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# Chen inequalities for submanifolds of a Riemannian manifold of nearly quasi-constant curvature

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**Abstract.** The object of the present paper is to study Chen first inequality and $k$-Ricci curvatures for submanifolds of a Riemannian manifold of nearly quasi-constant curvature.

## 1. Introduction

Let $(M, g)$ be a Riemannian manifold. If its curvature tensor satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) = a[&g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ b[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) \\ &+ g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)], \end{aligned} \tag{1.1}$$

where $a, b$ are scalar functions and $A$ is a 1-form defined by

$$g(X,P) = A(X), \tag{1.2}$$

$P$ is a unit vector field, then we say that $(M, g)$ is a Riemannian manifold of *quasi-constant curvature* [10]. If $b = 0$ then the manifold reduces to a space of constant curvature.

A non-flat Riemannian manifold $(M^n, g)$ $(n > 2)$ is defined to be a *quasi-Einstein manifold* if its Ricci tensor satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y),$$

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where $a, b$ are scalar functions and $A$ is a non-zero 1-form such that $g(X,U) = A(X)$ for every vector field $X$ and $U$ is a unit vector field. If $b = 0$ then the manifold reduces to an Einstein manifold. It can be easily seen that every Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

In 2009, A. K. Gazi and U. C. De [12] introduced the notion of a Riemannian manifold of *nearly quasi-constant curvature* as a Riemannian manifold with the curvature tensor satisfying the condition

$$\begin{aligned} R(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + q[g(X,W)B(Y,Z) - g(X,Z)B(Y,W) \\ &\quad + g(Y,Z)B(X,W) - g(Y,W)B(X,Z)] \end{aligned} \tag{1.3}$$

where $p$, $q$ are scalar functions and $B$ is a non-zero symmetric tensor of type $(0,2)$.

A non-flat Riemannian manifold $(M^n, g)$ $(n > 2)$ is defined to be a *nearly quasi-Einstein manifold* if its Ricci tensor satisfies the condition

$$S(X,Y) = ag(X,Y) + bE(X,Y),$$

where $a$ and $b$ are non zero scalar functions and $E$ is a non-zero symmetric tensor of type $(0,2)$ [11]. It can be easily seen that every Riemannian manifold of nearly quasi-constant curvature is a *nearly quasi-Einstein manifold*.

It is known that the outer product of two covariant vectors is a covariant tensor of type $(0,2)$ but the converse is not true, in general. Hence a Riemannian manifold of quasi-constant curvature is a manifold of nearly quasi-constant curvature, but there are existence of manifolds of nearly quasi-constant curvature which are not of quasi-constant curvature. It can be easily seen that a conformally flat manifold of dimension $> 3$ is a manifold of nearly quasi-constant curvature since the Ricci tensor $S$ is a symmetric $(0,2)$ tensor. But the converse is not necessarily true, in general. On the other hand, a manifold of quasi-constant curvature is conformally flat. Also, we can construct examples of a manifold of nearly quasi-constant curvature which is not a manifold of quasi-constant curvature. Hence, a Riemannian manifold of nearly quasi-constant curvature is a more general idea than a Riemannian manifold of quasi-constant curvature.

*Example 1.1.* Let us consider a Riemannian metric $g$ on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, \quad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{\frac{1}{3}}$$

$$R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{\frac{2}{3}}},$$

$$R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{\frac{2}{3}}$$

and the components obtained by the symmetry properties.

The non-vanishing components of the Ricci tensors are:

$$R_{11} = R_{22} = R_{33} = \frac{2}{3(x^4)^{\frac{2}{3}}}, \quad R_{44} = -\frac{2}{3(x^4)^2}.$$

The scalar curvature of the resulting manifold $(\mathbb{R}^4, g)$ is

$$g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44} = \frac{4}{3(x^4)^2},$$

which is non-vanishing and non-constant.

