi_p(x) = \frac{x^p-1}{x-1}$

$$4\Phi_p(x) = A_p(x)^2 - (-1)^{(p-1)/2} p B_p(x)^2,$$

where  $A_p(x), B_p(x) \in \mathbb{Z}[x]$ . This identity goes back to Gauss, nevertheless the formulae describing  $A_p(x)$  and  $B_p(x)$  were given recently in [2]. Using this identity we show that the existence of an integer solution to the equation in question implies the existence of a proper integer solution to some auxiliary Diophantine equation.

## 2. Factorization of cyclotomic polynomials

For an odd square-free integer  $n > 1$ , and  $|x| \leq 1$  define

$$f_n(x) = \sum_{j=1}^{\infty} \left(\frac{j}{n}\right) \frac{x^j}{j},$$

where  $\left(\frac{j}{n}\right)$  is the Jacobi symbol of  $j \bmod n$ . We state Theorem 1 of [2].

**Theorem 2.1.** *Let  $n > 3$  be an odd square-free integer. Consider the Gauss's identity  $4\Phi_n(x) = A_n(x)^2 - (-1)^{(n-1)/2} n B_n(x)^2$ , where  $A_n(x), B_n(x) \in \mathbb{Z}[x]$ . If  $n \equiv 1 \pmod{4}$ , then*

$$A_n(x) = 2\sqrt{\Phi_n(x)} \cosh\left(\frac{\sqrt{n}}{2} f_n(x)\right), \quad B_n(x) = 2\sqrt{\frac{\Phi_n(x)}{n}} \sinh\left(\frac{\sqrt{n}}{2} f_n(x)\right).$$

If  $n \equiv 3 \pmod{4}$ , then

$$A_n(x) = 2\sqrt{\Phi_n(x)} \cos\left(\frac{\sqrt{n}}{2}f_n(x)\right), \quad B_n(x) = 2\sqrt{\frac{\Phi_n(x)}{n}} \sin\left(\frac{\sqrt{n}}{2}f_n(x)\right).$$

### 3. An auxiliary Diophantine equation

The results of this section are motivated by Proposition 8.1 of [4].

By a proper solution  $(x_0, y_0, z_0)$  to the Diophantine equation  $ax^p + by^q = cz^r$ , we mean three integers  $x_0, y_0, z_0$  such that  $ax_0^p + by_0^q = cz_0^r$  and  $\gcd(x_0, y_0, z_0) = 1$ .

We state the following result on local solutions to  $cy^l = x^2 \pm pz^2$  where  $c, p$  are distinct odd primes and  $l \geq 2$ .

**Proposition 3.1.** *There are proper local solutions to*

$$\alpha^2 cy^l = x^2 \pm pz^2, \quad \alpha \in \{1, 2\},$$

*at every prime if and only if the Legendre symbol  $\left(\frac{\pm p}{c}\right) = 1$ ; and, when  $l$  is even we have  $\left(\frac{c}{p}\right) = 1$ .*

PROOF. The given conditions are clearly necessary. Now we need to prove they are sufficient. We use the fact that if  $q \nmid 2cp$ , then there are  $q$ -adic integer solutions to  $x^2 \pm pz^2 = \alpha^2 c$ , so take  $(x, 1, z)$ . For the prime  $c$ , since  $\left(\frac{\pm p}{c}\right) = 1$ , there are  $c$ -adic integer solutions to  $x^2 = \mp p$ , so take  $(x, 0, 1)$ . For the prime  $p$ , if  $l$  is odd, take  $(\alpha c^{(l+1)/2}, c, 0)$ ; if  $l$  is even, hence  $\left(\frac{c}{p}\right) = 1$ , then there is a  $p$ -adic integer satisfying  $x^2 = \alpha^2 c$ , and we take  $(x, 1, 0)$ . For the prime 2, the equation becomes  $x^2 - z^2 = \alpha^2 y^l$ , so we can lift the solution  $(1, 0, 1) \pmod{2}$  to a 2-adic integer solution.  $\square$

**Proposition 3.2.** *Let  $p, c$  be distinct odd primes, and  $l \geq 2$  be an integer. Set  $\delta = (-1)^{(p-1)/2}$ . If the Diophantine equation*

$$\alpha^2 cy^l = x^2 - \delta pz^2, \quad \alpha \in \{1, 2\},$$

*has a proper solution with  $y$  being odd and  $\gcd(x, y) = 1$ , then there exist coprime ideals  $I, J$  in  $\mathbb{Q}(\sqrt{\delta p})$  with  $IJ = (\alpha^2 c)$ , whose ideal classes are  $l$ -th powers inside the class group of  $\mathbb{Q}(\sqrt{\delta p})$ .*

PROOF. Suppose  $(x, y, z)$  is a proper solution to  $\alpha^2 cy^l = x^2 - \delta pz^2$  where  $y$  is odd and  $\gcd(x, z) = 1$ . Now considering the latter as ideal equation, we have

$$(\alpha^2 c)(y)^l = (x - \sqrt{\delta p} z)(x + \sqrt{\delta p} z).$$

Now the ideal  $\mathfrak{a} = (x - \sqrt{\delta p} z, x + \sqrt{\delta p} z) \mid (2x, 2\sqrt{\delta p}, \alpha^2 cy^l) = (2, \alpha)$ .