Let us now consider the associated scalars as follows:

$$p = -\frac{2}{9(x^4)^2}, \quad q\frac{1}{3(x^4)^{\frac{2}{3}}}. \tag{1.4}$$

We choose the associated nonzero symmetric $(0,2)$ tensor $B$ as follows:

$$\begin{aligned} B_{ij}(x) &= (x^4)^2, \qquad \text{for } i = j = 1,2,3 \\ &= -(x^4)^{\frac{2}{3}}, \quad \text{for } i = j = 4, \end{aligned}$$

for any point $x \in \mathbb{R}^4$.

In terms of local coordinates, the defining condition of a nearly quasi-constant curvature can be written as

$$R_{ijkl} = p[g_{jk}g_{il} - g_{ik}g_{jl}] + q[g_{jk}B_{il}g_{il}B_{jk} - g_{ik}B_{jl} - g_{jl}B_{ik}], \tag{1.5}$$

for $i, j, k, l = 1, 2, 3, 4$.

By virtue of (1.4) and choice of the $(0,2)$ tensor $B$, it can be easily seen that equation (1.5) holds for $i, j, k, l = 1, 2, 3, 4$. Therefore, $(\mathbb{R}^4, g)$ is a manifold of nearly quasi-constant curvature [11].

We shall now show that this manifold is not a manifold of quasi-constant curvature.

If possible, suppose this manifold is of quasi-constant curvature. Then in terms of local coordinates, the curvature tensor $R$ of type $(0,4)$ can be written as

$$Rijkl = p[g_{jk}g_{il} - g_{ik}g_{jl}] + q[g_{jk}A_iA_l + g_{il}A_jA_K - g_{ik}A_jA_l - g_{jl}A_iA_k],$$

for $i, j, k, l = 1, 2, 3, 4$, where $p$, $q$ are scalars of which $q \neq 0$ and $A$ is a non-zero 1-form.

Now, for $i = l = 1$, $j \neq k$ and $j, k \neq 1$, we have

$$R_{1jk1} = p[g_{jk}g_{11} - g_{1k}g_{1j}] + q[g_{jk}A_1A_1 + g_{11}A_jA_K - g_{1k}A_jA_1 - g_{j1}A_1A_k],$$

which implies,

$$0 = 0 + qA_jA_k,$$

which is a contradiction. Hence, the assumption is wrong. So, the manifold is not a manifold of quasi-constant curvature.

*Example 1.2.* Let $\tilde{\nabla}$ be a linear connection in an $n$-dimensional differentiable manifold $M$. The torsion tensor $T$ is given by

$$T\,(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y].$$

The connection $\tilde{\nabla}$ is symmetric if its torsion tensor $T$ vanishes, otherwise it is non-symmetric. If there is a Riemannian metric $g$ in $M$ such that $\tilde{\nabla} g = 0$, then the connection $\tilde{\nabla}$ is a metric connection, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi–Civita connection.

A linear connection $\tilde{\nabla}$ is said to be a *semi-symmetric connection* [16] if its torsion tensor $T$ is of the form

$$T(X, Y) = \omega(Y)X - \omega(X)Y, \tag{1.6}$$

where the 1-form $\omega$ is defined by

$$\omega(X) = g(X, U),$$

and $U$ is a vector field.

If $\nabla$ is the Levi–Civita connection of a Riemannian manifold $M$, then we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X, Y)U, \tag{1.7}$$

where

$$\omega(X) = g(X, U),$$

and $X$, $Y$, $U$ are vector fields on $M$ [16]. Let $R$ and $\widetilde{R}$ denote the Riemannian curvature tensor of $\nabla$ and $\widetilde{\nabla}$, respectively. Then from [16] we know that

$$\begin{aligned}\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) &- \theta(Y, Z)g(X, W) + \theta(X, Z)g(Y, W) \\ &- g(Y, Z)\theta(X, W) + g(X, Z)\theta(Y, W),\end{aligned} \tag{1.8}$$

where

$$\theta(X,Y)=g(AX,Y)=(\nabla_X\omega)Y-\omega(X)\omega(Y)+\frac{1}{2}g(X,Y). \tag{1.9}$$

Assume that $\omega$ is a closed 1-form. Then $(\nabla_X\omega)Y=(\nabla_Y\omega)X$. Hence $\theta$ is a symmetric $(0,2)$-tensor field. Now let $M(c)$ be a Riemannian space of constant curvature $c$. If $M(c)$ has a semi-symmetric metric connection with closed associated 1-form $\omega$, then the curvature tensor of $M(c)$ with respect to the semi-symmetric metric connection is

$$\begin{aligned}\widetilde{R}(X,Y,Z,W)&=c(g(Y,Z)g(X,W)-g(X,Z)g(Y,W))\\&\quad-\theta(Y,Z)g(X,W)+\theta(X,Z)g(Y,W)\\&\quad-g(Y,Z)\theta(X,W)+g(X,Z)\theta(Y,W).\end{aligned}$$

Then $M(c)$ is a space of nearly quasi-constant curvature with respect to the semi-symmetric metric connection.

In [5]–[8], B. Y. Chen established some sharp inequalities between intrinsic invariants like Ricci curvatures and the squared mean curvatures, an extrinsic invariant in a submanifold immersed in a Riemannian manifold. Afterwards, various authors studied the inequality in different ambient spaces, for example, see [1], [2], [13], [14] and references therein.

Recently, in [15], the first author studied Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature. In the present paper, we generalize the results of the paper [15] to submanifolds of a Riemannian manifold of nearly quasi-constant curvature.

## 2. Preliminaries

Let $M$ be an $n$-dimensional submanifold of an $(n+m)$-dimensional Riemannian manifold $N^{n+m}$. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_XY=\nabla_XY+h(X,Y)\quad\text{and}\quad\widetilde{\nabla}_XN=-A_NX+\nabla^\perp_XN$$

for all $X,Y\in TM$ and $N\in T^\perp M$, where $\widetilde{\nabla}$, $\nabla$ and $\nabla^\perp$ are the Riemannian, induced Riemannian and normal connections in $\widetilde{M}$, $M$ and the normal bundle $T^\perp M$ of $M$, respectively, and $h$ is the second fundamental form related to the shape operator $A$ by $g(h(X,Y),N)=g(A_NX,Y)$. The equation of Gauss is given by

$$\begin{aligned}\widetilde{R}(X,Y,Z,W)&=R(X,Y,Z,W)-g(h(X,W),h(Y,Z))\\&\quad+g(h(X,Z),h(Y,W))\end{aligned} \tag{2.1}$$

for all $X, Y, Z, W \in TM$, where $R$ is the curvature tensor of $M$. The mean curvature vector $H$ is given by $H = \frac{1}{n}\,\mathrm{trace}(h)$.

Using (1.3), the Gauss equation for the submanifold $M^n$ of a Riemannian manifold of nearly quasi-constant curvature is

$$\begin{aligned} R(X,Y,Z,W) = {} & p\,[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q[g(X,W)B(Y,Z) \\ & - g(X,Z)B(Y,W) + g(Y,Z)B(X,W) - g(Y,W)B(X,Z)] \\ & + g\,(h(X,W),h(Y,Z)) - g\,\,(h(X,Z),h(Y,W))\,. \end{aligned} \tag{2.2}$$

Let $\pi \subset T_xM^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of $M^n$. For any orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space $T_xM^n$, the scalar curvature $\tau$ at $x$ is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

We recall the following algebraic lemma:

**Lemma 2.1** ([4]). *Let $a_1, a_2, \dots, a_n, b$ be $(n+1)$ $(n \ge 2)$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

*Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3 = \dots = a_n$.*

Let $M^n$ be an $n$-dimensional Riemannian manifold, $L$ a $k$-plane section of $T_xM^n$, $x \in M^n$, and $X$ a unit vector in $L$.

We choose an orthonormal basis $\{e_1, \dots, e_k\}$ of $L$ such that $e_1 = X$.

Ones define [6] the *Ricci curvature* (or *$k$-Ricci curvature*) of $L$ at $X$ by

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \dots + K_{1k},$$

where $K_{ij}$ denotes, as usual, the sectional curvature of the 2-plane section spanned by $e_i$, $e_j$. For each integer $k$, $2 \le k \le n$, the Riemannian invariant $\Theta_k$ on $M^n$ is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} \mathrm{Ric}_L(X), \quad x \in M^n,$$

where $L$ runs over all $k$-plane sections in $T_xM^n$ and $X$ runs over all unit vectors in $L$.