1) If  $\alpha = 1$ , then

$$(x - \sqrt{\delta p} z) = IL_1^l, (x + \sqrt{\delta p} z) = JL_2^l,$$

where  $IJ = (c)$  and  $L_1L_2 = (y)$ . This implies that the ideal classes of  $I$  and  $J$  are both  $l$ -th powers inside the class group of  $\mathbb{Q}(\sqrt{\delta p})$ .

2) If  $\alpha = 2$ , then both  $x, z$  are odd. This will yield a contradiction when  $p \equiv \pm 1 \pmod{8}$ . This follows from the fact that  $4cy^l = x^2 - \delta pz^2 \equiv 0 \pmod{8}$  when  $p \equiv \pm 1 \pmod{8}$ .

When  $p \equiv \pm 5 \pmod{8}$ , the ideal (2) is prime inside  $\mathbb{Q}(\sqrt{\delta p})$  because  $\delta p \equiv 5 \pmod{8}$ . If  $\mathfrak{a} = (2)$ , then  $2 \mid (x - \sqrt{\delta p}z)$  which implies that  $2 \mid x, z$ , a contradiction. Thus  $\mathfrak{a} = 1$ , and we argue like in the first case.  $\square$

#### 4. The equation $cy^l = \frac{x^p-1}{x-1}$

We start by stating the following elementary lemma.

**Lemma 4.1.** *Let  $a \in \mathbb{Z}$  and  $p$  be an odd prime.*

- i)  $\Phi_p(a)$  is odd.
- ii) Set  $d = \gcd(A_p(a), B_p(a))$ . Then  $d \in \{1, 2\}$ . If  $p \equiv \pm 1 \pmod{8}$ , then  $d = 2$ .

PROOF. i) Since  $\Phi_p(a) \equiv 1 \pmod{a}$ , hence if  $a$  is even,  $\Phi_p(a)$  is odd. If  $a$  is odd, then  $\Phi_p(a) \equiv \Phi(1) = p \pmod{2}$ .

ii) Assume that  $q \mid d$ , where  $q > 1$  is an odd prime. We will write  $\tilde{a}$  for the reduction of  $a \pmod{q}$ .

If  $q \neq p$ , then  $(x - \tilde{a}) \mid A_p(x), B_p(x) \pmod{q}$  because  $q \mid A_p(a)$  and  $B_p(a)$ . Hence  $(x - \tilde{a})^2 \mid \Phi_p(x) \pmod{q}$ . The latter statement contradicts the fact that  $x^p - 1$  has no multiple factors  $\pmod{q}$  when  $\gcd(q, p) = 1$ .

If  $q = p$ , then  $p^2 \mid \Phi_p(a)$ . In particular  $\tilde{a}^p \equiv 1 \pmod{p}$ . Fermat's Little Theorem yields that there is a  $\lambda \in \mathbb{Z}$  such that  $a = 1 + \lambda p$ . So

$$\Phi_p(a) = \sum_{i=0}^{p-1} a^i = \sum_{i=0}^{p-1} (1 + \lambda p)^i \equiv p + \lambda p \sum_{i=0}^{p-1} i \equiv p \pmod{p^2},$$

which contradicts that  $p^2 \mid \Phi_p(a)$ . We conclude that  $d \mid 2$ .

Now we assume  $p \equiv \pm 1 \pmod{8}$ . Assume on the contrary that  $2 \nmid d$ . This implies that both  $A_p(a)$  and  $B_p(a)$  are odd as  $4 \mid A_p^2(a) - (-1)^{(p-1)/2} p B_p^2(a)$ . A direct calculation shows that if  $A_p(a), B_p(a)$  are both odd, then

$$4\Phi_p(a) = A_p^2(a) - (-1)^{(p-1)/2} p B_p^2(a) \equiv 1 - (-1)^{(p-1)/2} p \equiv 0 \pmod{8},$$

which contradicts (i).  $\square$

**Corollary 4.2.** *Let  $p, c$  be distinct odd primes. Let  $l \geq 2$  be an integer. Assume that  $(a, b)$  is an integer solution to the Diophantine equation  $cy^l = \Phi_p(x)$ . Then there exists an integer solution  $(x, y, z)$ , where  $\gcd(x, z) = 1$  and  $y$  is odd, to a Diophantine equation of the form*

$$\alpha^2 cy^l = x^2 - (-1)^{(p-1)/2} pz^2, \quad \alpha \in \{1, 2\}.$$

In the case  $p \equiv \pm 1 \pmod{8}$ , one has  $\alpha = 1$ .