## 3. Chen first inequality

In this section, we study submanifolds of a Riemannian manifold of nearly quasi-constant curvature and find Chen first inequality.

**Theorem 3.1.** *Let $M^n, n \geq 3$, be an $n$-dimensional submanifold of an $(n+m)$ -dimensional Riemannian manifold of nearly quasi-constant curvature $^{n+m}$. Then we have:*

$$\tau - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{p}{2}\right] + q\left[(n-2)\lambda + \operatorname{trace} B|_{\pi^\perp}\right], \quad (3.1)$$

*where $\pi$ is a 2-plane section of $T_xM^n, x \in M^n$ and $\lambda = \operatorname{trace} B$. The equality case of inequality (3.1) holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $T_xM^n$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{n+m}\}$ of $T_x^\perp M^n$ such that the shape operators of $M^n$ in $N^{n+m}$ at $x$ have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a+b=\mu,$$

$$A_{e_{n+i}} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 2 \leq i \leq m,$$

*where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n+1 \leq r \leq n+m$.*

Proof. Let $x \in M^n$ and $\{e_1, e_2, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{n+m}\}$ be an orthonormal basis of $T_xM^n$ and $T_x^\perp M^n$, respectively. For $X = W = e_i, Y = Z = e_j$, from the Gauss equation (2.2) it follows that

$$\begin{aligned} R(e_i, e_j, e_j, e_i) &= p\left[g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\right] \\ &\quad + q\left[g(e_i, e_i)B(e_j, e_j) - g(e_i, e_j)B(e_j, e_i)\right. \\ &\quad \left.+ g(e_j, e_j)B(e_i, e_i) - g(e_j, e_i)B(e_i, e_j)\right] \\ &\quad + g\left(h(e_i, e_i), h(e_j, e_j)\right) - g\left(h(e_i, e_j), h(e_j, e_i)\right). \end{aligned} \quad (3.2)$$

By summation after $1 \leq i, j \leq n$, from the previous relation we get

$$2\tau + \|h\|^2 - n^2 \|H\|^2 = 2q(n-1)\lambda + (n^2 - n)p, \tag{3.3}$$

where

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

One takes

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1} \|H\|^2 - (n^2 - n)p - 2q(n-1)\lambda. \tag{3.4}$$

Then, from (3.3) and (3.4) we get

$$n^2 \|H\|^2 = (n-1)\big(\|h\|^2 + \varepsilon\big). \tag{3.5}$$

Let $x \in M^n$, $\pi \subset T_x M^n$, $\dim \pi = 2$, $\pi = sp\{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and using (3.5) we obtain:

$$\left( \sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = (n-1) \left( \sum_{i,j=1}^{n} \sum_{r=n+1}^{n+m} (h_{ij}^r)^2 + \varepsilon \right),$$

or equivalently,

$$\left( \sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = (n-1) \left\{ \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+m} (h_{ij}^r)^2 + \varepsilon \right\}. \tag{3.6}$$

By the use of Lemma 2.1 in view of (3.6):

$$2h_{11}^{n+1} h_{22}^{n+1} \geq \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+m} (h_{ij}^r)^2 + \varepsilon. \tag{3.7}$$

Gauss equation for $X = W = e_1$, $Y = Z = e_2$ gives us

$$\begin{aligned} K(\pi) = R(e_1, e_2, e_2, e_1) &= p + q\,[B(e_1, e_1) + B(e_2, e_2)] \\ &+ \sum_{r=n+1}^{m} [h_{11}^r h_{22}^r - (h_{12}^r)^2] \geq p + q\,[B(e_1, e_1) + B(e_2, e_2)] \\ &+ \frac{1}{2} \left[ \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+m} (h_{ij}^r)^2 + \varepsilon \right] \end{aligned}$$

$$+\sum_{r=n+2}^{n+m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+m} (h_{12}^r)^2 = p + q\,[B(e_1,e_1) + B(e_2,e_2)]$$