PROOF. One has  $4cb^l = 4\Phi_p(a) = A_p(a)^2 - (-1)^{(p-1)/2} pB_p(a)^2$ , where  $A_p(x), B_p(x) \in \mathbb{Z}[x]$  and  $d = \gcd(A_p(a), B_p(a)) \mid 2$ , Lemma 4.1. If  $d = 1$ , then  $(A_p(a), b, B_p(a))$  is a proper solution to  $4cy^l = x^2 - (-1)^{(p-1)/2} z^2$ . If  $d = 2$ , then  $(A_p(a)/2, b, B_p(a)/2)$  is a proper solution to  $cy^l = x^2 - (-1)^{(p-1)/2} z^2$ . Observe that if  $p \equiv \pm 1 \pmod{8}$ , then  $d = 2$ , Lemma 4.1 (ii).  $\square$

Now we state our main result which says that there is an infinite number of triples  $(c, p, l)$  such that  $cy^l = \Phi_p(x)$  has no integer solution.

**Theorem 4.3.** *Let  $p, c$  be distinct odd primes, and  $l \geq 2$  an integer. Set  $\delta = (-1)^{(p-1)/2}$ . If the triple  $(p, c, l)$  satisfies one of the following conditions:*

- i)  $\left(\frac{\delta p}{c}\right) = -1$ ;
- ii)  $\left(\frac{c}{p}\right) = -1$ , and  $l$  is even;
- iii) *There exist no ideals  $I, J$  whose ideal classes are  $l$ -th powers in the class group of  $\mathbb{Q}(\sqrt{\delta p})$  and satisfy  $(\alpha^2 c) = IJ$ , where*

$$\alpha \in \begin{cases} \{1\} & \text{if } p \equiv \pm 1 \pmod{8} \\ \{1, 2\} & \text{if } p \equiv \pm 3 \pmod{8} \end{cases}$$

then the Diophantine equation

$$cy^l = \Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$$

has no integer solutions.

PROOF. Assume on the contrary that there exists a proper integer solution to  $cy^l = \Phi_p(x)$ . This implies the existence of a proper integer solution to  $\alpha^2 cy^l = x^2 - \delta pz^2$ , see Corollary 4.2. Hence we have a contradiction in (i) and (ii), see Proposition 3.1. Furthermore one has a contradiction in case (iii) obtained using Proposition 3.2.  $\square$

Parts (i) and (ii) of the above theorem provide an infinite family of Diophantine equations with no integer solutions. For example

$$13y^{2l} = \Phi_{137}(x) = x^{136} + \cdots + x + 1$$

has no integer solutions because  $\left(\frac{13}{137}\right) = -1$ .

In the following example we show that (iii) of Theorem 4.3 can be used to find explicit triples  $(c, l, p)$  such that the Diophantine equation  $cy^l = \Phi_p(x)$  has no integer solutions.

*Example 4.4.* The Diophantine equation

$$3y^{5k} = \Phi_{47}(x) = x^{46} + x^{45} + \cdots + x + 1, \quad k \geq 1,$$

has no integer solutions. We have  $47 \equiv -1 \pmod{8}$  and  $(3) = \mathfrak{p}\mathfrak{p}'$  in the ring of integers of  $\mathbb{Q}(\sqrt{-47})$ . The class number of  $\mathbb{Q}(\sqrt{-47})$  is 5. The ideal class  $[\mathfrak{p}]$  of  $\mathfrak{p}$  can not be a fifth power inside the ideal class group of  $\mathbb{Q}(\sqrt{-47})$  because  $[\mathfrak{p}]$  generates the ideal class group.

## References

- [1] Y. BEGEAUD, M. MIGNOTTE, Y. ROY and T. SHOREY, The equation  $\frac{x^n-1}{x-1} = y^q$  has no solution with  $x$  square, *Math. Proc. Camb. Phil. Soc. (to appear)*.
- [2] R. BRENT, On computing factors of cyclotomic polynomials, *Math. Comp.* **61**(203) (1993), 131–149.
- [3] Y. BUGEAUD, On the Diophantine equation  $a\frac{x^n-1}{x-1} = y^q$ , in: *Number Theory conference held in Turku, De Gruyter*, 19–24, 2001.
- [4] HENRI DARMON and ANDREW GRANVILLE, On the equations  $x^p + y^q = z^r$  and  $z^m = f(x, y)$ , *Bull. London Math. Soc.* **27** (1995), 513–543.
- [5] P. MIHAILESCU, Class number conditions for the diagonal case of the equation of Nagell–Ljunggren, *preprint*.
- [6] P. MIHAILESCU, New bounds and conditions for the equation of Nagell–Ljunggren, *J. Number Theory* **124**(2) (2007), 380–395.
- [7] T. N. SHOREY, The equation  $a\frac{x^n-1}{x-1} = by^q$  with  $ab > 1$ , *Number Theory in Progress*, Vol. 1, *Walter de Gruyter, Berlin*, 1999.

MOHAMMAD SADEK  
DEPARTMENT OF MATHEMATICS  
AND ACTUARIAL SCIENCES  
AMERICAN UNIVERSITY IN CAIRO

*E-mail:* mmsadek@aucegypt.edu

(Received September 9, 2011; revised May 9, 2012)