$$+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{n+m}(h_{ij}^r)^2 + \frac{1}{2}\varepsilon + \sum_{r=n+2}^{n+m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+m}(h_{12}^r)^2$$

$$= p + q\,[B(e_1,e_1) + B(e_2,e_2)] + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{n+m}\sum_{i,j>2}(h_{ij}^r)^2$$

$$+\frac{1}{2}\sum_{r=n+2}^{n+m}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2}\varepsilon$$

$$\geq p + q\,[B(e_1,e_1) + B(e_2,e_2)] + \frac{\varepsilon}{2},$$

which implies

$$K(\pi) \geq p + q\,[B(e_1,e_1) + B(e_2,e_2)] + \frac{\varepsilon}{2}. \tag{3.8}$$

By the use of (3.4), from (3.8), we find

$$K(\pi) \geq \tau - (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{p}{2}\right] - q[(n-2)\lambda + \operatorname{trace} B|_{\pi^\perp}]$$

hence we obtain (3.1).

The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

$$h_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$h_{ij}^r = 0, \quad \forall i \neq j, i, j > 2,\ r = n+1, \ldots, n+m,$$

$$h_{11}^r + h_{22}^r = 0, \quad \forall r = n+2, \ldots, n+m,$$

$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \quad \forall j > 2,$$

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}.$$

We may chose $\{e_1,e_2\}$ such that $h_{12}^{n+1} = 0$ and we denote by $a = h_{11}^r, b = h_{22}^r, \mu = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}$.

It follows that the shape operators take the desired forms. □

## 4. *k*-Ricci curvature

In this section, we consider the $k$-Ricci curvature which is an intrinsic invariant and find a relation with the squared mean curvature $\|H\|^2$.

**Theorem 4.1.** *Let $M^n, n \geq 3$, be an $n$-dimensional submanifold of an $(n+m)$ -dimensional space of nearly quasi-constant curvature $N^{n+m}$. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - p - \frac{2q}{n}\lambda. \tag{4.1}$$

PROOF. Let $x \in M^n$ and $\{e_1, e_2, \ldots, e_n\}$ be orthonormal basis of $T_xM^n$. From (3.3) we can write

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - 2q(n-1)\lambda - (n^2 - n)p. \tag{4.2}$$

We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}\}$ at $x$ such that $e_{n+1}$ is parallel to the mean curvature vector $H(x)$ and $e_1, \ldots, e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix}, \tag{4.3}$$

$$A_{e_r} = (h_{ij}^r),\ i, j = 1, \ldots, n;\ r = n+2, \ldots, n+m, \operatorname{trace} A_r = 0. \tag{4.4}$$

From (4.2), we get

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - 2q(n-1)\lambda - (n^2 - n)p. \tag{4.5}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = \left( \sum_{i=1}^{n} a_i \right)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2, \tag{4.6}$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2 .$$

From (4.5) we get

$$n^2 \|H\|^2 \geq 2\tau + n \|H\|^2 - 2q(n-1)\lambda - (n^2 - n)p \tag{4.7}$$

or, equivalently,

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - p - \frac{2q}{n}\lambda,$$

this proves the theorem. □

**Theorem 4.2.** *Let $M^n, n \geq 3$, be an $n$-dimensional submanifold of an $(n+m)$ -dimensional Riemannian manifold of nearly quasi-constant curvature $N^{n+m}$. Then for any integer $k$, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2 \geq \Theta_k(x) - p - \frac{2q}{n}\lambda. \tag{4.8}$$

Proof. Let $\{e_1, \dots e_n\}$ be an orthonormal basis of $T_x M$. Denote by $L_{i_1 \dots i_k}$ the $k$-plane section spanned by $e_{i_1}, \dots, e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1 \dots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} \mathrm{Ric}_{L_{i_1 \dots i_k}}(e_i),$$

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \tau(L_{i_1 \dots i_k}).$$

From (4.1) and the above relations, one derives

$$\tau(x) \geq \frac{n(n-1)}{2} \Theta_k(x),$$

which implies (4.8). □

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